Math 52 0 - Linear algebra, Spring Semester 2012-2013
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Review: we understood linear transformations and matrices.

2.1 Matrix operations

Matrices behave in many ways like vectors:

you can add two matrices of the same dimensions term by term:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11} + b_{11}) & \cdots & (a_{1n} + b_{1n}) \\ \vdots & \ddots & \vdots \\ (a_{m1} + b_{m1}) & \cdots & (a_{mn} + b_{mn}) \end{bmatrix}$$

or multiply by a scalar:

$$c\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix}$$

do this now→ Example:

$$2\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right) =$$

There is a more delicate operation - matrix multiplication. The rule for multiplying matrices AB is similar to that of multiplying a vector $A\mathbf{x}$:

First the matrix B should have as many **rows** as A has **columns**. So you multiply an $m \times n$ matrix A by an $n \times k$ matrix B $\leftarrow$draw it

If the columns of B are $\mathbf{b}_1, \ldots \mathbf{b}_k$ then

$$AB = \begin{bmatrix} A\mathbf{b}_1 & \ldots & A\mathbf{b}_k \end{bmatrix}.$$

"the j-th column of AB is A multiplied by the jth column of B"

do it now→ Example

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} =$$

Justification for definition: think about composing linear transformations.

This is the only way we get

$A(B\mathbf{x}) = (AB)\mathbf{x}$ ←prove it

Other rules, in particular **the Row column rule**

draw it→

$$(AB)_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$$

Also the row rule:

"The i-th row of AB is the i-th row of A multiplied by B."

Wonderful:

Theorem

$$A(BC) = (AB)C$$
$$A(B + C) = AB + AC$$
$$(A + B)C = AC + BC$$
$$c(AB) = (cA)B = A(cB)$$

The proof of the first is interesting! $\leftarrow$do it

The idenity matrix I_n

The square matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

behaves a bit like the number 1:

Theorem:

$$AI_n = A, \quad I_nB = B$$

whenever the sizes fit!

The transpose of a matrix

A^T is obtained by flipping around the diagonal.

←draw it

$$\begin{bmatrix} & & a_{1j} & & & \\ & & \vdots & & & \\ & & \vdots & & & \\ a_{i1} & \cdots & \boxed{a_{ij}} & \cdots & \cdots & a_{in} \\ & & \vdots & & & \\ & & a_{mj} & & & \end{bmatrix}^T$$

$$= \begin{bmatrix} & & & a_{i1} & & \\ & & & \vdots & & \\ a_{1j} & \cdots & \cdots & \boxed{a_{ij}} & \cdots & a_{mj} \\ & & & \vdots & & \\ & & & \vdots & & \\ & & & a_{in} & & \end{bmatrix}$$

so $(A^T)_{ji} = A_{ij}$

Example

do it now→

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}^T =$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}^T =$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}^T =$$

Theorem:

$$(AB)^T = B^T A^T$$

2.2 The inverse of a square matrix It is interesting to find an analogue of a reciprocal of a matrix. Can only do it if the matrix is square:

Defnition: an $n \times n$ matrix A is invertible if there is a matrix V such that

temporary name→

$$AV = VA = I_n.$$

Also known as nonsingular (if not invertible it is "singular")

The matrix V is then unique, and is denoted

$$A^{-1}$$

For 2×2 there is a simple formula, which you get by solving algebraic equations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In fact the matrix is invertible precisely when the 2×2 **determinant**

$$\boxed{\det A = ad - bc}$$

is **nonzero**.

In higher dimensions, determinants exist, are important, but are a terrible computational tool in general.

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = ?$$

do these now→

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^{-1} = ?$$

Theorem Suppose A is invertible. Then the matrix equation $A\mathbf{x} = \mathbf{b}$ always has solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Solve

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

←do these now

Proof.

Theorem

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} =$$

what is it?$\rightarrow$

$$(A^T)^{-1} = (A^{-1})^T$$

Elementary matrices: what you get if you do **one** row operation to I_n:

e.g. a replacement

$$E_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{5} & 0 & \cdots & 1 \end{bmatrix}$$

a switch

$$E_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \boxed{0} & \cdots & \boxed{1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \boxed{1} & \cdots & \boxed{0} \end{bmatrix}$$

rescaling

$$E_3 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \boxed{3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Key observation Consider elementary matrix E of a certain row operation. Then EA is the matrix obtained from A by the same row operation.

E.G.:

do it→

$$\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$$

Elementary matrices are invertible!

do E_i→

Theorem a $n \times n$ matrix A is invertible if and only if it is row equivalent to I_n.

Proof ←two ways!

Algorithm:

A is invertible if and only if the matrix

$$[\, A \;\; I \,]$$

is row reduced to the form

$$[\, I \;\; A^{-1} \,].$$

(same as solving $A\mathbf{x} = \mathbf{e}_i$ for all $\mathbf{e}_i$)

do it→ $$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix} \Rightarrow$$

2.3
When is a matrix invertible?

Theorem. T.F.A.E. for $n \times n$ matrix A:

(1) A is invertible
(2) A is row equivalent to I_n
(3) A has n pivot positions
(4) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$
(5) $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent
(6) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is 1-to-1.
(7) $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^n$.
(8) $Span(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbb{R}^n$
(9) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.
(10) There is C such that $CA = I_n$
(11) There is D such that $AD = I_n$
(12) A^T is invertible

one of many paths→

Instructor's notes:

(1),(2),(3) are equivalent from the last theorem.

(3),(4),(5) are equivalent as discussed about independence

(6),(7),(8) are equivalent as discussed about linear combinations

(1) and (12) are equivalent by transpose theorem

(3) and (4) are equivalent by linear equations

(3) and 7) are equivalent by linear equations

(1) implies (10), and (10) implies (4) by plugging in

(12) implies (11) by transposing twice.

do them→

(11) implies (7) by plugging in.

Theorem A linear transformation is invertible if an only if the associated matrix is.

Proof.

Block matrices / partitioned matrices:

If two matrices are subdivided in a way that can be multiplied, you can multiply as written:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]$$

Note: keep the order!

Most important case: the inverse of a block diagonal matrix:

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{bmatrix}$$

where A_{jj} are square (and all the 0s are really matrices!). Then

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk}^{-1} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} =$$

My architect friends use linear algebra without knowing it:

they need to describe at the very least

(1) rotations
(2) reflections
(3) rescaling
(4) Shears - occur as more exotic transformations for special effects.
(5) parallel projections
(6) translations
(7) Stereographic (perspective) projections

rotations in the plane:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotation around, say y axis in space:

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

(Other axes come with the theory of eigenvalues!)

Reflection, say through x axis in plane:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

through $x = y$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Through $x = y$ in space:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

parallel projection on $x-y$ plane in space:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

what about translation etc??

Trick: add new coordinate 1 (put everything in a plane at distance 1).

These are "homogeneous" or "projective" coordinates.

The plane: $(x, y) \mapsto (x, y, 1)$.

Then translation by (a, b) becomes a linear transfomation:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \boxed{\qquad} \\ \boxed{\qquad} \\ \boxed{\qquad} \end{bmatrix}$$

Can do the same in space - move 1 away in 4th dimension!

The other transformation don't really change - only another row/column with 1 in corner.

Geometers think about $(x, y, 1)$ as representing the whole line of view (X, Y, H) where $x = X/H$ and $y = Y/H$.

Read about perspective projections. This is the beginning of **projective geometry.**

Projective geometry is the beginning of **algebraic geometry,** my subject.