# MA 254 notes: Diophantine Geometry

## (Distilled from [Hindry-Silverman])

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# E.1 The theorem

$C/K$ a curve over a number field of genus $g > 1$. It's Jacobian is $J$, with symmetric divisor $\Theta$ with resulting Height pairing $\langle x, y \rangle$ and norm $\hat{h}_\Theta(x) = |x|^2 = \langle x, x \rangle$. We embed $C$ in $J$ somehow, for instance using a rational point (or $j_A$ defined below).

**Theorem (Faltings)**

*$C(K)$ is finite*

# E.1 Vojta's inequality

The heart of the proof presented is

**Theorem (Vojta)**

*There exist* $\kappa_1, \kappa_2 > 1$ *such that if* $z, w \in C(\bar{K})$ *with* $|z| > \kappa_1$ *and* $|w| > \kappa_2 |z|$, *then*

$$\langle z, w \rangle \leq \frac{3}{4} |z||w|.$$

**Proposition**

*Vojta's inequality implies Faltings's Theorem*

# E.1 Proof of implication

- It suffices to show that the image of $C(K)$ in $J(K)/torsion$ is finite. Let $r$ be the rank.
- Suppose $C(K)$ is infinite. So the norms $|x|$ are unbounded.
- Inductively pick $z_i \in C(K)$ so that $|z_1| > \kappa_1$ and $|z_i| > \kappa_2 |z_{i-1}|$, so that Vojta's inequality applies to all.
- The inequality implies that the angle between $z_i$ and $z_j$ is $\geq \alpha := \arccos(3/4)$.
- In particular the spherical open balls $B(u_i, \alpha/2)$ of angle $\alpha/2$ around $u_i = z_i/|z_i|$ are disjoint.
- But $\operatorname{vol} S_{r-1} \geq \operatorname{vol} \coprod B(u_i, \alpha/2) = \sum \operatorname{vol} B(u_i, \alpha/2) = \infty$, since $\operatorname{vol} B(u_i, \alpha/2) = \operatorname{vol} B(u_j, \alpha/2) > 0$. ♠

# E.2 Compatible divisors

- Choose a divisor $A$ on $C$ such that $\deg_C A = 1$ and $(2g-2)A \sim K_C$.
- Possibly after extending $K$ the divisor $A \in Div(C/K)$.
- Get an embedding $j_A : C \to J$ by $x \to (x) - A$.
- Will chose $\Theta_A = \sum_{i=1}^{g-1} j_A C$.
- Write $\mathcal{P}_A = sum^*\Theta_A - p_1^*\Theta_A - p_2^*\Theta_A$.

**Lemma**

(1) $\Theta_A$ *is symmetric;*

(2) $j_A^*\Theta_A \sim gA$

(3) $(j_A \times j_A)^*\mathcal{P}_A \sim -\Delta_C + p_1^*A + p_2^*A$.

# E.2 Compatible divisors - proof

(1) The divisor $\Theta^{g-1} \subset Pic^{g-1}(C)$ is characterized by $H^0(O(D)) \neq 0$, equivalently $H^1(O(D)) \neq 0$ or $H^0(K_C - D) \neq 0$, so symmetric under $x \mapsto K_C - x$. Now $\Theta_A = \Theta^{g-1} - (g-1)A = K_C - \Theta^{g-1} - (g-1)A = (g-1)A - \Theta^{g-1} = [-1]^*\Theta_A$.

(2) $j_A^*\Theta_A = \{p \in C | h^0(p + (g-2)A) \geq 0\}$
$= \{p \in C | h^0(gA - p) \geq 0\} \sim gA$.

(3) Restricting $\sum^* \Theta_A$ to $q \times C$ we get $\{p \in C | h^0(p + q + (g-3)A) \geq 0\} = \{p \in C | h^0((g+1)A - q - p) \geq 0\} \sim (g+1)A - q$. The restriction of $p_1^*\Theta_A \sim 0$ and of $p_2^*\Theta_A \sim gA$ as before, so $\mathcal{P}_A$ restricts to $A - q$. On the right hand side the restriction of $\Delta_C$ is $q$, the restriction of $p_1^*A \sim 0$ and of $p_2^*A \sim A$, also giving in total $A - q$. Symmetry and the seesaw principle give the result.

