MA 254 notes: Diophantine Geometry

(Distilled from [Hindry-Silverman], [Manin], [Serre])

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Mordell Weil and Weak Mordell-Weil

Theorem (The Mordell-Weil theorem)

Let A/k be an abelian variety over a number field. Then $A(k)$ is finitely generated.

We deduce this from

Theorem (The weak Mordell-Weil theorem)

Let A/k be as above and $m \geq 2$. Then $A(k)/mA(k)$ is finite.

Mordell-Weil follows from Weak Mordell-Weil

The reduction comes from the following.

Lemma (Infinite Descent Lemma)

Let G be an abelian group, $q : G \to \mathbb{R}$ a quadratic form. Assume

(i) *for all $C \in \mathbb{R}$ the set $\{x \in G | q(x) \leq C\}$ is finite, and*

(ii) *there is $m \geq 2$ such that G/mG is finite.*

Then G is finitely generated. In fact if $\{g_i\}$ are representatives for G/mG and $C_0 = \max\{q(g_i)\}$ then the finite set $S := \{x \in G | q(x) \leq C_0\}$ generates G.

indeed the canonical Néron-Tate height $\hat{h}_D$ on $A(k)$ associated to an ample symmetric D gives a quadratic form satisfying (i), and (ii) follows from Weak Mordell-Weil. The Mordell-Weil theorem follows. ♠

Proof of the Infinite Descent Lemma for $m \geq 3$

- $q(x) \geq 0$ otherwise the elements $\{nx\}$ would violate (i).
- Write $|x| = \sqrt{q(x)}, \quad c_0 = \max\{|g_i|\}$,
$$S_{\sqrt{m}^k c_0} = \{x \in G : |x| \leq \sqrt{m}^k c_0\}.$$
- Clearly $G = \cup S_{\sqrt{m}^k c_0}$, so enough to show $S_{\sqrt{m}^k c_0} \subset \langle S \rangle$ for all k.
- We use induction on $k \geq 0$. Clearly $S_{c_0} = S \subset \langle S \rangle$.
- Assume $S_{\sqrt{m}^k c_0} \subset \langle S \rangle$ and let $x \in S_{\sqrt{m}^{k+1} c_0} \setminus S_{\sqrt{m}^k c_0}$.
- $x \in g_i + mG$ for some i so there is x_1 such that $m\, x_1 = x - g_i$. So
$$|x_1| = \frac{|x - g_i|}{m} \quad \leq \quad \frac{|x| + |g_i|}{m} \quad \leq \quad \frac{\sqrt{m}^{k+1} + 1}{m} c_0 \quad \leq \sqrt{m}^k c_0$$
since $m \geq 3$. ♠

Proof of Weak Mordell-Weil - Finite Kernel Lemma

Lemma (Finite Kernel Lemma)

Fix a finite k'/k. Then $\operatorname{Ker}\ \left(A(k)/mA(k) \to A(k')/mA(k')\right)$ *is finite.*

Proof of lemma

- Note this kernel is $B_{k'/k} := (A(k) \cap mA(k'))/mA(k)$.
- Since for a further extension k''/k' we have $B_{k'/k} \subset B_{k''/k}$, we may replace k' by a further extension.
- In particular we may assume k'/k Galois, with Galois group G. The lemma now follows from the following, since G and $A_m(\bar{k})$ are finite:

Sublemma

There is an injective function $B_{k'/k} \to Maps(G, A_m(\bar{k}))$.

