Math 161 0 - Probability, Fall Semester 2012-2013
Dan Abramovich

Expected values. Consider a random variable $X : \Omega \to \mathbb{R}$.

Definition 0.0.1. The *expected value* of X is

$$E(X) \quad := \quad \int_\Omega X(\omega)\ f(\omega) d\omega$$

(continuous case) or

$$E(X) \quad := \quad \sum_\Omega X(\omega)\ m(\omega)$$

(discrete case), provided it converges absolutely.

One can think of it as "the average outcome of X".

Lemma

$$E(X) = \int_{x \in X(\Omega)} x f_X(x) dx.$$

(continuous)

$$E(X) = \sum_{x \in X(\Omega)} x P(X = x).$$

(discrete)

Example: one fair coin toss, $X =$ 1 if heads, 0 otherwise

$$E(X) = (1/2) \times 1 + 0 = 1/2.$$

General Bernoulli trial with probability p:

$$E(X) = p \times 1 + q \times 0 = p.$$

result of one throw of fair die:

$$E(X) = 1/6(1+2+3+4+5+6) = 3.5$$

Uniform density on $[a, b]$:

$$\begin{aligned} E(X) &= \int_a^b \frac{1}{b-a} x \, dx \\ &= \frac{1}{b-a} \left(x^2/2 \right|_a^b \\ &= \frac{b+a}{2}. \end{aligned}$$

Theorem. $E(X+Y) = E(X) + E(Y)$ and $E(cX) = cE(X)$.

Proof (discrete):

$$
\begin{aligned}
E(X+Y) &= \sum (X(\omega) + Y(\omega))m(\omega) \\
&= \sum \left(X(\omega)m(\omega) + Y(\omega)m(\omega)\right) \\
&= \sum X(\omega)m(\omega) + \sum Y(\omega)m(\omega) \\
&= E(X) + E(Y)
\end{aligned}
$$

and similarly for the rest.

Note: holds whether or not X, Y independent.

Binomial: $S_n = X_1 + \cdots + X_n$ where X_i are bernoulli with probability p. So

$$E(S_n) = \sum E(X_i) = \sum p = \boxed{np}.$$

Fixed points of a random permutation of size n: $F = \sum_{i=1}^{n} X_i$ where $X_i = 1$ if i is fixed and 0 otherwise, so Bernoulli with probability $1/n$.

$E(F) = E(\sum X_i) = \sum E(X_i) = \sum 1/n = \boxed{1}$.

records in a random permutation $R = \sum R_j$ where R_j counts j being a record, Bernouli with probability $1/j$. So

$$E(R) = \sum_{1}^{n} E(R_j) = \boxed{\sum_{1}^{n} 1/j}.$$

Geometric: want to compute

$$E(T) = \sum_{k=1}^{\infty} kq^{k-1}p.$$

Claim: $E(T) = \boxed{1/p}$.

First derivation: $X_i = 1$ if and only if $T \geq i$. Then $T = \sum X_i$. Each X_i is Bernoulli with probability q^{i-1}. So $E(T) = \sum E(X_i) = \sum q^{i-1} = 1/(1-q) = 1/p$

Second derivarion: $1/(1-x) = \sum_{i=0}^{\infty} x^i$. So $1/(1-x)^2 = \sum_{i=1}^{\infty} ix^{i-1}$. So $\sum_{k=1}^{\infty} kq^{k-1}p = p\sum_{k=1}^{\infty} kq^{k-1} = p/(1-q)^2 = 1/p$

Negative binomial:

$S_{k,p}$ = time for k successes in a sequence of p trials.

Write $T_i = S_i - S_{i-1}$. Then T_i are independent exponentials with p. So

$$E(S_k) = \sum_{i=1}^{k} E(T_i) = \boxed{k/p}.$$

Hypergeometric: H = number of red balls in n pulls out of an urn of k red and N_k blue. $X_i = 1$ if the i-th ball is red.

$E(X_i) = k/N$,

so $E(H) = E(\sum X_i) = \boxed{nk/N}$.

