Math 161 0 - Probability, Fall Semester 2012-2013
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Sums of random variables.

We have used the notations S_n, A_n, and S_n^* for a sum, average, or normalized sum or IID random variables.

Two main theorems: assume $X_i : \Omega \to \mathbb{R}$ are IID with expected value μ and finite *positive* variance. Write δ_μ for the discrete RV which takes value μ with probability 1. Then

Law of large numbers:

$$A_n \xrightarrow[n\to\infty]{P} \delta_\mu.$$

Central limit theorem:

$$S_n^* \xrightarrow[n\to\infty]{d} \mathrm{Normal}(0,1)$$

The meaning of the first convergence is: for any $\epsilon > 0$ we have

$$\lim_{n\to\infty} P(|A_n - \mu| < \epsilon) = 1.$$

The meaning of the second convergence is: for any a, b we have

$$\lim_{n\to\infty} P(a < S_n^* < b) = P(a < X_{0,1} < b).$$

(The law of large number actually holds in a stronger meaning)

Here is an indication that the law of large numbers is natural and important (Borel's law of large numbers):

Corollary: Say you make a series of n samples S_i from a sample space Ω. Let n_E be the number of times the sample lies in an event $E \subset \Omega$. Then

$$\text{“}\lim_{n\to\infty} \frac{n_E}{n} \;=\; P(E).\text{”}$$

Proof: Define $X_i = \begin{cases} 1 & \text{if } S_i \in E \\ 0 & \text{otherwise.} \end{cases}$

Then X_i are bernoulli with probability $P(E)$.

So $A_n = n_E/n$. Then for every ϵ, the probability that

$$\left|\frac{n_E}{n} - P(E)\right| \;<\; \epsilon$$

approaches 1.

Need to talk about:

- sums of independent random variables
- convergence
- proof
- applications

Suppose X, Y are independent, with integer values, and $Z = X + Y$.

$$\begin{aligned} P(Z = n) &= P(X + Y = n) \\ &= \sum_{k=-\infty}^{\infty} P(X = k, Y = n - k) \\ &= \sum_{k=-\infty}^{\infty} P(X = k) P(Y = n - k) \end{aligned}$$

by independence.

This motivates the definition of a convolution: given distribution functions m_1, m_2 on the integers, we define

$$m_1 * m_2(n) := \sum_{k=-\infty}^{\infty} m_1(k) \cdot m_2(n - k).$$

Example of a die: $m(i) = 1/6$).

$$m * m(n) = \begin{cases} \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 \le n \le 12. \end{cases}$$

Try out the simulation in the book's web pages!

Suppose now X, Y are continuous independent with $f(x,y) = f_X(x)f_Y(y)$. Then

$$f_X * f_Y(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,dx$$
$$= \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)\,dy$$

Theorem 0.0.1. *If* $Z = X + Y$ *then* $f_Z(z) = f_X * f_Y(z)$.

Proof: $P(Z < z)$
$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_X(x) f_Y(y)\, dx$. So its derivative is
$\int_{x=-\infty}^{\infty} \frac{d}{dz} \left(\int_{y=-\infty}^{z-x} f_X(x) f_Y(y)\ dx \right)$.

By the fundamental theorem of calculus we get what we wanted.

Uniform [0.1]:

$$u * u(z) = \begin{cases} z & 1 > z \\ 2 - z & 1 < z < 2 \end{cases}$$

Two exponentials:

$$(\lambda e^{-\lambda x}) * (\lambda e^{-\lambda x})(z) = \lambda^2 z e^{-\lambda z}$$

Challenge: do it for different exponentials!

$$\lambda\mu(e^{-\mu} - e^{-\lambda})/(\lambda - \mu)$$

Two normals:

$N_{0,1} * N_{0,1} = (1/2\pi) \int e^{-(z-y)^2/2} e^{-y^2/2} dy$

Note that

$(z-y)^2 + y^2 = 2(y - z/2)^2 + z^2/4.$

So

$N_{0,1} * N_{0,1} = (1/2\pi) e^{-z^2/4} \int e^{-(y-z/2)^2} dy$

The integral is $\sqrt{\pi}$ times $\int N(2y, \sqrt{2}) dy$

So $f_Z(z) = \frac{1}{2\sqrt{\pi}} e^{-z^2/4}$, namely $N(0, \sqrt{2})$.

Challenge: you know it!