MA 254 notes: Diophantine Geometry

(Distilled from [Hindry-Silverman])

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Height defined by a morphism

Definition

Let V be projective and $\phi : V \to \mathbb{P}^n$ a morphism. Define $h_\phi(P) := h(\phi(P))$.

Theorem (Linear equivalence propoerty)

If $\phi_i : V \to \mathbb{P}^{n_i}$ and $\phi_i^* H_{\mathbb{P}^{n_i}}$ are linearly equivalent, then $h_{\phi_1}(P) = h_{\phi_2}(P) + O(1)$.

Height defined by a morphism

Proof of Linear Equivalence property

- suffices to consider $\phi_1 = \lambda : V \to \mathbb{P}^N$ associated to a complete linear series, $\phi_1 = \phi$. So there is a projection $\pi : \mathbb{P}^N \to \mathbb{P}^n$ which is regular on $\lambda(V)$ such that $\phi = \pi \circ \lambda$
- By functoriality theorem $h(\pi(Q)) = h(Q) + O(1)$ for $Q \in \lambda(V)(\bar{\mathbb{Q}})$.
- Substituting $Q = \lambda(P)$ we get

$$\begin{aligned} h_\phi(P) = h(\phi(P)) = h(\pi(\lambda(P))) = h(\lambda(P)) + O(1) \\ = h_\lambda(P) + O(1). \quad \spadesuit \end{aligned}$$

Theorem (Weil height machine)

There is a way to associate to a variety V over a number field k and a divisor $D \in Div(V)$ a function $h_{V,D} : V(\bar{k}) \to \mathbb{R}$ so that:

(a) (Normalization) $h_{\mathbb{P}^n,H}(P) = h(P) + O(1)$

(b) (Functoriality) $h_{V,\phi^* D}(P) = h_{W,D}(P) + O(1)$

(c) (Additivity) $h_{V,D+E}(P) = h_{V,D}(P) + h_{V,E}(P) + O(1)$

(d) (Linear equivalence) $D \sim_{lin} E \Rightarrow h_{V,D}(P) = h_{V,E}(P) + O(1)$

(e) (Positivity) $h_{V,D}(P) \geq O(1)$ for $P \in (V \smallsetminus Bs|D|)(\bar{k})$.

(f) (Algebraic equivalence) D ample, $E \sim_{alg} 0$ then
$$\lim_{h_{V,D}(P)\to\infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0.$$

(g) (Finiteness) D ample, $[k' : k] < \infty, B \in \mathbb{R}$
$\Rightarrow$ the set $\{P \in V(k') | h_{V,D}(P) < B\}$ is finite.

(h) (Uniqueness) $h_{V,D}$ determined up to $O(1)$ by Normalization, Functoriality for projective embeddings, and additivity.

Theorem (Quick restatement in terms of line bundles)

To a variety V over a number field k there is a unique homomorphism $h_V : Pic(V) \to \frac{\{\text{functions } f : V(\bar{k}) \to \mathbb{R}\}}{\text{bounded functions}}$ such that if $\mathcal{L} \in Pic(V)$ very ample with $\phi_{\mathcal{L}} : V \to \mathbb{P}^n$, then $h_{V,\mathcal{L}} = h \circ \phi_{\mathcal{L}}$.

These have the additional properties:

(b) (Functoriality) $h_{V,\phi^*\mathcal{L}} = h_{W,\mathcal{L}} \circ \phi$

(e) (Positivity) $h_{V,\mathcal{L}}(P) \geq O(1)$ on $(V \smallsetminus Bs|\mathcal{L}|)(\bar{k})$.

(f) (Algebraic equivalence) $\mathcal{L}$ ample, $\mathcal{M} \sim_{alg} 0$ then
$$\lim_{h_{V,\mathcal{L}}(P) \to \infty} \frac{h_{V,\mathcal{M}}(P)}{h_{V,\mathcal{L}}(P)} = 0.$$

(g) (Finiteness) $\mathcal{L}$ ample, $[k' : k] < \infty, B \in \mathbb{R}, \tilde{h}_{V,\mathcal{L}}$ a representative of $h_{V,\mathcal{L}}$
$\Rightarrow$ the set $\{P \in V(k') | \tilde{h}_{V,\mathcal{L}}(P) < B\}$ is finite.

Construction of $h_{V,D}$

- If $|D|$ is base point free with morphism $\phi_{|D|} : V \to \mathbb{P}^n$, define $h_{V,D}(P) = h_{\phi_{|D|}}(P)$.

Note: this is independent of choice of $\phi_{|D|}$

by the linear equivalence property of $h_{\phi_{|D|}}$.

- In general write $D = D_1 - D_2$ with D_i base point free, and write $h_{V,D}(P) = h_{V,D_1}(P) - h_{V,D_2}(P)$.
- need to show ths is a well defined function modulo $O(1)$, but patience!