Poisson. Since
$Poisson(\lambda) = \lim Binomial(n, \lambda/n)$
we expect that
$E(Poisson(\lambda)) = \lim n\lambda/n = \lambda.$

This can be proven by convergence theorems. Or directly:

$$\sum_{k=0}^{\infty} k \cdot e^{-\lambda}\lambda^k/k!$$

$$= e^{-\lambda}\sum_{k=1}^{\infty} \cdot\lambda^k/(k-1!$$

$$= \lambda e^{-\lambda}\sum_{k=1}^{\infty} \lambda^{k-1}/(k-1)!$$

$$= \lambda e^{-\lambda}e^{\lambda} = \boxed{\lambda}$$

Theorem. Assume X, Y are independent. Then $E(X \cdot Y) = E(X) \cdot E(Y)$.

Proof (discrete):

$$E(X \cdot Y) = \sum_{x,y} xyP(X = x, Y = y)$$

$$= \sum_{x,y} xyP(X = x)P(Y = y)$$

$$= \sum_{x,y} (xP(X = x))\,(yP(Y = y))$$

$$= \sum_{x} xP(X = x) \sum_{y} yP(Y = y)$$

$$= E(X)E(Y).$$

Compare two independent coins with one coin!

Conditional expectation.

$$E(X|F) = \sum x_i P(X = x_j|F).$$

Theorem. If

$$\Omega = F_1 \sqcup \cdots \sqcup F_r$$

then

$$E(X) = \sum E(X|F_j)P(F_j).$$

Proof: exchanging summations. **Example:** in a game you either roll a die and get the number, or toss 7 coins and get the number of heads, each case with probability 1/2. What's your expected value?

$E(X) = E(X|D)P(D) + E(X|C)P(C)$
$= 3.5 \times 1/2 + (7 \times 1/2) \times 1/2 = 3.5$

Martingales In playing heads or tails, winning 1 with heads and losing 1 with tails,

$E(S_n \mid S_{n-1} = a) = 1/2 \times (a + 1) + 1/2(a - 1) = a.$

This is called a *fair game* or **Martingale**.

Examples

Variance

If $E(X) = \mu$ we define $V(X) = E\left((x-\mu)^2\right)$ and $\sigma(X) = \sqrt{(V(X))}$ - the variance and standard deviation of X.

Example

$V(0-1 \text{ fair coin toss})$
$= 1/2 \times (0-1/2)^2$
$+ 1/2 \times (1-1/2)^2 = 1/4.$
So $\sigma = 1/2$.

Theorem. $V(X) = E(X^2) - \mu^2$.

Proof:

$$\begin{aligned} V(X) &= E\left((x-\mu)^2\right) \\ &= E(X^2 - 2\mu X + \mu^2) \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2\mu^2 + \mu^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

Example Roll of die: $\mu = 3.5$, $E(X^2) = (1^2 + \cdots + 6^2)/6 = 91/6$, so $V = 91/6 - 49/4 = 35/12$.

Theorem $V(c\,X) = c^2\,V(X); V(X+c) = V(X)$.

Proof: easy!

Theorem If X, Y are **independent** then $V(X+Y) = V(X) + V(Y)$.

Proof.

$$
\begin{aligned}
V(X+Y) &= E\left((X+Y)^2\right) - E(X+Y)^2 \\
&= E(X^2) + 2E(XY) + E(Y^2) \\
&\quad - (E(X) + E(Y))^2 \\
&= E(X^2) + \boxed{2E(X)E(Y)} + E(Y^2) \\
&\quad - E(X)^2 - \boxed{2E(X)E(Y)} - E(Y)^2 \\
&= \boxed{E(X^2) - E(X)^2} + \boxed{E(Y^2) - E(Y)^2} \\
&= V(X) + V(Y)
\end{aligned}
$$

Theorem. if X_i are independent with same μ, σ and if $S_n = \sum_{i=1}^{n} X_i$ (sum), $A_n = S_n/n$ (average) and $S_n^* = \frac{S_n - n\mu}{\sqrt{n}\sigma}$ (normalized sum) then

$$E(S_n) = n\mu;\ V(S_n) = n\sigma^2$$
$$E(A_n) = \mu;\quad V(A_n) = \sigma^2/n.$$
$$E(S_n^*) = 0;\quad V(S_n^*) = 1$$

Proof.

$\bullet E(S_n) = E(\sum X_i)$
$= \sum E(X_i) = \sum \mu = \boxed{n\mu}$.