Proof of Weak Mordell-Weil - reduction - sublemma

Proof of sublemma

- For $x \in B_{k'/k}$ fix $y \in A(k')$ such that $[m]y$ represents x.
- Consider the function $f_x : G \to A(k')$ where $f(\sigma) = y^\sigma - y$.
- Note $[m]f_x(\sigma) = [m](y^\sigma - y) = [my]^\sigma - [m]y = x^\sigma - x = 0$.
- This defines $B_{k'/k} \to Maps(G, A_m(\bar{k}))$.
- We claim it is injective.
- Assume $f_x = f_{x'}$, and let y, y' be the chosen points in $A(k')$.
- So $y'^\sigma - y' = y^\sigma - y$, hence $(y' - y)^\sigma = y' - y$ for all σ.
- So $y' - y \in A(k)$ and $[m](y' - y) \in mA(k)$.
- Hence $x - x' = 0$ in $B_{k'/k}$, as needed. ♠♠

Theorem (Chevalley-Weil+Hermite for $[m]$)

Fix an abelian variety A/k and an integer m. There is a finite extension k'/k such that if $x \in A(k)$ then there is $y \in A(k')$ such that $[m]y = x$.

Proof of Weak Mordell-Weil assuming Chevalley-Weil+Hermite

Let k' be as in Chevalley-Weil+Hermite for $[m]$. Then the map $A(k)/mA(k) \to A(k')/mA(k')$ is zero, hence by the Finite Kernel Lemma $A(k)/mA(k)$ is finite. ♠

We will present two proofs of Chevalley-Weil+Hermite: one quick and dirty using Scheme Theory. One going through with rings and differentials explicitly.

Hermite's theorem(s)

The following classical theorem appears in a first course:

Theorem (Hermite 1)

For any real B there are only finitely many number fields k with $|d_k| \leq B$. ♠

This one might not be as visible:

Theorem (Hermite 2)

For any integer n and finite set of primes S there are only finitely many number fields k unramified outside S with $[k : \mathbb{Q}] \leq n$.

Hermite 2 follows from Hermite 1 given the following:

Proposition (Discriminant bound (Serre), not proven here)

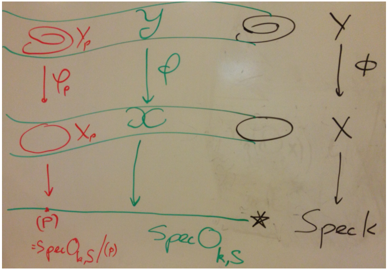
Write $[k : \mathbb{Q}] = n$ and $N = \prod_{p|d_k} p$. Then $|d_k| \leq (N \cdot n)^n$.

Why number-theorists want to understand schemes

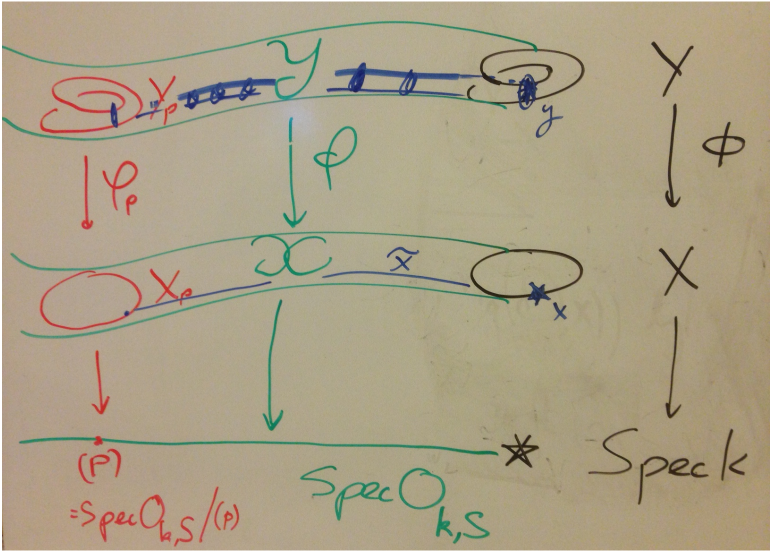
What I stated as "Chevalley-Weil+Hermite" is a consequence of

Proposition (Spreading out)

Let $\phi : Y \to X$ be a finite unramified morphism of projective varieties, all over k. Then there is a finite set of primes S, projective schemes $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{k,S}$, $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}_{k,S}$ and a finite unramified morphism $\varphi : \mathcal{Y} \to \mathcal{X}$, whose restriction to $\operatorname{Spec} k$ is $\phi : Y \to X$.



