MA 252 notes: Commutative algebra
(Distilled from [Atiyah-MacDonald])

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Recall: Modules, module homomorphisms, submodules, quotients, isomorphism theorems.

\[ \text{Hom}_A(M, N) \] is an \( A \)-module.

\( aM \subset M \) the submodule generated by products.

\( (N : P) \) an ideal, \( (0 : P) = \text{Ann}(P) \).

In general \( (N : P) = \text{Ann}(P/(N \cap P)) = \text{Ann}((N + P/N)) \).

\( M \) is a **faithful** module if \( \text{Ann}(M) = 0 \).

Any \( A \)-module \( M \) is naturally a faithful \( A/\text{Ann}(M) \)-module.

\( x \in M \Rightarrow Ax \subset M \) submodule.

\( M = \sum Ax_i \Rightarrow \{x_i\} \) generates \( M \).

direct sums, direct products of modules.

\( A = a_1 \oplus \cdots \oplus a_n \) as a module \( \iff A \simeq \prod A/b_i \) (where \( b_i = \bigoplus_{i \neq j} a_j \)).
**Finite generation and ideals**

\[ M \text{ is finitely generated as module } \iff M \text{ quotient of } A^r. \]

**Proposition**

\[ M \text{ finitely generated, } \mathfrak{a} \text{ ideal, and } \phi \in \text{End}_A(M) \text{ with } \phi(M) \in \mathfrak{a}M. \text{ Then there is } f \in A[x], \text{ monic with all other coefficients in } \mathfrak{a}, \text{ such that } f(\phi) = 0. \]

Let \( x_i \) generate \( M \) and \( \phi(x_i) = \sum a_{ij}x_j \) (in essence lifting \( \phi \) to \( A^n \)). The element \( f(t) = \det(tI - B) \) has \( f(B) = 0 \) so \( f(\phi) = 0 \). See sleek argument in the book which includes Cayley-Hamilton.

**Corollary**

\[ \text{If } M \text{ finitely generated and } \mathfrak{a}M = M \text{ then there is } x \equiv 1 \mod \mathfrak{a} \text{ such that } xM = 0. \]

Taking \( \phi = id \) get \( f(id) = 1 + a_1 + \cdots \) with \( a_i \in \mathfrak{a} \).
Proposition

If \( M \) finitely generated, \( \mathfrak{a} \subset \mathfrak{R}(A) \), and \( \mathfrak{a}M = M \), then \( M = 0 \).

We get \( xM = 0 \) where \( x \equiv 1 \mod \mathfrak{R}(A) \). We have seen \( x \in A^\times \).
So \( M = 0 \).

Corollary

\( M \) finitely generated, \( N \subset M \), \( \mathfrak{a} \subset \mathfrak{R}(A) \), and \( M = \mathfrak{a}M + N \). Then \( M = N \).

Apply proposition to \( M/N \): note that \( \mathfrak{a}(M/N) = (\mathfrak{a}M + N)/N \).

Theorem (Nakayama)

\( M \) finitely generated, \( (A, \mathfrak{m}) \) local, \( \bar{x}_i \) generate \( M/\mathfrak{m}M \). Then \( x_i \) generate \( M \).

If \( N = \sum A x_i \subset M \) then \( M = \mathfrak{m}M + N \) so \( N = M \).
Recall:

- complexes, exact sequences $\rightarrow M_{i-1} \rightarrow M_i \rightarrow M_{i+1} \rightarrow$.
- injective and surjective as exact sequences
- Short exact sequences
- Splitting an exact sequence into short exact sequences using $N_i = Im(f_i) = Ker(f_{i+1})$. 
Left exactness of $\text{Hom}$

**Proposition**

- $M' \to M \to M'' \to 0$ is exact if and only if for all $N$ the sequence $0 \to \text{Hom}(M'', N) \to \text{Hom}(M, N) \to \text{Hom}(M', N)$ exact.

- $0 \to N' \to N \to N''$ is exact if and only if for all $M$ the sequence $0 \to \text{Hom}(M, N') \to \text{Hom}(M, N) \to \text{Hom}(M, N'')$ exact.

For instance, if $\nu$ surjective, $f : M'' \to N$ and $f \circ \nu = 0$ then $\forall m f(\nu(m)) = 0$ so $\forall m'' f(m'') = 0$ so $f = 0$ $\nu^*$ injective. In the other direction take $f : M'' \to M''/\nu(M)$, so $f \circ \nu = 0$ so $\nu^* f = 0$ so $f = 0$ so $\nu(M) = M''$. Also if $g : M \to N$ such that $g \circ u = 0$ then $g = \bar{g} \circ \nu$ for well-defined $\bar{g}$ so $g \in \text{Im}(\nu^*)$. In the other direction take $g : M \to M/u(M')$, so $g \circ u = 0$ so $g \in \text{Im}(\nu^*)$ so $g = \bar{g} \circ \nu$ so $u(M') \supset \text{Ker}(\nu)$ and equality follows.
Snake lemma

Given commutative diagram of short exact sequences

\[
\begin{array}{cccccccc}
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & N' & \rightarrow & N & \rightarrow & N'' & \rightarrow & 0 \\
\end{array}
\]

get long exact sequence

\[
0 \rightarrow \text{Ker}(f') \rightarrow \text{Ker}(f) \rightarrow \text{Ker}(f'') \xrightarrow{d} \text{Coker}(f') \rightarrow \text{Coker}(f) \rightarrow \text{Coker}(f'') \rightarrow 0.
\]

see

https://en.wikipedia.org/wiki/It%27s_My_Turn_(film)

Homework: find a source on Ext and Tor and write a 4 page summary of construction and main properties, with examples.
Modules

Tensor products

- $M, N, P$ modules, one looks at $A$-bilinear maps $M \times N \to P$.
- The universal one is the tensor product $M \times N \to M \otimes_A N$.
- One constructs this as the quotient of the free module $\bigoplus_{M \times N} A(m, n)$ modulo the bilinear relations.
- It suffices to use just generators of $M, N$.
- The universal property gives associativity, commutativity, distributivity, unit $A \otimes_A M = M$.
- Tensor product is functorial.
- Can do multilinear tensor of many modules.
Finiteness of vanishing

Usually one does not use a particular construction. The following is an example where the book does. I’d like to see a nice proof avoiding this.

