MA 206 notes: introduction to resolution of singularities

Notes on Kollár 3.9, 3.13

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Going up

Giovanni defined

Definition

$\mathcal{I} \subset \mathcal{O}_X$ with $\operatorname{maxord}(\mathcal{I}) = m$ is D-balanced if for $i \leq m - 1$ we have $(\mathcal{D}^i(\mathcal{I}))^m \subset \mathcal{I}^{m-i}$.

Assume $\mathcal{I}$ as above is D-balanced and S a smooth hypersurface, $J = \mathcal{I}\mathcal{O}_S$, and assume $S \not\subset V(\mathcal{D}^{m-1}\mathcal{I})$. The goal was to prove:

Theorem (3.84)

A blowup sequence Π^S of order $\geq m$ for (J, m) gives rise to a blowup sequence Π of order m for $\mathcal{I}$.

Since the order at a point of a D-balanced ideal is either m or 0, Giovanni showed that the case of one blowup is immediate.
He also showed that after r steps

$$S_r \cap \operatorname{cosupp}(\Pi_*^{-1}(\mathcal{I}, m)) = \bigcap_{j=0}^{m} \operatorname{cosupp}(\Pi^S)_*^{-1}\left((D^j\mathcal{I})|_S, m - j\right).$$

Proof of going up

We need to show that $\mathrm{cosupp}(J_r, m) \subset \mathrm{cosupp}(I_r, m)$.

$$
\begin{aligned}
S_r \cap \mathrm{cosupp}(\Pi_*^{-1}(\mathcal{I}, m)) &= \bigcap_{j=0}^{m} \mathrm{cosupp}(\Pi^S)_*^{-1}\left((D^j\mathcal{I})|_S, m-j\right) \\
&= \bigcap_{j=0}^{m} \mathrm{cosupp}(\Pi^S)_*^{-1}\left((D^j\mathcal{I})^m|_S, m(m-j)\right) \\
&\supset \bigcap_{j=0}^{m} \mathrm{cosupp}(\Pi^S)_*^{-1}\left(\mathcal{I}^{m-j}|_S, m(m-j)\right) \\
&= \bigcap_{j=0}^{m} \mathrm{cosupp}(\Pi^S)_*^{-1}(\mathcal{I}|_S, m) \\
&= \mathrm{cosupp}(\Pi^S)_*^{-1}(\mathcal{J}, m) = \mathrm{cosupp}\,(\mathcal{J}_r, m)
\end{aligned}
$$

as needed ♠

3.10 Uniqueness

- Let (X, I, E) of maximal order m and maximal contact $j, j' : H, H' \hookrightarrow X$ having normal crossings with E.
- We want to apply a lower dimensional procedure $\mathcal{B}(H, I_H, m, E_H)$, which Giovanni showed gives a procedure $j_*\mathcal{B}(H, I_H, m, E_H)$ for X.
- However to make sure these glue we need to have $j_*\mathcal{B}(H, I_H, m, E_H) = j_*\mathcal{B}(H', I_{H'}, m, E_{H'})$.
- This would be OK if we had an automorphism ϕ of X sending H to H',
- such that $\phi^* I = I$ but also fixing $V(\mathcal{I}, m)$ and inductively fixing $V(\Pi_*^{-1}\mathcal{I}, m)$.

Formal equivalence

Fix a point $p \in X$ such that $\operatorname{maxord}(\mathcal{I}) = \operatorname{ord}_p \mathcal{I}$.

Definition

A formal equivalence at p of H, H' with respect to $(X, \mathcal{I}, E)$ is an automorphism ϕ of $\hat{X}_p$ such that (1) $\phi^* \hat{H} = \hat{H}'$, (2) $\phi^* \hat{\mathcal{I}} = \hat{\mathcal{I}}$, (3) $\phi^* \hat{E}^i = \hat{E}^i$ and (4) $h - \phi^* h \in MC(\hat{\mathcal{I}}) \forall h \in \hat{\mathcal{O}}_x$.

Definition

An étale equivalence at p of H, H' with respect to $(X, \mathcal{I}, E)$ is a pair of étale surjections $\psi, \psi' : U \to X_x$ such that (1) $\phi^* H = \psi'^* H'$, (2) $\psi^* \mathcal{I} = \psi'^* \mathcal{I}$, (3) $\psi^* E^i = \psi'^* E^i$ and (4) $\psi^* h - \psi'^* h \in MC(\psi^* \mathcal{I}) \forall h \in \mathcal{O}_x$.

Definition

$\mathcal{I}$ is MC-invariant if $MC(\mathcal{I})D(\mathcal{I}) \subset \mathcal{I}$.

Theorem (3.92)

Suppose $X, \mathcal{I}, E, H, H'$ as above with $\mathcal{I}$ MC-invariant, $p \in H \cap H'$. Then H, H' are étale equivalent at p with respect to $(X, \mathcal{I}, E)$.

We will use special types of automorphisms.

Definition

An automorphism ψ of $k[\![x_1, \ldots, x_n]\!]$ is of form $\mathbf{1} + B$, where $B \subset \mathfrak{m}$, if $\psi(x_i) - x_i \in B$. We say $\mathcal{I}$ is $\mathbf{1} + B$ if it is invariant under any automorphism of form $\mathbf{1} + B$.