# E.2 Specific heights

- Choose $N$ so that $NA$ is very ample (e.g. $N = 2g - 1$). Let $\phi_{NA} : C \to \mathbb{P}^n$ be a fixed embedding.
- This provides a height $h_{C,NA}(z)$ and $h_{C,\xi NA}(z) := \xi h_{C,NA}(z)$ - in particular $h_{C,NA}(z) = Nh_{C,A}(z)$ - more to come.
- We assume that $\phi_{NA}(C)$ does not meet the intersection planes of hyperplanes.
- We assume further that any three of the basis sections define a birational map $C \to \mathbb{P}^2$.
- Choose $M$ large so that $B = (M+1)p_1^*A + (M+1)p_2^*A - \Delta_C$ is very ample. This gives an embedding $\phi_B : C \times C \to \mathbb{P}^m$, providing for a height $h_{C\times C,B}(z,w)$. We extend to $h_{C\times C,dB}(z,w) = dh_{C\times C,B}(z,w)$.

# E.2 Specific heights - continued

- For all $\delta_1, \delta_2 > 0$ consider $D_{\delta_1,\delta_2} = \delta_1 p_1^* A + \delta_2 p_2^* A$. We fix $h_{C\times C, D_{\delta_1,\delta_2}}(z,w) = \delta_1 h_{C,\delta_1 A}(z) + \delta_2 h_{C,\delta_2 A}(w)$.
- For $d_1, d_2, d$ define the Vojta divisor

$$\Omega(d_1,d_2,d) = (d_1 - d)p_1^*A + (d_2 - d)p_2^*A + d\Delta_C,$$

where we assume $gd^2 < d_1 d_2 < g^2 d^2$. It is deliberately going to be somewhat lopsided: $d_1 > d > d_2$.
- We require $N | d_1, d_2, d$ and fix $\delta_i = (d_i + Md)/N$. Then

$$\Omega(d_1,d_2,d) = D_{\delta_1 N, \delta_2 N} - dB.$$

- We use the corresponding height $h_{C\times C,\Omega(d_1,d_2,d)}(z,w) = \delta_1 h_{C,NA}(z) + \delta_2 h_{C,NA}(w) - d h_{C\times C,B}(z,w)$.
- We require $d$ to be large enough that

$$H^0(C\times C, O(B))^{\otimes d} \to H^0(C\times C, O(dB))$$

is surjective.

# E.4 Mumford's estimate

**Proposition**

*There is a constant $c_1$ so that for all $d_1, d_2, d$ and all $z, w \in C(\bar{K})$ we have*

$$h_{\Omega(d_1,d_2,d)}(z,w) \leq \frac{d_1}{g}|z|^2 + \frac{d_2}{g}|w|^2 - 2d\langle z, w\rangle + (d_1 + d_2 + d)c_1.$$

Taking $d_1 = d_2 = d = 1$ gives Mumford's gap principle:

$$\cos \measuredangle(z,w) \quad \leq \quad (|z|/|w| + |w|/|z|)/2g \quad + \quad c'/|z||w|.$$

Proof: Since $\Omega(d_1, d_2, d) = D_{d_1,d_2} - d(D_{1,1} - \Delta_C)$, the height machine gives

$$\begin{aligned} h_{\Omega(d_1,d_2,d)}(z,w) =& d_1 h_{C,A}(z) + d_2 h_{C,A}(w) \\ & - d h_{C\times C, -\Delta_C + D_{1,1}}(z,w) + O(d_1 + d_2 + d). \end{aligned}$$

# E.4 Mumford's estimate - concluded

Recall $j_A^*\Theta_A \sim gA$, so

$$h_{C,A}(u) = \frac{1}{g}|u|^2 + O(1).$$

Recall $(j_A \times j_A)^*\mathcal{P}_A \sim -\Delta_C + p_1^*A + p_2^*A$, so

$$\begin{aligned} h_{C\times C,-\Delta_C+D_{1,1}}(z,w) \\ &= \hat{h}_{J,\Theta_A}(z+w) - \hat{h}_{J,\Theta_A}(z) - \hat{h}_{J,\Theta_A}(w) + O(1) \\ &= |z+w|^2 - |z|^2 - |w|^2 + O(1) = 2\langle z,w\rangle + O(1). \end{aligned}$$

