MA 254 notes: Diophantine Geometry

(Distilled from [Hindry-Silverman])

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Theorem (Canonical heights - Néron-Tate)

Given k a number field, V/k smooth projective variety, $D \in Div(V), \phi : V \to V$ a morphism, suppose $\phi^* D \sim \alpha D$ for $\alpha > 1$. There is a unique $\hat{h}_{V,\phi,D} : V(\bar{k}) \to \mathbb{R}$ such that

(i) $\hat{h}_{V,\phi,D} = h_{V,D} + O(1)$.

(ii) $\hat{h}_{V,\phi,D} \circ \phi = \alpha \hat{h}_{V,\phi,D}$.

Further, $\hat{h}_{V,\phi,D}(P) = \lim_{n\to\infty} h_{V,D}(\phi^n(P))/\alpha^n$.

Proof of Uniqueness

- Say $\hat{h}, \hat{h}'$ both satisfy (i),(ii).
- Then by (i) $g := \hat{h} - \hat{h}'$ is bounded: $|g(P)| < C'$,
- and by (ii) $g \circ \phi = \alpha g$, giving $g(\phi^n(P)) = \alpha^n g(P)$.
- So $|g(P)| = |g(\phi^n(P))|/\alpha^n < C'/\alpha^n \to 0$,
- hence $g = 0$. ♠

Proof of convergence

- We show the latter limit exists by showing the sequence is Cauchy.
- By functoriality $|h_{V,D}(\phi(Q)) - \alpha h_{V,D}(\phi(Q))| \leq C$.
- For $n > m$ we have

$$
\begin{aligned}
(*) \qquad & |h_{V,D}(\phi^n(P))/\alpha^n - h_{V,D}(\phi^m(P))/\alpha^m| \\
&= \left| \sum_{i=m+1}^{n} (h_{V,D}(\phi^i(P)) - \alpha h_{V,D}(\phi^{i-1}(P)))/\alpha^i \right| \\
&\leq \sum_{i=m+1}^{n} \left|(h_{V,D}(\phi^i(P)) - \alpha h_{V,D}(\phi^{i-1}(P)))/\alpha^i\right| \\
&\leq \sum_{m+1}^{n} C/\alpha^i = C(\alpha^{-m} - \alpha^{-n})/(\alpha - 1) \to 0 \quad \spadesuit
\end{aligned}
$$

We thus define $\hat{h}_{V,\phi,D}(P) = \lim_{n\to\infty} h_{V,D}(\phi^n(P))/\alpha^n$.

Proof of (i)

Taking $m = 0$ and $n \to \infty$ in (*) gives
$\left|\hat{h}_{V,\phi,D}(P) - h_{V,D}(P)\right| \leq C/(\alpha - 1)$ ♠

Proof of (ii)

$$\begin{aligned}\hat{h}_{V,\phi,D}(\phi(P)) &= \lim_{n\to\infty} h_{V,D}(\phi^n(\phi(P)))/\alpha^n \\ &= \alpha \lim_{n\to\infty} h_{V,D}(\phi^{n+1}(P))/\alpha^{n+1} = \hat{h}_{V,\phi,D}(P). \quad ♠\end{aligned}$$

Thus the theorem is proved. ♠

Preperiodic points and Northcott's theorem

Definition

$P \in V$ is preperiodic for ϕ if $\{\phi^n(P)\}$ is finite.

Proposition

Suppose D ample and $\phi^* D \sim \alpha D$ with $\alpha > 1$.

(a) $\hat{h}_{V,\phi,D}(P) \geq 0$, *with equality* $\Longleftrightarrow$ *P is preperiodic.*

(b) *The set of preperiodic k-rational points is finite.*

Proof.

