Math 52 0 - Linear algebra, Spring Semester 2012-2013
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Determinants

The *determinant* of a *square* matrix A determines whether or not it is invertible.

We have seen the 2×2 determinant:
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

In multivariable calculus we teach about 3×3 determinants:

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$+ a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

If we write A_{ij} for the matrix with row i and column j removed, this reads:

$$\det A =$$

examples and pictures→

$$a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}.$$

In general one defines for an $n \times n$ matrix:

$\det A =$

$a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n}$
$= \sum (-1)^{1+j} a_{1j} \det A_{1j}$. ←do 4×4

you can expand by any row. It is convenient to define the **cofactors**

$C_{ij} = (-1)^{i+j} \det A_{ij}$

and then

$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$

Also

$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$

To prove this, you can go through a formula using generalized diagonals, which we will skip.

You almost never calculate large determinants by expansion! We'll see how to do it faster.

The one and only easy case:

If A is triangular, then

$$\det A = a_{11} \cdot a_{22} \cdots a_{nn}$$

This works for echelon forms, and two types of elementary matrices!

Determinants and row operations

Theorem 3a

Start with A.

If you make a replacement - add a multiple of row i to another row j, getting B, then

$$\det B = \det A$$

In terms of matrices:

$$\det(EA) = \det A$$

Note that $\det E = 1$ so

$$\det(EA) = \det E \ \det A$$

Theorem 3b

Start with A.

If you switch row i and row j getting B, then

$$\det B = -\det A$$

In terms of matrices:

$$\det(EA) = -\det A$$

this time $\det E = -1$ so

$$\det(EA) = \det E \;\; \det A$$

Theorem 3c

Start with A.

If you multiply row i by constant c getting B, then

$$\det B = c \det A$$

In terms of matrices:

$$\det(EA) = c \det A$$

this time $\det E = c$ so

$$\det(EA) = \det E \ \det A$$

Calculating the determinant

Say you bring A to echelon form U, no rescalings. Say you used r switches precisely.

Then $\det A = (-1)^r \det U$.

So:

- If A is singular, $\det A = 0$.

- If A is invertible,

$\det A = (-1)^r$(product of pivots in U)

Two beautiful theorems:

Theorem A is invertible if and only if $\det A \neq 0$.

Theorem $\det AB = \det A \det B$.

Proof:

Cramer's rule for $A\mathbf{x} = \mathbf{b}$.

Denote by $A_i(b)$ the matrix where the i-th column of A is replaced by $\mathbf{b}$.

example $I_i(\mathbf{b})$→

Theorem. Assume A invertible. Then

$$\boxed{x_i = \frac{\det A_i(\mathbf{b})}{\det A}}$$

example→

Proof

a formula for the inverse

$$(A^{-1})_{ij} = \frac{\det A_i(\mathbf{e}_j)}{\det A}$$

but $\det A_i(\mathbf{e}_j) = C_{ji}$ ←draw it

Write adj(A) for the matrix whose ij-entry is C_{ji}
(**transpose** the matrix of C's)
Then

$$\boxed{A^{-1} = \frac{1}{\det A}\mathrm{adj}(A)}$$

What does the determinant signify geometrically?

2×2 :

example→ $\left| \det A \right|$ signifies area of parallelogram.

3×3 :

draw them→ $\left| \det A \right|$ signifies area of whatsit.

explain!→ **Reason: elementary matrices!**

Consequence: S a finite area plane region, T a linear transformation. Then

$$Area(T(S)) \quad = \quad \det T \cdot Area(S).$$

←explain

This is the reason for the change of variable formula!