Additivity: base point free case

Assume $|D|, |E|$ base point free of dimensions n, m. Then

$$D + E \sim \left(S_{n,m} \circ (\phi_{|D|} \times \phi_{|E|})\right)^* H$$

Hence

$$\begin{aligned} h_{V,D+E} &= h_{S_{n,m} \circ (\phi_{|D|} \times \phi_{|E|})} + O(1) \\ &= h_{\phi_{|D|}} + h_{\phi_{|E|}} + O(1) = h_{V,D} + h_{V,E} + O(1). \quad \spadesuit \end{aligned}$$

Height is well defined - general case

Assume $D = D_1 - D_2 = E_1 - E_2$. Then

$$\begin{aligned} h_{V,D_1} + h_{V,E_2} &= h_{V,D_1+E_2} + O(1) \\ &= h_{V,D_2+E_1} + O(1) = h_{V,D_2} + h_{V,E_1} + O(1). \end{aligned}$$

Hence

$$h_{V,D_1} - h_{V,D_2} = h_{V,E_1} - h_{V,E_2} + O(1). \quad \spadesuit$$

(a) Normalization

follows since $h_{\mathbb{P}^n,H} = h_{|H|} = h_{id_{\mathbb{P}^n}} = h$.

(b) Functoriality

follows in general since if $D = D_1 - D_2$ with $|D_i|$ base point free, then $|\phi^* D_i|$ are base point free, so

$$\begin{aligned} h_{V,\phi^* D} &= h_{V,\phi^* D_1} - h_{V,\phi^* D_2} + O(1) = h_{|\phi^* D_1|} - h_{|\phi^* D_2|} + O(1) \\ &= h_{W,D_1} \circ \phi - h_{W,D_2} \circ \phi + O(1) = h_{W,D} \circ \phi + O(1). \quad \spadesuit \end{aligned}$$

(c) Additivity: general case

Write $D = D_1 - D_2$ and $E = E_1 - E_2$ with $|D_i|, |E_i|$ base point free, hence $|D_i + E_i|$ base point free.

$$\begin{aligned} h_{V,D+E} &= h_{V,D_1+E_1} - h_{V,D_2+E_2} + O(1) \\ &= h_{V,D_1} + h_{V,E_1} - h_{V,D_2} + h_{V,E_2} + O(1) \\ &= h_{V,D} + h_{V,E} + O(1) \quad \spadesuit \end{aligned}$$

(h) Uniqueness

By (a),(b), for D ample $h_{V,D} = h \circ \phi_{|D|} + O(1)$ is uniquely determined. By (c) for $D = D_1 - D_2$ with D_i ample we have $h_{V,D} = h_{V,D_1} - h_{V,D_2} + O(1)$ uniquely determined. ♠

(d) Linear equivalence

Assume $D = D_1 - D_2 \sim E_1 - E_2 = E$ as before. Then

$$h_{\phi_{D_1+E_2}} = h_{\phi_{D_2+E_1}} + O(1),$$

hence

$$\begin{aligned} h_{V,D_1} + h_{V,E_2} &= h_{V,D_1+E_2} + O(1) \\ &= h_{V,D_2+E_1} + O(1) = h_{V,D_2} + h_{V,E_1} + O(1). \end{aligned}$$

So

$$h_{V,D_1} - h_{V,D_2} = h_{V,E_1} - h_{V,E_2} + O(1). \quad ♠$$

(e) Positivity

We first show that if $D_1 > D_2$ are base point free then $h_{V,D_1} \geq h_{V,D_2} + O(1)$ for points not in $D = D_1 - D_2$.

- Note that $|D_2| + D \subset |D_1|$, hence there is a linear projection $\pi : \mathbb{P}^n \to \mathbb{P}^m$ such that $\phi_{|D_2|} = \pi \circ \phi_{|D_1|}$, well defined away from the base locus D of $|D_2| + D$.
- By functoriality of heights for projective spaces we have

$$\begin{aligned} h_{V,D_2} = h \circ \phi_{|D_2|} &= h \circ \pi \circ \phi_{|D_1|} \\ &\leq h \circ \phi_{|D_1|} + O(1) = h_{V,D_1} + O(1) \end{aligned}$$

  for points not on D.

Applying this to finitely many D whose intersection is $Bs|D|$ we get the result. ♠

Existence of Picard schemes (or Nakai-Moishezon) and semicontinuity implies:

Lemma

If D ample and $E \sim_{alg} 0$ then there is $m \gg 0$ such that for all n we have $mD + nE$ effective (even very ample).