(a)
- Can take $h_{V,D} > 0$ hence $\hat{h}_{V,\phi,D} \geq 0$.
- Assume P preperiodic. $\{\phi^n(P)\}$ is finite so $\{\hat{h}_{V,\phi,D}(\phi^n(P))\}$ finite hence bounded. But then $\hat{h}(P) = \hat{h}(\phi^n(P))/\alpha^n \to 0$, so $\hat{h}(P) = 0$.
- Assume $\hat{h}(P) = 0$. Extend the field so that $P \in V(k)$, hence $\phi^n(P) \in V(k)$. So $h_{V,D}(\phi^n(P)) = \hat{h}_{V,\phi,D}(\phi^n(P)) + O(1) = \alpha^n \hat{h}_{V,\phi,D}(P) + O(1) = O(1)$. So by the finiteness property the set $\{\phi^n(P)\}$ is finite.

(b) By (a) the rational preperiodic points satisfy $\hat{h}_{V,\phi,D}(P) = 0$ hence $h_{V,D}(\phi^n(P)) = O(1)$. By the finiteness property the set is finite.

♠

Canonical heights on abelian varieties

Theorem (Néron-Tate)

Given abelian variety A over number field k and divisor D with $O_A(D)$ symmetric, there is $\hat{h}_{A,D} : A(\bar{k}) \to \mathbb{R}$ with

(a) $\hat{h}_{A,D} = h_{A,D} + O(1)$.

(b) $\hat{h}_{A,D}([m]P) = m^2 \hat{h}_{A,D}(P)$.

(c) $\hat{h}_{A,D}(P + Q) + \hat{h}_{A,D}(P - Q) = 2\hat{h}_{A,D}(P) + 2\hat{h}_{A,D}(Q)$

(d) $\hat{h}_{A,D}$ *is a quadratic form on the group* $A(\bar{k})$.

(e) $\hat{h}_{A,D}$ *is independent of the choice of* $D \in |D|$. *It is uniquely determined by (a) and (b) applied to any* $m > 1$.

Construction and proof of (a)

We define $\hat{h}_{A,D} := \hat{h}_{A,[2],D}$. This is possible since $[2]^*D \sim 4D$. Explicitly $\hat{h}_{A,D}(P) = \lim h_{A,D}([2^n]P)/4^n$. Thus (a) follows from (i).

Proof of (c)

- We have seen $h_{A,D}(P+Q) + h_{A,D}(P-Q) = 2h_{A,D}(P) + 2h_{A,D}(Q) + O(1)$.
- The same $O(1)$ applies to $[2^n]P, [2^n]Q$.
- Dividing by 4^n the $O(1)$ evaporates, giving the result in the limit.

End of proof.

A function on an abelian group satisfying the parallelogram law (c) is a quadratic form, giving (b) and (d) (see the book for a direct proof of (b)). Uniqueness of $\hat{h}_{A,[m],D}(P)$ implies (e). ♠

Definition (polarized height pairing)

We define the height pairing $\langle P, Q\rangle_D = \frac{\hat{h}_{A,D}(P+Q)-\hat{h}_{A,D}(P)-\hat{h}_{A,D}(Q)}{2}$.
By (d) it is bilinear, by (b) we have $\langle P, P\rangle_D = \hat{h}_{A,D}(P)$.

Proposition (real positivity of canonical heights)

Assume further D is ample.

(a) $\hat{h}_{A,D}(P) \geq 0$, *with equality* $\Leftrightarrow$ P *torsion.*

(b) *The real extension* $q : A(\bar{k}) \otimes \mathbb{R}$ *of* $\hat{h}_{A,D}$ *is a positive definite quadratic form.*

(b') *If* $x_i \in A(\bar{k}) \otimes \mathbb{R}$ *linearly independent then the hight regulator* $\det(\langle x_i, x_j\rangle_{ij}) > 0$.

Proof of real positivity of canonical heights, first steps

- If $[2^m]P = [2^n]P$ for $m \neq n$, then $[2^m - 2^n]P = O$, hence a preperiodic point is torsion. Part (a) of Northcott gives (a).
- Consider $x \in A(\bar{k}) \otimes \mathbb{R}$ with $\hat{h}_{A,D}(x) = 0$, and write $x = \sum a_i P_i$, with $a_i \in \mathbb{R}$. Extending k we may assume $P_i \in A(k)$.
- Write $\Lambda = \sum \mathbb{Z} P_i \subset A(\bar{k}) \otimes \mathbb{R}$ and $V = \sum \mathbb{R} P_i$. Note that $V = \Lambda \otimes \mathbb{R}$, so $\Lambda \subset V$ is a lattice.
- Note that $q|_\Lambda \geq 0$ with equality only for $\lambda = 0$, since $\ker(A(\bar{k}) \to A(\bar{k}) \otimes \mathbb{R}) = A(\bar{k})_{torsion}$. This is (i) below.

