MA 252 notes: Commutative algebra

(Distilled from [Atiyah-MacDonald])

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Rings of fractions

- Let $S \subset (A, \times)$ be a submonoid (multiplicative subset of A).
- Say $(a, s) \sim (b, t) \in (A \times S)$ if $\exists u \in S : u(at - bs) = 0$.
- S submonoid $\Rightarrow$ equivalence relation.
- Define a set $S^{-1}A = A[S^{-1}] = (A \times S)/\sim$.
- The class of (a, s) is denoted a/s.
- Define $a/s\ b/t = (ab)/(st)$, $a/s + b/t = (at + bs)/(st)$.
- This defines an A-algebra via $a \mapsto a/1$ sending S to $A[S^{-1}]^\times$.

Theorem

If $f : A \to B$ such that $F(S) \subset B^\times$ there is a unique h making

$$\begin{array}{ccc} A & \longrightarrow & A[S^{-1}] \\ & \searrow & \downarrow h \\ & & B \end{array}$$

commutative.

Facts on rings of fractions

- $A[S^{-1}] = 0 \Leftrightarrow 0 \in S$.
- $A \to A[S^{-1}]$ injective $\Leftrightarrow$ S contains no zero divisors.
- $A[S^{-1}] = A[(A^\times S)^{-1}]$.
- $\mathfrak{p} \subset A$ prime $\Leftrightarrow$ $S = A \smallsetminus \mathfrak{p}$ a multiplicative subset.
- If $\mathfrak{p} \subset A$ prime, one writes $A_\mathfrak{p} := A[S^{-1}]$.
- If $A = k[x_1, \ldots, x_n]$ and $\mathfrak{p} = (x_1, \ldots, x_n)$ then $A_\mathfrak{p}$ is the ring of rational functions which are well-defined near the origin. Thus "localization".
- If $A = k[x_1, \ldots, x_n]$ and $\mathfrak{p} \subset A$ prime then $A_\mathfrak{p}$ is the ring of rational functions which can be restricted as rational functions to $Z(\mathfrak{p})$.
- For $f \in A$ write $S = \{f^i : i \geq 0\}$, and $A[S^{-1}] = A[f^{-1}] = A[x]/(fx - 1)$.
- I'll never write A_f (just thing of $p \in \mathbb{Z}$).

Modules of fractions

- Let M and A-module, $S \subset A$ multiplicative subset.
- Say $(m, s) \sim (n, t) \in (N \times S)$ if $\exists u \in S : u(mt - ns) = 0$.
- S submonoid $\Rightarrow$ equivalence relation.
- Define a set $S^{-1}M = M[S^{-1}] = (M \times S)/\sim$.
- The class of (m, s) is denoted m/s.
- Define $a/s\ m/t = (am)/(st)$, $m/s + n/t = (mt + ns)/(st)$.
- This defines an $A[S^{-1}]$-module $M[S^{-1}]$.
- Functor: $M \to N \rightsquigarrow M[S^{-1}] \to N[S^{-1}]$ compatible with compositions. Additive (compatible with direct sums).

Exactness of fractions

Proposition

$M \mapsto M[S^{-1}]$ an exact functor.

By functoriality since $M' \to M \to M''$ a complex (factors through 0) then $M'[S^{-1}] \to M[S^{-1}] \to M''[S^{-1}]$ a complex.
If $g(m/s) = 0$ then there is $u \in S$ with $ug(m) = g(um) = 0$, so $um = f(m')$, so $m/s = f(m'/us)$.

Corollary

$M \mapsto M[S^{-1}]$ preserves sums of submodules, intersections of submodules, quotients.

(Look at the kernel and image of the difference map $N \oplus P \to M$.)

Flatness of $A[S^{-1}]$

- Define $A[S^{-1}] \times M \to M[S^{-1}]$ by $(a/s, m) \mapsto am/s$.
- It is well-defined, surjective and A-bilinear,
- giving $A[S^{-1}] \otimes_A M \to M[S^{-1}]$, an $A[S^{-1}]$-module homomorphism.
- Taking common denominators, every element of $A[S^{-1}] \otimes_A M$ is of the form $1/s \otimes m$.
- If its image $m/s = 0$ then $um = 0$, so $1/s \otimes m = 1/(su) \otimes um = 0$, giving

Proposition

$A[S^{-1}] \otimes_A M \simeq M[S^{-1}]$.

Theorem

$A[S^{-1}]$ is a flat A-module.

