Math 52 0 - Linear algebra, Spring Semester 2012-2013
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Eigenvectors and eigenvalues

Fix an $n \times n$ matrix A.

Definition 0.0.1. A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is an **eigenvector** with eigenvalue λ if

←examples, stretching

$$A\mathbf{v} = \lambda\mathbf{v}$$

What are all the eigenvectors with eigenvalue 0?

What are all the eigenvectors with eigenvalue λ?

Definition 0.0.2. The **eigenspace** of A corresponding to λ is the set of eigenvectors with eigenvalue λ.

This is $\mathbf{Null}\ (A - \lambda I)$.

example→ It is a verctor space!

Theorem 0.0.3. *The eigenvalues of a triangular matrix are the diagonal elements.*

We'll see this easily in a couple of slides.

Theorem 0.0.4. *If* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *are eigenvectors with distinct eigenvalues* $\lambda_1, \ldots, \lambda_k$ *then* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *are linearly independent.*

be prepared!→

How do we find these?

Eigenvectors are non-zero elements in **Null** $(A - \lambda I)$. So $(A - \lambda I)$ is **singular.**

So

$$\boxed{\det(A - \lambda I) = 0}$$

Note that $\det(A - \lambda I)$ is a polynomial in λ of degree n it is called the **characteristic polynomial** of A.

Theorem 0.0.5. *A scalar λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial.*

$$\boxed{\det(A - \lambda I) = 0}$$

examples: diagonal, triangular, 2×2, $3 \times 3 \rightarrow$

Once we have an eigenvalue λ, we solve for $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Similarity

A matrix A is **similar** to a matrix B if there is an invertible matrix P such that $B = P^{-1}AP$, equivalently $A = PBP^{-1}$.

Why do we care?

Theorem 0.0.6. *If A and B are similar they have the same characteristic polynomial.* ←invertibility etc

Proof. Note that

$$P^{-1}(A - \lambda I)P = P^{-1}AP - P^{-1}\lambda IP$$
$$= B - \lambda I.$$

So $\det(B - \lambda I) =$

$$= \det(P^{-1}(A - \lambda I)P)$$
$$= \det(P^{-1}) \det(A - \lambda I) \det P$$
$$= (\det P)^{-1} \det(A - \lambda I) \det P$$
$$= \det(A - \lambda I)$$

Definition 0.0.7. A square matrix A is diagonalizable if it is similar to a diagonal matrix D.

Theorem 0.0.8. *An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.*

prove using $AP = PD$. $\rightarrow$

Suppose these are $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Write

$$D = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$$

and

$$P = [\mathbf{v}_1 \cdots \mathbf{v}_n].$$

Then

$$A = PDP^{-1}.$$

Theorem 0.0.9. *An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.*

←lots of examples

Raising a matrix to k-th power:

If $A = PDP^{-1}$ then

$$A^k = P\,D^k\,P^{-1}.$$

For instance for Fibonacci:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Eigenvalues are

$\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$

$P = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix}$

$P^{-1} = \frac{-1}{\sqrt{5}} \begin{bmatrix} \beta & -1 \\ -\alpha & 1 \end{bmatrix}$

Need to worry about:

- missing (complex) roots
- repeated roots

With repeated roots a matrix might fail to be diagonalizable. For instance

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

For non-diagonalizable matrices we have a **Jordan canonical form**: $A = PJP^{-1}$ where J has eigenvalues on the diagonal, some 1's right above the diagonal, and zeros otherwise. We won't study these.

Complex numbers:

$\mathbb{C} = \{a + bi\}$ where $i^2 = -1$. ←draw them up

"real" and "imaginary" parts

rule for addition: like vectors!

rule for multiplication:

$(a + bi)(c + di) =$ ←do it, examples

Absolute value: $z = a + bi$, then $|z| =$

Conjugate: $\bar{z} = a - bi$, so $|z|^2 = z\bar{z}$.

non-real solutions of a real quadratic are complex conjugate: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

and vice versa: $z^2 - 2bz + (a^2 + b^2) = 0$

Angle: called ϕ

What is the absolute value of $\cos\phi + i\sin\phi$?

Every complex number can be written as

$z = |z|(\cos\phi + i\sin\phi)$.

Why don't mathematicians remember the triple angle formula?

