Math 52 0 - Linear algebra, Spring Semester 2012-2013
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Review

We saw: an **algorithm** - row reduction, bringing to reduced echelon form - answers the questions:

- **consistency** (no pivot on right)
- **dimension** (count free variables)
- **write** all solutions:

• free variables **arbitrary**

• **basic** variable x_i appearing as pivot in row k: the equation says

←reduced echelon...

$$x_i + a_{k\,i+1}x_{i+1} + \dots a_{kn}x_n = b_k$$

$$x_i = b_k - (a_{k\,i+1}x_{i+1} + \dots a_{kn}x_n).$$

Theorem said: reduced echelon form is **unique**.

people asked if we get the same thing!!→

Say x_j is a free variable. Set $x_j = 1$ and all other free variables to be 0.

we may!→

This determines all the basic variables. We get the equation

$$x_i + 0 + \cdots + 0 + \boxed{a_{kj} \cdot 1} + 0 + \cdots = b_k,$$

$$x_i + \boxed{a_{kj} \cdot 1} = b_k,$$

which means

$$\boxed{a_{kj}} = b_k - x_i.$$

This determines U, **given which variables are free!**

and which basic→

To determine that it is best to think in terms of *vectors*.

Vectors

A *column* vector is a matrix with one column. Here is a 3-vector:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The order of the entries does matter.

Basic operations:

1. addition - you add two vectors of the same size and get again a vector of the same size:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} e \\ f \\ g \end{bmatrix} = \begin{bmatrix} a+e \\ b+f \\ c+g \end{bmatrix}$$

do example→

2. rescaling, or multiplication by a scalar:

$$d \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} da \\ db \\ dc \end{bmatrix}$$

do example→

Notation issues: different texts and different people use different notations. I grew up with $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Vectors are also denoted in different ways:

$$\vec{x}, \overline{x}, \underline{x} \text{ or } \mathbf{x}.$$

Get used to it!

Given a coordinate system, 2-vectors correspond to points in the plane. 3-vecotrs to points in space. We can interpret the basic operations of addition and rescaling - you may have seen this in calculus.

examples!→

The vector of all zeros is the zero vector, denoted **0**

but we will imagine→

n-vectors? we give up imaging them.

Physicists know that they do appear in nature. So do computer scientists.

The operations have the usual properties (p27):

$$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$$
$$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$$
$$\mathbf{u}+\mathbf{0}=\mathbf{u}=\mathbf{0}+\mathbf{u}$$
$$c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$$
$$(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$$
$$c(d\mathbf{u})=(cd)\mathbf{u}$$
$$1\mathbf{u}=\mathbf{u}.$$

we'll see these as axioms for a **vector space** a most important notion→

linear combination.

we say that the vector

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$$

is a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

example!→

The set of *all* linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is their **span**.

Example:

All plane vectors are linear combinations of $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Question: is the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ a linear combination of $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{a}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

Namely are there x_1, x_2 so that we have the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}?$$

←do this

A vector equations

$$x_1\mathbf{a}_1 + \cdots x_n\mathbf{a}_n = \mathbf{b}$$

is equivalent to the system of linear equations with augmeted matrix

$$\begin{bmatrix} \vdots & \vdots & & \vdots & \vdots \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix}$$

It answer the question:

In what ways is $\mathbf{b}$ a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$?

Uniqueness of reduced echelon form, one more step:

We need to determine basic variables, so pivot columns.

In an echelon form U, the k-th column is a pivot column exactly if it is not a linear combination of the previous columns.

still need to show: the same holds on A!!

The equation $A\mathbf{x} = \mathbf{b}$

Definition

Say $m \times n$ matrix A has columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

Then $A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$

Do this:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Note: matching n, result is an m vector.

justify notation→

The system of linear equations with augmeted matrix

$$\begin{bmatrix} \vdots & \vdots & & \vdots & \vdots \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{b} \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix}$$

Is equivalent to the vector equation

$$\boxed{A\mathbf{x} = \mathbf{b}.}$$

Do this

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Theorem. A an $m \times n$ matrix. The following are equivalent:

(1) $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$

(2) every $\mathbf{b}$ is a linear combination of columns of A

columns span $\mathbb{R}^m \rightarrow$

(3) The echelon form of A has a pivot in every row.