(f) Algebraic Equivalence (to be revisited)

We have

$mh_{V,D}(P) \pm nh_{V,E}(P) = h_{V,mD\pm nE}(P) \geq O(1) > -c(D,E,m,n)$

for all P. In other words $\frac{m}{n} + \frac{c}{nh_{V,D}(P)} \geq \frac{h_{V,E}(P)}{h_{V,D}(P)} \geq -\frac{m}{n} - \frac{c}{nh_{V,D}(P)}$

Taking $h_{V,D}(P) \to \infty$ we get

$\frac{m}{n} \geq \limsup_{h_{V,D}(P)\to\infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} \geq \liminf_{h_{V,D}(P)\to\infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} \geq -\frac{m}{n}.$

Taking $n \to \infty$ we get $\lim_{h_{V,D}(P)\to\infty} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0.$ ♠

(g) Strong finiteness property

Since $h_{V,mD} = mh_{V,D} + O(1)$ we may assume D very ample with embedding ϕ. This gives
$h_{V,D} = h_{V,\phi^* H} = h_{\mathbb{P}^n,H} \circ \phi + O(1) = h \circ \phi + O(1)$. Thus

$$\left\{ P \in V(\bar{k}) \middle| [k(P) : k] \le d,\ h_D(P) \le B \right\}$$
$$\subset \quad \left\{ P \in \mathbb{P}^n(\bar{\mathbb{Q}}) \middle| [k(P) : \mathbb{Q}] \le d,\ h(P) \le B + O(1) \right\}$$

which is finite by the strong finiteness property for projective spaces. ♠

This finally completes the proof of the Weil Height Machine Theorem. ♠

Corollary: Weil heights on abelian varieties

k number field, A/k abelian variety, $D \in Div(A)$.

(a) $h_{A,D}([m]P) = \frac{m^2+m}{2} h_{A,D}(P) + \frac{m^2-m}{2} h_{A,D}(-P) + O(1)$.

(b) If D symmetric

(b1) $$h_{A,D}([m]P) = m^2 h_{A,D}(P) + O(1)$$

and

(b2)
$$h_{A,D}(P+Q) + h_{A,D}(P-Q) = 2h_{A,D}(P) + 2h_{A,D}(Q) + O(1).$$

(c) If D antisymmetric

(c1) $$h_{A,D}([m]P) = m\, h_{A,D}(P) + O(1)$$

and

(c2) $$h_{A,D}(P+Q) = h_{A,D}(P) + h_{A,D}(Q) + O(1).$$

Geometric input for the corollary

(1) (Mumford's formula)

$$[m]^*D \quad \sim \quad \frac{m^2+m}{2}D + \frac{m^2-m}{2}[-1]^*D.$$

- So if D symmetric $[m]^*D \sim m^2D$,
- and if antisymmetric $[m]^*D \sim m\,D$.

(2) (Consequence of the Theorem of the Square) If D symmetric on A then on $A \times A$ we have

$$(\text{sum})^*D + (\text{difference})^*D \quad \sim \quad 2(\pi_1^*D + \pi_2^*D).$$

(3) (Consequence of the Seesaw Principle) If D antisymmetric then on $A \times A$ we have

$$(\text{sum})^*D \quad \sim \quad \pi_1^*D + \pi_2^*D.$$

Proof of the corollary on Weil heights on abelian varieties

- (a) as well as (b1),(b2) follow from Mumford's formula, applying functoriality for $[m] : A \to A$ (in particular $h_{A,[-1]^*D}(P) = h_{A,D}(-P)$).
- (b2) follows from (2), applying functoriality to the sum, difference, and projection morphisms.
- (b3) follows similarly from (3).

Proposition (height proportionality on curves)

C/k a curve.

(a) *Let $D, E \in Div(C)$, $\deg D \geq 1$. Then* $\displaystyle\lim_{h_D(P)\to\infty} \frac{h_E(P)}{h_D(P)} = \frac{\deg E}{\deg D}$.

(b) *Let $f, g \in K(C)$, $f \notin k$. Then* $\displaystyle\lim_{h(f(P))\to\infty} \frac{h(g(P))}{h(f(P))} = \frac{\deg g}{\deg f}$.

Proof. (NB: see direct argument in book)

(a) Replacing E by a multiple, we may assume $\deg E / \deg D =: m$ an integer, in which case $E \sim mD + E'$ with $E' \sim_{alg} 0$. So by the Algebraic Equivalence property

$$\lim_{h_D(P)\to\infty} \frac{h_E(P)}{h_D(P)} = \lim_{h_D(P)\to\infty} \frac{m\, h_D(P) + h_{E'}(P) + O(1)}{h_D(P)} = m = \frac{\deg E}{\deg D}.$$

(b) Write $f^*(x) = D_x$ and $g^*(x) = E_x$, so f is a base-point-free pencil in $|D_x|$, and g is a base-point-free pencil in $|E_x|$. So $h(f(P)) = h_{D_x}(P) + O(1)$, $h(g(P)) = h_{E_x}(P) + O(1)$, $\deg f = \deg D_x$, and $\deg g = \deg E_x$. Now apply (a). ♠