Proof of real positivity of canonical heights, conclusion

Real positivity of canonical heights follows from these two results:

Lemma

The set $\{\lambda \in \Lambda | q(\lambda) \leq B\}$ *is finite.*

Proposition (Real positivity of forms)

Let $q : \mathbb{Z}^r \to \mathbb{R}$ *be a quadratic form such that*

(i) $q(\lambda) \geq 0$ *with equality only for* $\lambda = 0$*, and*

(ii) *for all* B *the set* $\{\lambda \in \mathbb{Z}^r | q(\lambda) \leq B\}$ *is finite.*

Then the extension of $q_{\mathbb{R}}$ *to* $\mathbb{R}^n$ *is positive definite.*

Indeed, by the lemma we have (ii), and (i) was shown before, so the proposition applies giving (b). Statement (b') is an immediate conseuence of (b). ♠

Proof of Lemma

- Say λ is the image of $Q \in A(k)$.
- Since $|\hat{h} - h| < C$ for some C we have $q(\lambda) \leq B \Rightarrow h_{A,D}(Q) \leq B + C$.
- So the set $\{\lambda \in \Lambda | q(\lambda) \leq B\}$ is the image of the finite set $\{Q \in A(K) | h_{A,D}(Q) \leq B + C\}$, as needed. ♠

Lemma (This I won't prove)

- *Let* $q_{s,t} : \mathbb{R}^r \to \mathbb{R}, q(x_1, \ldots, x_r) = \sum_{i=1}^{s} x_i^2 - \sum_{i=s+1}^{s+t} x_i^2$ *be the standard quadratic form of signature* (s, t).
- *Let* $U(L, M) = \{(x_1, \ldots, x_r) | \sum_{i=1}^{s} x_i^2 < L, \sum_{i=s+1}^{s+t} x_i^2 < M.\}$.

Then

(i) $\mathrm{Vol}(U(L, M)) < \infty \Leftrightarrow s + t = r$

(ii) *in which case* $\mathrm{Vol}(U(L, M)) = V_s V_t L^{s/2} M^{t/2}$, *where* V_s *is the volume of the unit ball in* $\mathbb{R}^s$. ♠♠

Proof of real positivity of forms

- By (ii) there are finitely many elements of $\Lambda \smallsetminus \{0\}$ with bounded $q(\lambda)$, and by (i) these values of $q(\lambda)$ are > 0, so they have a minimum $L = \inf\{q(\lambda) | \lambda \in \Lambda, \lambda \neq 0\}$.
- Say the signature of $q_{\mathbb{R}}$ is (s,t), choose an isomorphism with the standard form and consider $U(\delta, \epsilon)$.
- Since $L > 0$ we have that $U(L/2, M) \cap \Lambda = \{0\}$ for all $M > 0$.
- But $U(L/2, M)$ is symmetric and convex. By Minkowski $\mathrm{Vol}(U(L/2, M)) \leq 2^r \,\mathrm{Vol}(V/\Lambda)$ is bounded for all M.
- By the Lemma part (i) $s + t = r$ and by part (ii) $t = 0$, hence $s = r$ and $q_{\mathbb{R}}$ is positive definite. ♠

Antisymmetric canonical height

Theorem

Given A/k as before and divisor D with $O_A(D)$ antisymmetric, there is $\hat{h}_{A,D} : A(\bar{k}) \to \mathbb{R}$ with

(a) $\hat{h}_{A,D} = h_{A,D} + O(1)$ *and*

(c) $\hat{h}_{A,D}(P+Q) = \hat{h}_{A,D}(P) + \hat{h}_{A,D}(Q)$,

so $\hat{h}_{A,D} : A(\bar{k}) \to \mathbb{R}$ is a homomorphism. Furthermore such $\hat{h}_{A,D}$ is unique.

