The Jordan-Hölder Theorem

Lemma. Let $G$ be a group with $A \neq B$ normal in $G$ such that $G/A, G/B$ are simple then:

$$G/A \simeq B/(A \cap B) \quad G/B \simeq A/(A \cap B)$$

Proof. Suppose that $A \subset B$ then $B/A$ is normal in the simple group $G/A$. Since $A$ is not equal to $B$ the quotient is not trivial, and by the assumption that $G/B$ is simple neither is it the whole group. This is a contradiction, so we can assume $A \not\subset B$ and by symmetry $B \not\subset A$.

Consider $AB$ a normal subgroup of $G$, its image under the quotient map, $AB/A$ will be a normal subgroup of $G/A$. However from $B \not\subset A$ we have that $AB/A \neq \{e\}$ and so since $G/A$ is simple we must have $AB/A = G/A$. Finally from the second isomorphism theorem we conclude:

$$B/(A \cap B) \simeq AB/A = G/A$$

By symmetry that $A/(A \cap B) \simeq G/B$.

\[ \square \]

Theorem. Let $G$ be a group and assume $G$ has a decomposition series. Let

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_r = \{e\}$$
$$G = H_0 \triangleright H_1 \triangleright \ldots \triangleright H_s = \{e\}$$

Be any two decomposition series for $G$ then $r = s$ and there exists $\sigma \in S_r$ such that $\forall ~ k$:

$$G_k/G_{k+1} \simeq H_{\sigma(k)}/H_{\sigma(k)+1}$$

Proof. We use induction over the length of shortest decomposition series for $G$. It is sufficient to show that any decomposition series is equivalent to a minimal series, and therefore that any two series are equivalent. If $G$ is simple then it has a unique decomposition series $G \triangleright \{e\}$. For the inductive case assume that:

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_r = \{e\}$$

is a minimal composition series for $G$. Suppose that $G_1 = H_1$ then by induction the series starting from $G_1$ will be equivalent to the series starting from $H_1$, and therefore the whole series will be as well. Otherwise let $K = H_1 \cap G_1$ which is normal in $G$. By the lemma we have that $G_1/K \simeq G/H_1$ and $H_1/K \simeq G/G_1$ are simple.

Let $K_i := K \cap G_i$ then $G_i \triangleright K_i$ and $K_i \triangleright K_{i+1}$. Consider the homomorphism $K_i \to G_i/G_{i+1}$ given by the quotient map. The image is normal and the kernel is $K_{i+1}$, therefore by the isomorphism theorems we have that $K_i/K_{i+1}$ is a normal subgroup of $G_i/G_{i+1}$. Furthermore since $G_i/G_{i+1}$ is simple for each $K_i, K_{i+1}$ either $K_i = K_{i+1}$ or the quotient $K_i/K_{i+1}$ is simple. By removing duplicates we get two decomposition series for $G_1$:

$$G_1 \triangleright G_2 \triangleright \ldots \triangleright G_r = \{e\}$$
$$G_1 \triangleright K_1 \triangleright \ldots \triangleright K_r = \{e\}$$

By induction on $G_1$ these series are equivalent, and in particular must have the same length, $r - 1$, so exactly one of the groups $K_i/K_{i+1}$ is trivial.
We have already shown that $H_1 \triangleright K_1$ with a simple quotient and therefore we also have two composition series for $H_1$:

$$H_1 \triangleright H_2 \triangleright \ldots \triangleright H_s = \{e\}$$

$$H_1 \triangleright K_1 \triangleright \ldots \triangleright K_r = \{e\}$$

Since exactly one of the groups $K_i/K_{i+1}$ is trivial we conclude that $H_1$ also has a decomposition series of length $r-1$ which is less than that of $G$. Therefore by induction these series are equivalent with $s-1 = r - 1$.

It is therefore sufficient to show that the series:

$$G \triangleright G_1 \triangleright K_1 \triangleright \ldots \triangleright K_r = \{e\}$$

$$G \triangleright H_1 \triangleright K_1 \triangleright \ldots \triangleright K_r = \{e\}$$

Are equivalent. By the lemma $G/G_1 \simeq H_1/K_1$ and $G/H_1 \simeq G_1/K_1$ and clearly $K_i/K_{i+1} \simeq K_i/K_{i+1}$ therefore this is the case.

\[\square\]

References