$\bullet V(S_n) = \boxed{V(\sum X_i)} = \boxed{\sum V(X_i)}$
$= \sum \sigma^2 = \boxed{n\sigma^2}$.

$\bullet E(A_n) = E(S_n/n) = E(S_n)/n =$
$n\mu/n = \boxed{\mu}$.

$\bullet V(A_n) = V(S_n/n) = V(S_n)/n^2 =$
$n\sigma^2/n^2 = \boxed{\sigma^2/n}$.

Bernoulli.

$E(B(p)) = p,$

$V(B(p)) = E(B(p)^2) - E(B(p))^2$

$= p - p^2 = \boxed{pq}.$

Binomial

$E(S_n) = np.$

$V(S_n) = nV(X_i) = \boxed{npq}.$

Hypergeometric challenge:

Calculate

$$V(H(N,k,n)) = \boxed{\frac{kn(N-k)(N-n)}{N^2(N-1)}}.$$

Geometric.

$E(T) = 1/p$.

$E(T^2) = \sum k^2 pq^{k-1} = p \sum k^2 q^{k-1}$.

Now $x/(1-x)^2 = \sum kx^k$. differentiation gives

$$\frac{1+x}{(1-x)^3} = \sum k^2 x^{k-1}.$$

So

$E(T^2) = p(1+q)/(1-q)^3 = \boxed{(1+q)/p^2}$.

So

$V(T) = (1+q)/p^2 - 1/p^2 = \boxed{q/p^2}$.

Negative binomial:

$V(S_k) = V(\sum T_i) = \boxed{kq/p^2}$

Poisson:

$$V(Poisson(\lambda)) = V(\lim(Binomial(n, \lambda/n))$$
$$= \lim(V(Binomial(n, \lambda/n))$$
$$= \lim(npq) = \boxed{\lambda},$$

assuming convergence theorems.

Directly:

$$E(Poisson(\lambda)^2)$$
$$= e^{-\lambda} \sum k^2 \lambda^k / k!$$
$$= e^{-\lambda} \left(\sum k(k-1)\lambda^k/k! + \sum k\lambda^k/k! \right)$$
$$= e^{-\lambda}(\lambda^2 + \lambda).$$

So

$$V(Poisson(\lambda)^2)$$
$$= \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Continuous: Again assuming absolute convergence we take

$$E(X) = \int_{-\infty}^{\infty} X(\omega) f(\omega) d\omega$$
$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

We also have, if $E(X) = \mu$,

$$V(X) = E\left((X-\mu)^2\right)$$
$$= \int_{-\infty}^{\infty} (X(\omega)-\mu)^2 f(\omega) d\omega$$
$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x)\, dx$$

Example X uniform on $[0,1]$ then $E(X) = 1/2$ and $E(X^2) = 1/3$ so $V(X) = 1/12$.

Example: density of the distance $\sqrt{x^2+y^2}$ from the origin on the uniform unit disk is $f_X(x) = 2x$. So $E(X) = \int_0^1 x(2x)dx = 2/3$.

Linearity still holds

Independence Theorem still holds: X, Y independent then

$$E(XY) = E(X)E(Y)$$

and

$$V(X+Y) = V(X) + V(Y).$$

Exponential: $f_X(x) = \lambda e^{-\lambda x}$.

$$E(X) = \int_0^\infty x \cdot \lambda e^{-\lambda x} \, dx$$

$$= -\left(\left(x + \frac{1}{\lambda} \right) e^{-\lambda x} \Big|_0^\infty \right) = \boxed{1/\lambda}$$

$$V(X) = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x} \, dx - 1/\lambda^2$$

$$= \ldots = \boxed{1/\lambda^2}$$

Standard Normal: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

$$\begin{aligned} E(X) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2}\, dx \\ &= \boxed{0} \end{aligned}$$

by symmetry.

$$\begin{aligned} V(X) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx - 0 \\ &= \ldots = \boxed{1} \end{aligned}$$

by integration by parts and using the fact that $\int f(x)dx = 1$.

General normal:

$X_{\mu,\sigma} = \sigma X_{0,1} + \mu$, so

$E(X_{\mu,\sigma}) = \mu$

$V(X_{\mu,\sigma}) = \sigma^2$.