Math 52 0 - Linear algebra, Spring Semester 2012-2013
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The diagonalization of symmetric matrices.

This is the story of the eigenvectors and eigenvalues of a symmetric matrix A, meaning $A = A^T$.

It is a beautiful story which carries the beautiful name *the spectral theorem*:

Theorem 1 (The spectral theorem)**.** *If A is an $n \times n$ symmetric matrix then*

(1) All eigenvalues of A are real.
(2) A is orthogonally diagonalizable: $A = PDP^T$ where P is an orthogonal matrix and D is real diagonal.

There are immediate important consequences:

Corollary 2. *If A is an $n \times n$ symmetric matrix then*

(1) A has an orthogonal basis of eigenvectors $\mathbf{u}_i$.
(2) (spectral decomposition)
$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$
(3) The dimension of the λ eigenspace is the multiplicity of λ as a root of $\det(A - \lambda I)$.
(4) Different eigenspaces are orthgonal to each other

In fact a matrix A is orthogonally diagonalizable if and only if it is symmetric.

(The name *the spectral theorem* is inspired by another story of the inter-relationship of math and physics.)

The first part is directly proved:

Proposition 3. *The eigenvalues of a symmetric matrix are real.*

Here are key geometric facts:

Proposition 4. *If A is symmetric and $\mathcal{B} = \{\mathbf{u}_1 \ldots, \mathbf{u}_n\}$ is an orthonormal basis then $[A]_{\mathcal{B}}$ is symmetric.*

Proposition 5. *If P and Q are orthogonal matrices then PQ is also orthogonal.*

Proof of Corollary 2(4). If $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$ then

$$(A\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_2^T A^T \mathbf{v}_1 = (A\mathbf{v}_1)^T \mathbf{v}_2 = \mathbf{v}_1^T A \mathbf{v}_2 = \mathbf{v}_1 \cdot (A\mathbf{v}_2)$$

so $\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \cdot \mathbf{v}_2$.
Since $\lambda_1 \neq \lambda_2$ we get $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. ♣

Proof of last statement in Corollary 2. Assume $A = PDP^T$ with P orthogonal. Then

$$A^T = (P^T)^T D^T P^T = PDP^T = A.$$

♣

In the declared pauses, do these as time permits

Orthogonally diagonalize $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Spectrally decompose $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

Orthogonally diagonalize $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & 0 \\ 5 & 0 & 1 \end{bmatrix}$

Orthogonally diagomalize $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$