**Corollary**

If \( x_i \in M, y_i \in N \) and \( \sum x_i \otimes y_i = 0 \in M \otimes_A N \) then there are finitely generated submodules \( \{x_i\} \subset M_0 \subset M, \{y_j\} \subset N_0 \subset N \), such that \( \sum x_i \otimes y_i = 0 \in M_0 \otimes_A N_0 \)

The element \( \sum (x_i, y_i) \in \bigoplus_{M \times N} A(m, n) \) lies in the submodule generated by the various bilinear relations. In particular it is the sum of finitely many such bilinear relations, involving finitely many \( m_j \in M \) and \( n_j \in N \). Let \( M_0 \) be the submodule generated by \( \{x_i, m_j\} \) and \( N_0 \) be the submodule generated by \( \{y_i, n_j\} \). Then \( \sum (x_i, y_i) \in \bigoplus_{M_0 \times N_0} A(m, n) \subset \bigoplus_{M \times N} A(m, n) \) and it is a combination of bilinear relations inside \( \bigoplus_{M_0 \times N_0} A(m, n) \), as needed.
Restriction and extension of scalars

If $f : A \to B$ a ring homomorphism, and $N$ a $B$-module, then it inherits the structure of an $A$-module by $(a, n) \mapsto f(a)n$. Sometime I denote this by $\_A N$. In particular $B =_A B$ is an $A$-module. Evidently we have

**Proposition**

if $B$ is a finitely generated $A$-module and $N$ a finitely generated $B$ module then $\_A N$ is a finitely generated $A$-module.

Let $M$ be an $A$-module. Then $M_B := B \otimes_A M$ is an $A$-module, with a compatible $B$-module structure by $b(b' \otimes m) = bb' \otimes m$. (Think of it in terms of multiplication $B \otimes_A B \to B$). Evidently we have

**Proposition**

If $M$ is a finitely generated $A$-module then $M_B$ is a finitely generated $B$-module.
Fundamental isomorphism

- By the universal property,
  \( \text{Hom}_A(M \otimes N, P) = \text{Bil}_A(M \times N, P) \) as \( A \)-modules.

- On the other hand by definition of bilinear maps
  \( \text{Bil}_A(M \times N, P) = \text{Hom}_A(M, \text{Hom}_A(N, P)) \) as \( A \)-modules.

- So the functors \( \bullet \otimes N \) and \( \text{Hom}(N, \bullet) \) are adjoint:
  \( \text{Hom}_A(M \otimes N, P) = \text{Hom}_A(M, \text{Hom}_A(N, P)) \).

- Counit: \( \text{Hom}(N, P) \otimes N \to P \); Unit: \( M \to \text{Hom}(N, M \otimes N) \).
Exactness of tensor products

**Proposition**

If \( M' \rightarrow M \rightarrow M'' \rightarrow 0 \) is exact and \( N \) an \( A \)-module then

\[ M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N \rightarrow 0 \]

is exact.

This can be proven by adjunction: since \( M' \rightarrow M \rightarrow M'' \rightarrow 0 \) is exact then for any \( P \)

\[ 0 \rightarrow \text{Hom}(M'', \text{Hom}(N, P)) \rightarrow \text{Hom}(M, \text{Hom}(N, P)) \rightarrow \text{Hom}(M', \text{Hom}(N, P)) \]

is exact, so for any \( P \)

\[ 0 \rightarrow \text{Hom}(M'' \otimes N, P) \rightarrow \text{Hom}(M \otimes N, P) \rightarrow \text{Hom}(M' \otimes N, P) \]

is exact, so \( M' \otimes N \rightarrow M \otimes N \rightarrow M'' \otimes N \rightarrow 0 \) is exact, as needed.

(A left adjoint is right exact and a right adjoint is left exact)
Flatness

In general $\otimes N$ is not an exact functor. For instance $\otimes \mathbb{Z}/2\mathbb{Z}$. A module for which $\otimes N$ is exact is called flat. The notion was introduced as a technical tool in a magnificent paper by Serre (1956). It took over like a wildfire.

Proposition

The following are equivalent: (1) $N$ flat, (2) $N \otimes \bullet$ preserves short exact sequences, (3) $N \otimes \bullet$ preserves injectivity, (4) $N \otimes \bullet$ preserves injectivity for $M' \hookrightarrow M$ with $M', M$ finitely generated.

To prove (4) $\Rightarrow$ (3), if $f : M' \to M$ injective and $\sum m'_i \otimes n_i$ such that $\sum f(m'_i) \otimes n_i = 0$. Write $M'_0 = \sum A m'_i \subset M'$. There is a finitely generated submodule $M_0 \subset M$ containing $f(m'_i)$ such that $\sum f(m'_i) \otimes n_i = 0 \in M_0 \otimes N$. Now $f_0 : M'_0 \to M_0$ still injective and $\sum f_0(m'_i) \otimes n_i = 0$, so by assumption $\sum m'_i \otimes n_i = 0$, as needed.