Proof.

Using $[m^*]D \sim mD$ we can again define $\hat{h}_{A,D} := \hat{h}_{A,[2],D}$ and the proof flows as before. ♠

Quadratic functions and combined height

Definition (Quadratic functions)

Let A and $\mathcal{R}$ be abelian groups, with $\mathcal{R}$ uniquely 2-divisible. A quadratic function $h : A \to \mathcal{R}$ is a function satisfying
$\sum_{I \subset \{1,2,3\}} (-1)^{\# I} h(\sum_I a_i) = 0$
for all triples of $a_i \in A$.

This includes constants, linear and quadratic forms (and that's really it).

Theorem

Let A/k as before and $D \in Div(A)$.

(a) $\exists!$ *quadratic* $\hat{h}_{A,D} : A(\bar{k}) \to \mathbb{R}$, *the* *canonical height*, *such that* $\hat{h}_{A,D} = h_{A,D} + O(1)$ *and* $\hat{h}_{A,D}(O) = 0$.

(b) *If* $D \sim D'$ *then* $\hat{h}_{A,D} = \hat{h}_{A,D'}$

(c) $\hat{h}_{A,D+E} = \hat{h}_{A,D} + \hat{h}_{A,E}$.

(d) *For* $\phi : B \to A$ *a morphism of abelian varieties* $\hat{h}_{B,\phi^* D} = \hat{h}_{A,D} \circ \phi - \hat{h}_{A,D}(\phi(O))$.

(e)
- *There is a unique quadratic form* $\hat{q}_{A,D}$ *and unique linear* $\hat{\ell}_{A,D}$ *such that* $\hat{h}_{A,D} = \hat{q}_{A,D} + \hat{\ell}_{A,D}$.
- $\hat{q}_{A,D} = \hat{h}_{A,D^+}/2$ *and* $\hat{\ell}_{A,D} = \hat{h}_{A,D^-}/2$, *where* $D^+ = D + [-1]^* D$ *and* $D^- = D - [-1]^* D$.
- *If* D *symmetric* $\hat{h}_{A,D} = \hat{q}_{A,D}$ *and if* D *antisymmetric* $\hat{h}_{A,D} = \hat{\ell}_{A,D}$.

Sketch proof (a)-(d) ((e) skipped)

(a)
- Define $\hat h_{A,D} := (\hat h_{A,D^+} + \hat h_{A,D^-})/2$.
- $\hat h_{A,D}$ is a quadratic function with $\hat h_{A,D}(O) = 0$.
- $2\hat h_{A,D} = \hat h_{A,D^+} + \hat h_{A,D^-} = h_{A,D^+} + h_{A,D^-} + O(1) = 2\, h_{A,D} + O(1)$ by additivity of Weil heights, hence $\hat h_{A,D} = h_{A,D} + O(1)$.
- If $\hat h'_{A,D}$ another, then $\hat h_{A,D} - \hat h'_{A,D}$ is a bounded quadratic with $f(O) = 0$ so $f = 0$, so $\hat h_{A,D}$ unique.

(b) $D \sim D'$ then $h_{A,D'} = h_{A,D} + O(1)$ so $\hat h_{A,D'}$ satisfies (a), and by uniqueness $\hat h_{A,D} = \hat h_{A,D'}$.

(c) $h_{A,D+E} = h_{A,D} + h_{A,E} + O(1)$ so $\hat h_{A,D} + \hat h_{A,E}$ satisfies (a) for $D + E$, and by uniqueness $\hat h_{A,D+E} = \hat h_{A,D} + \hat h_{A,E}$.

(d) Since ϕ is a translated homomorphism, the function $\hat h_{A,D} \circ \phi - \hat h_{A,D}(\phi(O)) = h_{A,D} \circ \phi + O(1)$, it is quadratic, vanishes at O, so by uniqueness agrees with $\hat h_{B,\phi^* D}$.