Combining, the result follows. ♠

# E.5 Sections interpreted (Also E.6)

Recall that $\Omega(d_1, d_2, d) = D_{\delta_1 N, \delta_2 N} - dB$. We have a bilinear $H^0(\Omega(d_1, d_2, d)) \otimes H^0(dB) \to H^0(D_{\delta_1 N, \delta_2 N})$.
Having required $d, \delta_1, \delta_2$ to be larger than what's required for Serre's theorem, we have surjective $\mathrm{Sym}^d H^0(B) \to H^0(dB)$ and $\mathrm{Sym}^{\delta_1} H^0(C, NA) \otimes \mathrm{Sym}^{\delta_2} H^0(C, NA) \to H^0(D_{\delta_1 N, \delta_2 N})$. Let $y_i \in H^0(B)$ and $x_i \in H^0(C, NA)$ be the section providing the embedding and heights.

## Lemma

*We have a bijection*

$$H^0(\Omega(d_1, d_2, d)) = \left\{ (F_i(x, x') \in k[x, x']_{\delta_1, \delta_2} / I_{C \times C}) \;\middle|\; \begin{array}{l} F_i(x, x') y_j^d|_{C \times C} \\ = F_j(x, x') y_i^d|_{C \times C} \end{array} \right\}.$$

# E.6 Sections interpreted and counted

Writing the rational functions $\frac{y_i^d}{y_0^d} = \frac{P_i(x,x')}{Q_i(x,x')}$ we can further represent $H^0(\Omega(d_1, d_2, d))$ by collections $(F_i(x, x') \in k[x, x']_{\delta_1,\delta_2})$ satisfying

$$(P_j Q_i)^d F_i(x, x')|_{C\times C} = (P_i Q_j)^d F_j(x, x')|_{C\times C}.$$

We will use Siegel's Lemma to find small sections. First we need the dimensions.

**Lemma**

(1) $\ell(\Omega(d_1, d_2, d)) \geq d_1 d_2 - gd^2 - (g-1)(d_1 + d_2)$.

(2) $\ell(D_{\delta_1 N,\delta_2 N}) = (N\delta_1 - g + 1)(N\delta_2 - g + 1)$.

# E.6 Intersection numbers

Part (2) follows since $\ell(p_1^* E_1 + p_2^* E_2) = \ell(E_1)\ell(E_2)$. Since $N$ was chosen large, Riemann-Roch for curves says $\ell(\delta_i NA) = \delta_i N - g + 1$. For Part (1) we need Riemann-Roch:

**Theorem (Surface Riemann-Roch)**

$\chi(\mathcal{O}_S(D)) = \frac{D\cdot(D-K_S)}{2} + \chi(\mathcal{O}_S)$.

We have $\chi(\mathcal{O}_{C\times C}) = (\chi(\mathcal{O}_C))^2 = (g-1)^2$ and intersection table

| | $p_1^*A$ | $p_2^*A$ | $\Delta_C$ |
|---|---|---|---|
| $p_1^*A$ | 0 | 1 | 1 |
| $p_2^*A$ | 0 | 1 | 1 |
| $\Delta_C$ | 1 | 1 | $2-2g$ |

# E.6 Proof of dimension estimate

- We now compute $\chi(\Omega(d_1, d_2, d)) = \frac{2d_1d_2 - 2gd^2 - (2g-2)(d_1+d_2)}{2} + (g-1)^2$,
- so $\ell(\Omega(d_1, d_2, d)) \geq d_1d_2 - gd^2 - (g-1)(d_1+d_2) + \ell(K_{C\times C} - \Omega(d_1, d_2, d))$.
- Since the intersection number $K - \Omega$ with the moving divisor $A_1$ is $(2g-2) - d_2$ and $d_2$ is large, $\ell(K - \Omega) = 0$, so our crude estimate holds. ♠

# E.7 summary: $\Omega$ has a small section

**Proposition**

*There is* $c_1(C/K, A, B) > 0$ *such that the following holds. Fix* $\gamma > 0$ *and assume* $d_1 d_2 - g d^2 \geq \gamma d_1 d_2$*. Then there is* $s \in \Gamma(C \times C, O(\Omega(d_1, d_2, d)))$ *corresponding to* $(F_i)$ *such that*

$$h((F_i)) \leq c_1 \frac{d_1 + d_2}{\gamma} + o(d_1 + d_2).$$

This is proven by an incredibly technical application of Sigel's Lemma. We'll come back to it.