Tensor of fractions

Given $f : A \to B$, we have $M_B \otimes_B N_B = (M \otimes_A N)_B$. Indeed the mapping $(b, m, b', n) \mapsto (bb' \otimes (m \otimes m)$ is A-multilinear and B-bilinear, and in the other direction $(b, m, n) \mapsto (b \otimes m) \otimes (1 \otimes n) = (1 \otimes m) \otimes (b \otimes n)$. Combining, we get

Proposition

$M[S^{-1}] \otimes_{A[S^{-1}]} N[S^{-1}] = (M \otimes_A N)[S^{-1}]$.

Local properties

A property $\mathcal{P}$ of a ring/module/module homomorphism is local if
$X \in \mathcal{P} \iff X_{\mathfrak{p}} \in \mathcal{P}\ \forall \mathfrak{p} \in \operatorname{Spec} A$.

Proposition

- $M = 0 \ \Leftrightarrow\ M_{\mathfrak{p}} = 0\ \forall \mathfrak{p} \ \Leftrightarrow\ M_{\mathfrak{m}} = 0\ \forall \mathfrak{m}$.
- $\phi : M \to N$ *injective* $\Leftrightarrow$ $\phi_{\mathfrak{p}}$ *injective* $\forall \mathfrak{p}$ $\Leftrightarrow$ $\mathfrak{m}_{\mathfrak{m}}$ *injective* $\forall \mathfrak{m}$.
- M *flat* $\Leftrightarrow$ $M_{\mathfrak{p}}$ *flat* $\forall \mathfrak{p}$ $\Leftrightarrow$ $M_{\mathfrak{m}}$ *flat* $\forall \mathfrak{m}$.

First, for $0 \neq x \in M$ we have $\mathfrak{a} = Ann(x) \neq A$. Let $\mathfrak{m} \supset \mathfrak{a}$ maximal. Then $0 = x/1 \in M_{\mathfrak{m}}$, so $ux = 0$ for some $u \in A \smallsetminus \mathfrak{m} \subset A \smallsetminus \mathfrak{a}$, contradiction.
Now $(\operatorname{Ker}\phi)_{\mathfrak{m}} = \operatorname{Ker}(\phi_{\mathfrak{m}})$ by exactness, so the second statement follows.
Finally, $N \hookrightarrow P \Rightarrow N_{\mathfrak{m}} \hookrightarrow P_{\mathfrak{m}} \Rightarrow N_{\mathfrak{m}} \otimes M_{\mathfrak{m}} \hookrightarrow P_{\mathfrak{m}} \otimes M_{\mathfrak{m}}$
$\Rightarrow (N \otimes M)_{\mathfrak{m}} \hookrightarrow (P \otimes M)_{\mathfrak{m}} \Rightarrow N \otimes M \hookrightarrow P \otimes M$.

Extending and contracting ideals

Write $f : A \to A[S^{-1}]$ and $\mathfrak{a} \mapsto \mathfrak{a}^e = \mathfrak{a}[S^{-1}]$, $\mathfrak{b}^c = f^{-1}\mathfrak{b}$, extended ideals $= E$, contracted ideals $= C$.

Proposition

- *Every* $\mathfrak{b} \in E$.
- $\mathfrak{a}^{ec} = \cup_S(\mathfrak{a} : s)$.
- $\mathfrak{a}^e = 1 \Leftrightarrow \mathfrak{a} \cap S \neq \emptyset$.
- $1 \neq \mathfrak{a} \in C \Leftrightarrow$ *no element of* S *is a zero divisor of* $A/\mathfrak{a}$.
- $\operatorname{Spec} A[S^{-1}] \leftrightarrow \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset\}$,
  - *in particular* $\operatorname{Spec} A_{\mathfrak{q}} \leftrightarrow \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \subset \mathfrak{q}\}$.
  - $\mathfrak{N}(A[S^{-1}]) = (\mathfrak{N}(A))[S^{-1}]$.
- $\mathfrak{a} \mapsto \mathfrak{a}[S^{-1}]$ *is compatible with finite sums, products, intersections, radicals.*