Proof.

alternative rules

row rule: $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$ then

$$A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1\mathbf{x} \\ \vdots \\ \mathbf{r}_m\mathbf{x} \end{bmatrix}$$

Completely explicitly

$$A\mathbf{x} = \begin{bmatrix} \sum a_{1j}x_j \\ \sum a_{2j}x_j \\ \vdots \\ \sum a_{nj}x_j \end{bmatrix}$$

linearity of operation

Theorem:

$$A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$$

$$A(c\mathbf{u}) = cA\mathbf{u}$$

Proof. (1)

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n\end{bmatrix} \left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right)$$

$$= \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n\end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

$$= (u_1 + v_1)\mathbf{a}_1 + \cdots + (u_n + v_n)\mathbf{a}_n$$

$$= (u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n)$$

$$+(v_1\mathbf{a}_1 + \cdots + v_n\mathbf{a}_n)$$

$$= A\mathbf{u} + A\mathbf{v}$$

Vector form of solutions

We looked at the system given by augmented matrix

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The original system is $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \\ 1 & 4 & 5 & -9 \end{bmatrix}; \; b = \begin{bmatrix} 9 \\ 1 \\ -1 \\ -7 \end{bmatrix}$$

Start with the associated **homogeneous** equation

$$A\mathbf{x}_h = \mathbf{0}.$$

which has reduced echelon form

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and solutions $x_1 = 3x_3,\ x_2 = -2x_3,\ x_4 = 0.$

In vector form:

$$\mathbf{x}_h = \begin{bmatrix} 3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$= x_3\mathbf{v}_1$$

at least there is always the trivial solution 0!

The original system has solutions $x_1 = 5+3x_3, x_2 = -3-2x_3, x_4 = 0.$

$$\mathbf{x} = \begin{bmatrix} 5+3x_3 \\ -3-2x_3 \\ x_3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$=: \boxed{\mathbf{p} + x_3\mathbf{v}_1}$$

This is called a papratmetric vector solution.

Do this: $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \left(x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

Always: "particular" + "general homogeneous".

Theorem. If $\mathbf{p}$ is one solution of $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}$ is a solution if and only if it is of the form $\mathbf{x} = \mathbf{p} + \mathbf{x}_h$, where $\mathbf{x}_h$ satisfies $A\mathbf{x}_h = \mathbf{0}$.

Proof.

Example: a chemical reaction combines a number of Al_2O_3 molecules and a number of C atoms resulting in an amount of Al and an amount of CO_2. What can we say about these amounts?

$$x_1 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{0}, \text{ where } A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 3 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

(Do this)

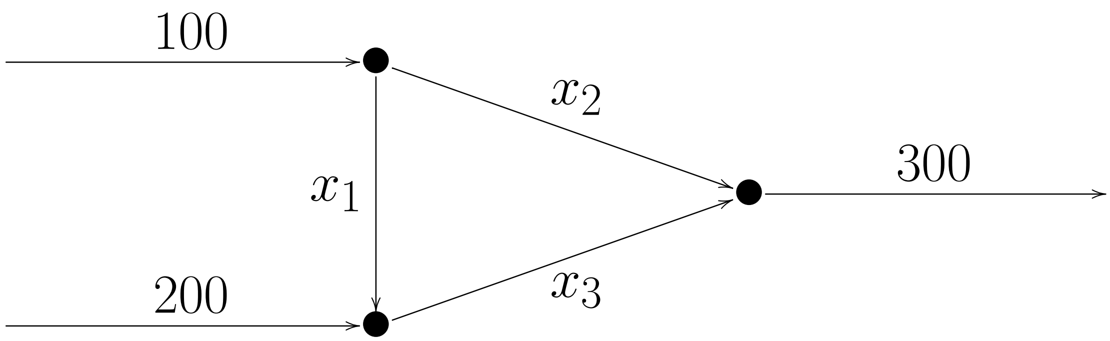
Reduced echelon: $\begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4/3 \end{bmatrix}$

so $\mathbf{x} = x_4 \begin{bmatrix} 2/3 \\ 1 \\ 4/3 \\ 1 \end{bmatrix}$

To get integers take $x_4 = 3$ and get

$2Al_2O_3 + 3C \quad \rightarrow \quad 4Al + 3CO_2.$

netweork:



$$\begin{aligned} x_1 + x_2 \quad\quad &= \quad 100 \\ x_1 \quad\quad - x_3 &= -200 \\ x_2 + x_3 &= \quad 300 \end{aligned}$$

The bottom two are the reduced echelon

$$\begin{aligned} x_1 \quad\quad - x_3 &= -200 \\ x_2 + x_3 &= \quad 300 \end{aligned}$$

$$\mathbf{x} = \begin{bmatrix} -200 \\ 300 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

INDEPENDENCE

Consider the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}.$$

Definition. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are

- **linearly independent** if the only solution is $\mathbf{x} = \mathbf{0}$, and
- **linearly dependent** if there is a nonzero solution.

←discuss explicitly

Examples: dependent or independent?

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}?$$

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}?$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}?$$

When is $\mathbf{v}$ linearly dependent?

What does it mean for $\mathbf{v}_1, \mathbf{v}_2$ to be linearly dependent?

Similar for $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to be linearly dependent?

Theorem

$\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent if and only if at least one of them is in the span of the others.

Proof.

←do it now

Columns of $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$?

Proposition. The columns of A are linearly independent if and only if

$$A\mathbf{x} = \mathbf{b}$$

has only the trivial solution.

do it now→ **Proof.**

Theorem If $\mathbf{v}_i$ are m-vectors, and $n > m$, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

Proof.

←do it now