# E.8 Lower bound on $h_\Omega(z,w)$: the index

We will fix $(z,w) \in C \times C$, align $d_1, d_2, d$ to $(z,w)$, and then choose a section $s \in \Gamma(\Omega)$. We need to consider the index of $s$ at $(z,w)$, with respect to weights $(\delta_1, \delta_2)$ For this we choose coordinates at $z, w$ and take derivatives with respect to parameters $\zeta, \zeta'$:

$$\mathrm{Ind}(s) := \min\left\{ \frac{i_1}{\delta_1} + \frac{i_2}{\delta_2} \middle| i_1, i_2 \geq 0, \partial_{i_1}\partial'_{i_2} f(z,w) \neq 0 \right\}.$$

We say $(i_1^*, i_2^*)$ is admissible if $\mathrm{Ind}(s) = \frac{i_1^*}{\delta_1} + \frac{i_2^*}{\delta_2}$ and $\partial_{i_1^*}\partial'_{i_2^*} f(z,w) \neq 0$.

**Lemma**

*If $(i_1^*, i_2^*)$ is admissible and if $g(z,w) \neq 0$ then* $\partial_{i_1^*}\partial'_{i_2^*} f(z,w) = \frac{\partial_{i_1^*}\partial'_{i_2^*}(f(z,w)g(z,w))}{g(z,w)}$.

# E.8 Implicit lower bound on $h_\Omega(z,w)$

## Proposition

*$s \in \Gamma(\Omega)$ given by $(F_i)$, $(i_1^*, i_2^*)$ is admissible at $(z,w)$, then*

$$h_\Omega(z,w) \geq -h((F_i)) - (i_1^* + i_2^* + 2\delta_1 + 2\delta_2 + 2n)$$
$$- \sum_v \max_{i_1+\cdots+i_{\delta_1}=i_1^*} \sum_{k=1}^{\delta_1} \max_{0\leq \ell \leq n} \min_j \log \left| \left( \partial_{i_k} \frac{x_\ell}{x_j} \right)(z) \right|_v$$
$$- \sum_v \max_{i'_1+\cdots+i'_{\delta_2}=i_2^*} \sum_{k=1}^{\delta_2} \max_{0\leq \ell \leq n} \min_{j'} \log \left| \left( \partial_{i'_k} \frac{x'_\ell}{x'_{j'}} \right)(z) \right|_v$$

The key is to unwind
$h_\Omega(z,w) = \delta_1 h_{NA}(z) + \delta_2 h_{NA}(w) - d h_B(z,w)$.
We'll get back to this.

# E.9 Derivative bounds

We need to bound $\left|\left(\partial_{i_k}\frac{x_\ell}{x_j}\right)(z)\right|_v$ and $\left|\left(\partial_{i'_k}\frac{x'_\ell}{x'_{j'}}\right)(z)\right|_v$. A rational function $\xi(\zeta) = x_\ell/x_j(\zeta)$ satisfies an equation $p(\xi,\zeta) = 0$ of degree bounded by the degree $D$ of $C$. We wish to estimate the derivatives of $\xi$ at the point $(a,0)$ corresponding to $z$. Also write

$$|x|_v^{\text{arch}} := \begin{cases} |v|_v & v \text{ archimedean} \\ 1 & \text{otherwise.} \end{cases}$$

## Proposition

$$|\partial_i\xi(0)|_v \leq (|2D|_v^{arch})^{11i}\left(\frac{|p|_v}{\partial p/\partial\xi(a,0)|_v}\right)^{2i-1}\max\{1,|a|_v\}^{2iD}.$$

# E.10 More explicit lower bound on $h_\Omega(z, w)$

**Proposition**

*$s \in \Gamma(\Omega)$ given by $(F_i)$, then there is a finite set $\mathcal{Z} \subset C$ such that for $z, w \notin \mathcal{Z}$ and $(i_1^*, i_2^*)$ is admissible at $(z, w)$,*

$$h_\Omega(z, w) \geq -h((F_i)) - c_{18}(i_1^*|z|^2 + i_2^*|w|^2) - c_{19}(i_1^* + i_2^* + \delta_1 + \delta_2 + 1).$$

(Here we fall back to $|z|$ being the canonical height in the jacobian.)