Answer 1:

$(\cos\phi + i\sin\phi)(\cos\psi + i\sin\psi) = \cos(\phi+\psi) + i\sin(\phi+\psi)$

Answer 2:

$e^{i\phi} = \cos\phi + i\sin\phi$

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix}$$ ←read book examples

$$D = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

If A is a real 2×2 matrix with complex eigenvalue $\lambda = a - bi, b \neq 0$ and eigenvector $\mathbf{v}$, then

$$A = QCQ^{-1}$$

where

$$Q = [\Re e\, \mathbf{v}, \Im m\, \mathbf{v}] \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

5.4 The matrix of a linear transformation

Recall:

If V is a vector space and $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ a basis, then the mapping $V \to \mathbb{R}^n$ which takes $\mathbf{v}$ and gives its coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ is an invertible linear transformation.

For instance it sends $\mathbf{b}_1$ to $\mathbf{e}_1$.

If V itself is $\mathbb{R}^n$ then the mapping $[\mathbf{v}]_{\mathcal{B}} \mapsto \mathbf{v}$ is given by $P_{\mathcal{B}} = [\mathbf{b}_1 \ldots \mathbf{b}_n]$ and the mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is given by $P_{\mathcal{B}}^{-1}$

We also learned what happened with two bases: the mapping $[\mathbf{v}]_{\mathcal{B}} \mapsto [\mathbf{v}]_{\mathcal{C}}$ is given by

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [[\mathbf{b}_1]_{\mathcal{C}} \ldots [\mathbf{b}_n]_{\mathcal{C}}] = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}.$$

Now assume we have a linear transformation $T : V \to W$ and bases $\mathcal{B}$ of V and $\mathcal{C}$ of W.

How can we describe T in coordinates?

$$\begin{tikzcd}
V \arrow[r, "T"] & W \\
\mathbb{R}^n \arrow[u, "P_{\mathcal{B}}"] \arrow[r, dotted, "?"] & \mathbb{R}^m \arrow[u, "P_{\mathcal{C}}"']
\end{tikzcd}$$

The mystery function is a linear transformation, right?

$P_{\mathcal{C}}^{-1} T P_{\mathcal{B}} \to$

So it is given by a matrix. What is it?

$$\begin{tikzcd} V \arrow[r, "T"] & W \\ \mathbb{R}^n \arrow[u, "P_{\mathcal{B}}"] \arrow[r, dotted, "?"] & \mathbb{R}^m \arrow[u, "P_{\mathcal{C}}"'] \end{tikzcd}$$

Follow the trail:

$$\mathbf{e}_i \mapsto \mathbf{b}_i \mapsto T(\mathbf{b}_i) \mapsto [T(\mathbf{b}_i)]_{\mathcal{C}}$$

so

$$\boxed{M = \big[[T(\mathbf{b}_1)]_{\mathcal{C}} \cdots [T(\mathbf{b}_n)]_{\mathcal{C}}\big]}$$

$\longleftarrow$example $\mathbb{P}_2 \to \mathbb{P}_3$

When $V = W$ we may take $\mathcal{C} = \mathcal{B}$ and then

example $\mathbb{P}_2 \to \mathbb{P}_2 \to$

$$\boxed{M = \big[[T(\mathbf{b}_1)]_{\mathcal{B}} \cdots [T(\mathbf{b}_n)]_{\mathcal{B}}\big]}$$

We write $M = [T]_{\mathcal{B}}$ and note

$$[T(\mathbf{v})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$$

Again

$$[T]_{\mathcal{B}} = P_{\mathcal{B}}^{-1} T P_{\mathcal{B}}$$

We can now interpret a diagonalization $A = PDP^{-1}$ or

$$D = P^{-1}AP.$$

D is the matrix $[A]_{\mathcal{B}}$ of A in terms of the basis of eigenvectors $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$!

5.6 Linear discrete dynamics.

We assume that a system is characterized by a state vector $\mathbf{x}_i$ at time i, and evolves by the rule

$$\mathbf{x}_{i+1} = A\mathbf{x}_i.$$

We have seen the purely mathematical Fibonacci:

$$\mathbf{x}_i = \begin{bmatrix} F_i \\ F_{i+1} \end{bmatrix}; \qquad \mathbf{x}_{i+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}_i.$$

Predator-Prey: this is a first attempt to model populations of Owls O_k and Rats R_k in month k.

$$\begin{aligned} O_{k+1} &= 0.7O_k + 0.4R_k \\ R_{k+1} &= -0.1O_k + 1.2R_k \end{aligned}$$