Extending and contracting: proofs

- Every $\mathfrak{b} \in E$: need $\mathfrak{b} \subset \mathfrak{b}^{ce}$. Let $y = x/s \in \mathfrak{b}$ then $x/1 \in \mathfrak{b}$ so $x \in \mathfrak{b}^c$ so $x/s \in \mathfrak{b}^{ce}$.
- $\mathfrak{a}^{ec}$: $x \in \mathfrak{a}^{ec} \Leftrightarrow x/1 \in \mathfrak{a}^e \Leftrightarrow x/1 = a/s \Leftrightarrow (xs - a)t = 0 \Leftrightarrow xst \in \mathfrak{a} \Leftrightarrow x \in \cup_S(\mathfrak{a} : s)$.
- $\mathfrak{a}^e = 1 \Leftrightarrow \mathfrak{a}^{ec} = 1 \Leftrightarrow 1 \in (\mathfrak{a} : s) \Leftrightarrow 1s \in \mathfrak{a}$.
- $\mathfrak{a} \in C \Leftrightarrow \mathfrak{a}^{ec} \subset \mathfrak{a} \Leftrightarrow (sx \in \mathfrak{a} \Rightarrow x \in \mathfrak{a}) \Leftrightarrow (\overline{sx} = 0 \in A/\mathfrak{a} \Rightarrow \bar{x} = 0 \in A/\mathfrak{a}) \Leftrightarrow$ no element of S is a zero divisor of $A/\mathfrak{a}$.
- $\mathfrak{p} \in C \Leftrightarrow$ no element of S is a zero divisor of the domain $A/\mathfrak{p} \Leftrightarrow$ no element of S is a zero modulo $\mathfrak{p} \Leftrightarrow \mathfrak{p} \cap S = \emptyset$.
- sums, products work for extension in general, intersections have seen, radicals: if $x/t \in r(\mathfrak{a}[S^{-1}])$ with $(x/t)^n - a/s$, then $(xs)^n = s^{n-1}a \in \mathfrak{a}$ so $x/t = (xs)/(ts) \in (r(\mathfrak{a}))[S^{-1}]$, the other direction works in general.

Fractions and finitely generated modules

Proposition

Let M be a finitely generated A-module, S multiplicative. Then $Ann(M[S^{-1}]) = (Ann(M))[S^{-1}]$.

Corollary

If $N, P \subset M$ and P finitely generated, then $(N : P)[S^{-1}] = (N[S^{-1}] : P[S^{-1}])$.

Proof of corollary.

$(N : P) = Ann((N + P)/N)$. ♠

Fractions and finitely generated modules - proof

- If $M = Ax$ then $M \simeq A/Ann(M)$, so $M[S^{-1}] \simeq A[S^{-1}]/(Ann(M))[S^{-1}]$, so $Ann(M[S^{-1}]) = (Ann(M))[S^{-1}]$ as needed.
- If $M = \sum_{i=1}^{n} Ax_i$ and $M' = \sum_{i=1}^{n-1} Ax_i$, $M'' = Ax_n$, we may assume by induction that the result holds for M'. Now

$$
\begin{aligned}
(Ann(M))[S^{-1}] &= (Ann(M') \cap Ann(M''))[S^{-1}] \\
&= (Ann(M'))[S^{-1}] \cap (Ann(M''))[S^{-1}] \\
&= Ann(M'[S^{-1}]) \cap Ann(M''[S^{-1}]) \\
&= Ann(M''[S^{-1}] + M''[S^{-1}]) = Ann(M'[S^{-1}]).
\end{aligned}
$$

Contracted prime ideals in general

Proposition

Let $f : A \to B$, $\mathfrak{p} \in \operatorname{Spec} A$.
Then $\mathfrak{p} = f^{-1}\mathfrak{q}$ with $\mathfrak{q}$ prime $\Leftrightarrow \mathfrak{p}^{ec} = \mathfrak{p}$.

- If $\mathfrak{p} = \mathfrak{q}^c$ then indeed $\mathfrak{p}^{ec} = \mathfrak{p}$.
- On the other hand assume $\mathfrak{p}^{ec} = \mathfrak{p}$ and consider the subsets $\mathfrak{p}^e$ and the multiplicative set $T = f(A \setminus \mathfrak{p})$.
- The assumption implies these are disjoint.
- So $(\mathfrak{p}^e)[T^{-1}] \subset B[T^{-1}]$ is a proper ideal.
- It lies in a maximal ideal $\mathfrak{m} \subset B[T^{-1}]$.
- If $g : B \to B[T^{-1}]$ let $\mathfrak{q} = g^{-1}\mathfrak{m}$, so $\mathfrak{q}$ is prime.
- We have $\mathfrak{q} \supset g^{-1}(\mathfrak{p}^e[T^{-1}]) \supset \mathfrak{p}^e$ so $\mathfrak{q}^c \supset \mathfrak{p}$,
- and $\mathfrak{q} \cap T = \emptyset$ so $\mathfrak{q}^c \subset \mathfrak{p}$.
- So $\mathfrak{q}^c = \mathfrak{p}$.