# E.11 Roth's lemma: two variable case

## Proposition

*Let $P \in \bar{\mathbb{Q}}[X_1, X_2]$ of bidegree at most $(r_1, r_2)$. Let $\beta_1, \beta_2 \in \bar{\mathbb{Q}}$. Let $1 \geq \omega > 0$ such that $r_1 \leq \omega r_1$ and*

$$h(P) + 4r_1 \leq \omega \min\{r_1 h(\beta_1), r_2 h(\beta_2)\}.$$

*Then there are $i_1, i_2 \geq 0$ such that $\frac{i_1}{\delta_1} + \frac{i_2}{\delta_2} \leq 4\sqrt{\omega}$ and $\partial_{i_1,i_2} P(\beta_1, \beta_2) \neq 0$.*

# E.11 Index bound

## Proposition

*Given $C, A, B$ there is a constant $c_{35}$ such that the following holds. Let $\epsilon, \delta > 0$ be small, and let $d_1, d_2, d$ and $(z, w)$ satisfy*

$$\epsilon^2 d_2 \geq d_2, \ \ \min\{d_1|z|^2, d_2|w^2\} \geq c_{35} d_1/(\gamma\epsilon^2), \ \ d_2 d_2 - d^2 \geq \gamma d_1 d_2.$$

*Let $s \in \Gamma(\Omega)$ be a small section of height $h((F_i)) = c_1 \frac{d_1+d_2}{\gamma} + o(d_1 + d_2)$. Then there is $(i_1^*, i_2^*)$ admissible at $(z, w)$ with*

$$\frac{i_1^*}{d_1} + \frac{i_2^*}{d_2} \leq 12N\epsilon.$$

# E.12: Vojta's inequality - first choices

- In a proposition from E.10 there was an "exceptional" finite set of points $\mathcal{Z} \subset C(\bar{K})$.
- We require $\kappa_1 > \max_{z \in \mathcal{Z}} h(z)$. There will be another lower bound required on $\kappa_1$ below.
- Take points $z, w \in C(\bar{K})$, where $|z| > \kappa_1$. We'll specify requirements on $\kappa_2$ later.
- Pick $1 \gg \epsilon, \nu > 0$ depending on $C$.
- We'll want $\kappa_1^2 > 1/\epsilon,\ \ \kappa_2/\sqrt{2} > 1/\epsilon$.
- We pick $D \gg 1$. This choice does depend on $z, w$: we need $D > |w|^2$, though we'll let it go to infinity.
- Set $d_1 = N\left\lfloor \sqrt{g+\nu}\frac{D}{|z|^2} \right\rfloor,\ d_2 = N\left\lfloor \sqrt{g+\nu}\frac{D}{|w|^2} \right\rfloor,\ d = N\left\lfloor \frac{D}{|z||w|} \right\rfloor.$

# E.12: Vojta's inequality - lower height bounds

- We proved, for $z, w \notin \mathcal{Z}$ and admissible $I_1^*, I_2^*$,
  $h_\Omega(z, w)$
  $\geq -h((F_i)) - c_{18}(i_1^*|z|^2 + i_2^*|w|^2) - c_{19}(i_1^* + i_2^* + \delta_1 + \delta_2 + 1)$.
- Recall that $\delta_1, \delta_2$ are positive linear combinations of $d_1, d_2, d$. Also we may, and will, assume $|z|, |w| \gg 1$. So
  $h_\Omega(z, w) \geq -h((F_i)) - c_{42}(i_1^*|z|^2 + i_2^*|w|^2) - c_{43}(d_1 + d_2 + d)$.
- For the last term we have $d_1 + d_2 + d \leq \frac{c_{44} D}{\kappa_1^2} \leq c_{44}\epsilon D$, so
  $h_\Omega(z, w) \geq -h((F_i)) - c_{42}(i_1^*|z|^2 + i_2^*|w|^2) - c_{46}\epsilon D$.