Spreading out: Picture with point



Spreading out implies Chevalley-Weil

The reduction works as follows:

- In our case take $X = Y = A$ and $\phi = [m]$.
- Since $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{k,S}$ pojective, a point $x \in X(k)$ extends to a morphism $\tilde{x} : \operatorname{Spec} \mathcal{O}_{k,S} \to \mathcal{X}$.
- Write $\mathcal{Y}_x = \operatorname{Spec} \mathcal{O}_{k,S} \times_{\mathcal{X}} \mathcal{Y}$.
- Since $\mathcal{Y} \to \mathcal{X}$ is unramified we have $\Omega_{\mathcal{Y}/\mathcal{X}} = 0$.
- $\Omega_{\mathcal{Y}_x/\operatorname{Spec} \mathcal{O}_{k,S}}$ is the pullback of this to $\mathcal{Y}_x$, so also 0.
- If $[m]y = x$ then $y \subset \mathcal{Y}_x$ and its closure $\tilde{y}$ is unramified over $\tilde{x}$.
- This means that $\mathcal{O}(\tilde{y})$ is finite unramified over $\mathcal{O}_{k,S}$, so $k(y)/k$ unramified away from S.
- By Hermite 2 there are only finitely many such $k(S)$. Let k' be the Galois closure of their compositum.
- so $y \in A(k')$. ♠

Exorcising schemes: Spreading Out explained

STEP 1: COORDINATES

- Say $Y \in \mathbb{P}^m$ and $X \in \mathbb{P}^n$. We may replace $Y \subset \mathbb{P}^m$ by the graph of ϕ.
- We now have $Y \subset \mathbb{P}^m \times \mathbb{P}^n$, and ϕ is the restriction of the natural projection $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$.
- For each coordinate X_i of $\mathbb{P}^n$ we have $U_i = X \setminus Z(X_i)$ affine, with coordinate ring $A_i = k[x_0, \ldots, \cancel{x_i}, \ldots, x_n]/(f_{1i}, \ldots, f_{r_i i})$.
- "ϕ finite" means: preimage of affine is affine, corresponding to a finite ring extension.
- So $V_i = \phi^{-1}U_i$ affine, with ring B_i finite over A_i. Let $\{y_{1i}, \ldots, y_{k_i i}\}$ be module generators and g_{ji} the module relations and $Y_{ji}Y_{j'i} = h_{jj'i}$ the ring relations, with $g_{ji}, h_{jj'i}$ A_i-linear in $Y_{1i}, \ldots, Y_{k_i i}$:

$$B_i = A_i[Y_{1i}, \ldots, Y_{k_i i}]/(\{g_{ji}\}, \{Y_{ji}Y_{j'i} - h_{jj'i}\}).$$

Exorcising schemes: Spreading Out explained

STEP 2: SHRINKING TO DEFINE RINGS AND MAINTAIN FINITENESS

- There are finitely many nonzero coefficients $\{a_\alpha\}$ in $f_{j,i}, g_{ji}, h_{jj'i}$, giving a subring $\mathcal{O}_{k,S_0} := \mathcal{O}_k[\{a_\alpha, a_\alpha^{-1}\}]$ of k in which a_α are units. Here S_0 is the set of places appearing in the factorizations of the a_α.
- Let $\mathcal{A}_i = \mathcal{O}_{k,S}[x_0, \ldots, \cancel{x_i}, \ldots, x_n]/(f_{1i}, \ldots, f_{r_i i})$ and $\mathcal{B}_i = \mathcal{A}_i[Y_{1i}, \ldots, Y_{k_i i}]/(\{g_{ji}\}, \{Y_{ji}Y_{j'i} - h_{jj'i}\})$.
- We clearly have $A_i = \mathcal{A}_i \otimes_{\mathcal{O}_{k,S}} k$ and $B_i = \mathcal{B}_i \otimes_{\mathcal{O}_{k,S}} k$.
- By construction $\mathcal{A}_i\langle Y_{ji}\rangle \to \mathcal{B}_i$ is a surjective module homomorphism.
- So $\mathcal{B}_i$ is a finite $\mathcal{A}_i$-algebra.