Canonical height pairing

Definition

Define the canonical height pairing $A(\bar{k}) \times \hat{A}(\bar{k}) \to \mathbb{R}$ by $[P, Q] = \hat{h}_{A,D_Q}(P)$ for any D_Q representing Q.

Theorem (Canonical height pairing)

- *This is bilinear, kernels are $A_{torsion}$ and $\hat{A}_{torsion}$*
- $[P, Q] = \hat{h}_{A \times \hat{A}, \mathcal{P}}(P, Q)$.

Consequence:

Theorem (Strong algebraic equivalence)

V/k variety over number field, D ample, $E \sim_{alg} 0$. There is c such that $h_{V,E} \leq c\sqrt{h_{V,D}(P) + 1}$

Proof of Strong algebraic equivalence, first steps

- Let $A = Alb(V)$ with $\pi : V \to A$. There is $E_1 \in Div(A), E_1 \sim_{alg} 0$, such that $E = \pi^* E_1$.
- If $D_1 \in Div(A)$ symmetric ample, then there is $a \in A$ such that $E_1 = t_a^* D_1 - D_1$.
- For $P \in V(\bar{k})$ write $Q = \pi(P)$.

$$\begin{aligned} h_{V,E}(P) &= h_{V,\pi^* E_1}(P) = h_{A,E_1}(Q) + O(1) \\ &= \hat{h}_{A,E_1}(Q) + O(1) = \hat{h}_{A,t_a^* D_1}(Q) - \hat{h}_{A,D_1}(Q) + O(1) \\ &= \hat{h}_{A,D_1}(t_a(Q)) - \hat{h}_{A,D_1}(t_a(O)) - \hat{h}_{A,D_1}(Q) + O(1) \\ &= \hat{h}_{A,D_1}(Q + a) - \hat{h}_{A,D_1}(a) - \hat{h}_{A,D_1}(Q) + O(1) \\ &= 2\langle A, a\rangle + O(1) \le 2\sqrt{\hat{h}_{A,D_1}(Q)\hat{h}_{A,D_1}(a)} + O(1) \end{aligned}$$

by Caushy-Schwartz.

Proof of Strong algebraic equivalence, conclusion

- On the other hand
$$\begin{aligned}\hat{h}_{A,D_1}(Q) &= \hat{h}_{A,D_1}(\pi(P))\\ &= \hat{h}_{V,\pi^* D_1}(P).\end{aligned}$$
- But $cD - D_1$ is an effective class since D ample, so $\hat{h}_{V,\pi^* D_1}(P) < c\hat{h}_{V,\pi^* D}(P)$. Plugging in we get the result. ♠

Proof of Canonical Height Pairing

- Recall that for any ample symmetric divisor class D we have a surjective homomorphism with torsion kernel
  $\lambda_D : A \to \hat{A} \qquad P' \mapsto Q := [t^*_{P'}D - D]$.
- Mumford shows $(id, \lambda_D)^*\mathcal{P} = m^*D - \pi_1^*D - \pi_2^*D$, so
  $$\begin{aligned}\hat{h}_{A\times\hat{A},\mathcal{P}}((P,Q)) &= \hat{h}_{A\times A,\ m^*D-\pi_1^*D-\pi_2^*D}((P,P')) \\ &= \hat{h}_{A,D}(P+P') - \hat{h}_{A,D}(P) - \hat{h}_{A,D}(P') = 2\langle P, P'\rangle_D\end{aligned}$$
- Hence $\hat{h}_{A\times\hat{A},\mathcal{P}}((P,Q))$ is bilinear with kernels on both sides being the torsion subgroups.
- Also we had
  $$\begin{aligned}\hat{h}_{A,D}(P+P') - \hat{h}_{A,D}(P) - \hat{h}_{A,D}(P') &= \hat{h}_{A,t^*_{P'}D}(P) - \hat{h}_{A,D}(P) \\ &= \hat{h}_{A,t^*_{P'}D-D}(P) \\ &= \hat{h}_{A,Q}(P) = [P,Q]_A \quad \spadesuit\end{aligned}$$