# E.12: Vojta's inequality - there is a small section

- we have seen in $E_7$ that if $d_1 d_2 - gd^2 \geq \gamma d_1 d_2$ there is a section $s$ of $\Omega(d_1, d_2, d)$ corresponding to $(F_i)$ such that

$$h((F_i)) \leq c_1 \frac{d_1 + d_2}{\gamma} + o(d_1 + d_2).$$

- $\lim\limits_{D\to\infty} \dfrac{d_1 d_2 - gd^2}{d_1 d_2} = 1 - \dfrac{g}{g+\nu} = \dfrac{\nu}{g+\nu} \;>\; \nu/(g+1)$. So $\gamma = \nu/3g$ will work for large $D$ and any $g$.
- For such gamma we get a section with

$$h((F_i)) \leq c_{47} \frac{d_1 + d_2}{\gamma} \leq c_{48}(d_1 + d_2) \leq c_{49}\epsilon D.$$

- This gives $h_\Omega(z, w) \geq -c_{50}(i_1^*|z|^2 + i_2^*|w|^2) - c_{51}\epsilon D$.

# E.12: Vojta's inequality - verifying index bound

- To bound the index we needed in E.11

$$\epsilon^2 d_2 \geq d_2, \ \min\{d_1|z|^2, d_2|w^2\} \geq c_{35} d_1/(\gamma\epsilon^2), \ d_2 d_2 - d^2 \geq \gamma d_1 d_2.$$

- Note that $d_2/d_1 \sim_{D\to\infty} |z|^2/|w|^2$. So for large $D$ we have

$$\frac{d_1}{d_2} \leq 2\frac{|z|^2}{|w|^2} \leq 2/\kappa_2^2 \leq \epsilon^2.$$

- $d_1|z|^2$ and $d_2|w|^2$ are "balanced": $\frac{d_2|w|^2}{d_1|z|^2} \sim_{D\to\infty} 1$. so for large $D$ we have $1/2 \leq \frac{d_2|w|^2}{d_1|z|^2} \leq 2$.
- Assuming further $\kappa_1 \geq 2c_{35}/(\gamma\epsilon^2)$ we get $\min\{d_1|z|^2, d_2|w^2\} \geq d_1|z|^2/2 \geq d_1\kappa_1^2/2 \geq c_{35} d_1/(\gamma\epsilon^2)$, as needed.

# E.12: Vojta's inequality - Mumford

- We now have the index bound

$$\frac{i_1^*}{d_1} + \frac{i_2^*}{d_2} \leq 12N\epsilon.$$

- Combining we get $h_\Omega(z, w) \geq -c_{53}(d_1|z|^2 + d_2|w|^2) - c_{54}\epsilon D$.
- The definitions of $d_1, d_2$ give $d_1|z|^2 \leq c_{55}D$, $d_2|w|^2 \leq c_{56}D$.
- So $h_\Omega(z, w) \geq -c_{57}\epsilon D$.
- Mumford gave in E.4
  $h_{\Omega(d_1,d_2,d)}(z, w) \leq \frac{d_1}{g}|z|^2 + \frac{d_2}{g}|z_2|^2 - 2d\langle z, w\rangle + (d_1 + d_2 + d)c_1$.
- Since $d_1 + d_2 + d \leq c_{44}\epsilon D$ we have

$$\frac{d_1}{g}|z|^2 + \frac{d_2}{g}|z_2|^2 - 2d\langle z, w\rangle \geq -c_{60}\epsilon D.$$

# E.12: Vojta's inequality - conclusion

- Plugging in we get $\left\lfloor\sqrt{g+\nu}\frac{D}{|z|^2}\right\rfloor\frac{|z|^2}{g} + \left\lfloor\sqrt{g+\nu}\frac{D}{|w|^2}\right\rfloor\frac{|w|^2}{g} - 2\left\lfloor\frac{D}{|z||w|}\right\rfloor\langle z,w\rangle \geq -c_{60}\epsilon D.$
- For $D\to\infty$ we get

$$\langle z,w\rangle \leq \left(\frac{\sqrt{g+\nu}}{g} + \frac{c_{61}\epsilon}{2}\right)|z||w|.$$

Since $g\geq 2$ we have $\lim_{\nu,\epsilon\to 0}\left(\frac{\sqrt{g+\nu}}{g}+\frac{c_{61}\epsilon}{2}\right) = 1/\sqrt{g} < 3/4$, so for small $\epsilon,\nu$, not depending on $z,w$,

$$\langle z,w\rangle \leq \frac{3}{4}|z||w|,$$

as required. ♠