Exorcising schemes: Spreading Out explained

STEP 3: SHRINKING TO EVADE RAMIFICATION

- The statement "$Y \to X$ unramified" is equivalent to "$V_i \to U_i$ unramified".
- This is equivalent to "$\Omega_{B_i/A_i} = 0$".
- Consider the finitely generated $\mathcal{B}_i$-module $\Omega_{\mathcal{B}_i/\mathcal{A}_i}$.
- We have $\Omega_{\mathcal{B}_i/\mathcal{A}_i} \otimes_{\mathcal{O}_{k,S}} k = \Omega_{B_i/A_i} = 0$.
- So $Ann_{\mathcal{O}_{k,S}}(\Omega_{\mathcal{B}_i/\mathcal{A}_i}) \neq 0$. In other words there is nonzero $c \in \mathcal{O}_{k,S}$ such that $c\Omega_{\mathcal{B}_i/\mathcal{A}_i} = 0$.
- Replacing $\mathcal{O}_{k,S}$ by $\mathcal{O}_{k,S}[c^{-1}] \subset k$ we may assume $\Omega_{\mathcal{B}_i/\mathcal{A}_i} = 0$,
- hence $\mathcal{B}_i$ is an unramified $\mathcal{A}_i$-algebra.

In the language of spreading out, $\mathcal{Y} := \cup \operatorname{Spec} \mathcal{B}_i$ and $\mathcal{X} := \cup \operatorname{Spec} \mathcal{A}_i$. ♠(Spreading Out)

Exorcising schemes: Chevalley-Weil explained

Proposition

Let $x \in X(k)$ and $\phi(y) = x$. Then $k(y)/k$ is unramified outside S.

- Fix a nonzero prime $\mathfrak{p} \subset \mathcal{O}_{k,S}$. We show $k(y)/k$ is unramified at $\mathfrak{p}$.
- Let $x = (\alpha_0, \ldots, \alpha_n) \in X$. Since $\mathcal{O}_{k,\mathfrak{p}}$ is principal we can take $\alpha_i \in \mathcal{O}_{k,\mathfrak{p}}$ relatively prime.
- Without loss of generality the uniformizer $\pi_\mathfrak{p} \not| \alpha_0$.
- Replacing α_i by α_i/α_0 we may assume $\alpha_0 = 1$.
- Consider the epimorphism $A_0 \to k$ given by the maximal ideal $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$.
- It gives $(\mathcal{A}_0)_\mathfrak{p} \stackrel{\tilde{x}}{\to} \mathcal{O}_{k,\mathfrak{p}}$ (the image is no bigger).

Exorcising schemes: Chevalley-Weil explained

Proof of the proposition

- Consider $\mathcal{C} := \mathcal{B}_0 \otimes_{\mathcal{A}_0} \mathcal{O}_{k,\mathfrak{p}}$.
- Since $\mathcal{A}_0 \to \mathcal{B}_0$ is finite and unramified, also $\mathcal{O}_{k,S} \to \mathcal{C}$ is finite and unramified.
- Consider the commutative diagram

$$
\begin{tikzcd}
\mathcal{B}_0 \arrow[r] \arrow[drr] & \mathcal{C} \arrow[drr, dotted] & & \\
\mathcal{A}_0 \arrow[r] \arrow[u] \arrow[drr] & \mathcal{O}_{k,\mathfrak{p}} \arrow[u] \arrow[drr] & B_0 \arrow[r] & k(y) \\
& & A_0 \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}
$$

- The universal property of tensor gives an arrow $\mathcal{C} \to k(y)$.
- Its image is a subring $R_{y,\mathfrak{p}} \subset k(y)$ finite unramified over $\mathcal{O}_{k,\mathfrak{p}}$, which must be $\mathcal{O}_{k(y),\mathfrak{p}}$. ♠