The matrix $A = \begin{bmatrix} 0.7 & 0.4 \\ -0.1 & 1.2 \end{bmatrix}$ has ←diagonalize

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The matrix $A = \begin{bmatrix} 0.7 & 0.4 \\ -0.1 & 1.2 \end{bmatrix}$ has

eigenvalue 1.1 with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

eigenvalue 0.8 with eigenvector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

If $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$

and $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ ←example

then $\mathbf{x}_k = A^k\mathbf{x}_0$

$= c_1(1.1)^k\mathbf{v}_1 + c_2(0.8)^k\mathbf{v}_2.$ ←draw trajectories

Three cases (positive eigenvalues):

1. Attractor: $0 < \lambda_1, \lambda_2 < 1$

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3. Saddle: $0 < \lambda_1 < 1 < \lambda_2$

Linear differential equations with constant coefficients

$$\begin{aligned}
x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n
\end{aligned}$$

becomes

$$\mathbf{x}' = A\mathbf{x}.$$

The equation is linear - both the derivative and matrix multiplication are.

so the sulutions form a vector space:

$$\operatorname{Ker}\left(\frac{d}{dt} - A\right)$$

One function: $x' = ax$
we solved in calculus:

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$x = ce^{at}$

Note: $c = x(0)$.

Diagonal:

$$
\begin{aligned}
x_1' &= 2x_1 \\
x_2' &= \qquad 3x_2
\end{aligned}
$$

Diagonal:

$$x_1' = 2x_1$$
$$x_2' = \qquad 3x_2$$

Solution:

$$x_1 = c_1 e^{2t}$$
$$x_2 = c_2 e^{3t}$$

so

$$\mathbf{x} = c_1 e^{2t}\mathbf{e}_1 + c_2 e^{3t}\mathbf{e}_2.$$ ←linear

$$\mathbf{x} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$ ←$c =$

Three cases for 2×2:

1. Attractor: $0 < \lambda_1, \lambda_2 < 1$
2. repeller: $1 < \lambda_1, \lambda_2$
3. Saddle: $0 < \lambda_1 < 1 < \lambda_2$

$$\mathbf{x}' = \begin{bmatrix} 0.7 & 0.4 \\ -0.1 & 1.2 \end{bmatrix} \mathbf{x}$$

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The matrix $A = \begin{bmatrix} 0.7 & 0.4 \\ -0.1 & 1.2 \end{bmatrix}$ has

eigenvalue 1.1 with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

eigenvalue 0.8 with eigenvector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

If

$$\mathbf{x}(t) = y_1(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2(t) \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

then

$$\mathbf{x}' = y_1' \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2' \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

also

$$A\mathbf{x} = y_1 A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2 A \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

so

$$A\mathbf{x} = 1.1\, y_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; + \; 0.8\, y_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$y_1' \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y_2' \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 1.1\, y_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.8\, y_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Since the vectors are independent we get

$$\begin{aligned} y_1' &= 1.1 y_1 \\ y_2' &= \qquad\qquad 0.8 y_2 \end{aligned}$$

In general: suppose $A = PDP^{-1}$ with $D = diag(\lambda_1, \ldots, \lambda_n)$ and $P = [\mathbf{v}_1 \cdots \mathbf{v}_n]$

You can write $\mathbf{x} = P\mathbf{y}$ and get $\mathbf{y}' = D\mathbf{y}$.

The solutions for $\mathbf{y}$ are spanned by

$$e^{\lambda_i t}\mathbf{e}_i$$

Then the solution for $\mathbf{x}$ are spanned by

$$e^{\lambda_i t}\mathbf{v}_i$$

...the solution for $\mathbf{x}$ are spanned by

$$e^{\lambda_i t}\mathbf{v}_i$$

If you write

$$e^{Dt} = diag(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$$

and

$$e^{At} = Pe^{Dt}P^{-1}$$

remark on complex→

then

$$\mathbf{x} = e^{At}\mathbf{x}_0.$$

The case of $\lambda = a \pm bi$:

A basis for the **real** solutions is

$$\mathbf{y}_1(t) = (\Re e\,\mathbf{v})e^{at}\cos(bt) - (\Im m\,\mathbf{v})e^{at}\sin(bt)$$

$$\mathbf{y}_2(t) = (\Re e\,\mathbf{v})e^{at}\sin(bt) + (\Im m\,\mathbf{v})e^{at}\cos(bt)$$