Taylor expansions

Proposition

The following are equivalent.

(1) $\mathcal{I}$ *is* $\mathbf{1} + B$*-invariant.*

(2) $BD(\mathcal{I}) \subset \mathcal{I}$.

(3) $B^j D^j(\mathcal{I}) \subset \mathcal{I}$ *for all* $j \geq 0$.

- Assuming (3) write $f(x+b) = f(x) + \sum b_i \frac{\partial f}{\partial x_i} + \cdots$.
- If $f \in \mathcal{I}$ then for any $s \geq 1$ this gives $\psi(f) \in \sum_{j=0}^{s} B^j D^j(\mathcal{I}) + \mathfrak{m}^{s+1} = \mathcal{I} + \mathfrak{m}^{s+1}$.
- By Krull $\psi(f) \in \mathcal{I}$ giving (1).
- Assuming (1) write $f(x_1 + \lambda_1 b_1, x_2, \ldots, x_n) = \sum_{j=0}^{s} \lambda^j b^j \frac{\partial^j f}{\partial x_1^j} \mathfrak{m}^{s+1} \in \mathcal{I} + \mathfrak{m}^{s+1}$.
- Applying this to $s = 1$ different λ and using Vandermonde we get in particular $\lambda b \frac{\partial f}{\partial x_1} \in \mathcal{I} + \mathfrak{m}^{s+1}$, so $BD(\mathcal{I}) \subset \mathcal{I}$ giving (2).
- (2) implies (3) by the product rule.

Constructing formal and étale equivalence

- Write $H = V(x_1), H' = V(x')$ for $x, x' \in MC(\mathcal{I})$.
- Choose $x_2, \ldots, x_n$ so that $x_1, x_2, \ldots, x_n$ and $x_1', x_2, \ldots, x_n$ are both coordinate systems and $E^i = V(x_{i+1})$.
- Writing $h = x_1 - x_1'$ and considering the automorphism $\phi(x_1') = x_1 = X_1' + h$ and $\phi(x_i) = x_i, i > 1$ we get an automorphism of the form $\mathbf{1} + MC(\mathcal{I})$, so we get a formal equivalence.
- To get an étale equivalence, consider the locus $x_1^1 = x_1^{2'}, x_2^1 = x_2^2, \ldots x_n^1 = x_n^2$ in $X \times X$.
- Its completion at $(p.p)$ is the graph of $\hat{\phi}$, so unramified at (p, p) over each projection.
- A neighborhood gives U étale over each projection.

Uniqueness of blowup sequences

We say that blowup sequences $\mathcal{B}, \mathcal{B}'$ are equivalent if at every point there is an étale equivalence $\psi, \psi' : U \to X$ of $(X, \mathcal{I}, E)$ such that $\psi^*\mathcal{B} = \psi'^*\mathcal{B}'$.

Theorem

If blowup sequences $\mathcal{B}, \mathcal{B}'$ of order $m = \operatorname{maxord}(\mathcal{I})$ for $(X, \mathcal{I}, E)$ are equivalent then they are identical.

By taking finitely many étale neighborhoods we get a covering with equivalence $\psi, \psi' : U \to X$. We claim inductively that

(1) $(X_i, \mathcal{I}_i) = (X'_i, \mathcal{I}'_i)$

(2) ψ, ψ' lift to $\psi_i, \psi'_i : U_i \to X_i$ with $\psi_i^* - \psi_i'^* \in (\Pi_i^U)_*^{-1}(MC(\mathcal{I}), 1)$

(3) $Z_{i-1} = Z'_{i-1}$

The base $i = 0$ follows by assumption. Assume things hold for i, so $Z_i \subset V((\Pi_i^U)_*^{-1}(MC(\mathcal{I}), 1))$, so $Z_i = \psi_i(Z_i^U) = \psi'_i(Z_i^U) = Z'_i$ giving $(3)_{i+1}$. This implies also $(1)_{i+1}$.

Lifting the equivalence

- The universal property gives $\psi_{i+1}, \psi'_{i+1} : U_{i+1} \to X_{i+1}$.
- We claim it is of form $\mathbf{1} + (\Pi^U_{i+1})_*^{-1}(MC(\mathcal{I}), 1)$.
- Choose coordinates so that $Z_i = V(x_1, \ldots, x_k)$.
- We have $\psi_i'^*(x_j) = \psi_i^*(x_j) - b_{ij}$,
- $b_{ij} \in (\Pi^U_i)_*^{-1}(MC(\mathcal{I}), 1)$, in particular vanish on Z^U_i.
- A generic blowup chart:
  $y_1 = x_1/x_k, \ldots, y_{k-1} = x_{k-1}/x_k, y_k = x_k, \ldots, y_n = x_n$.
- $(\pi^U_i)_*^{-1} b_{ij} = \psi^*_{i+1}(x_r) b_{i+1\,j}$, $b_{i+1\,j} \in (\Pi^U_{i+1})_*^{-1}(MC(\mathcal{I}), 1)$.
- Compute:
  $\psi'_{i+1}{}^*(y_j) = (\psi^*_{i+1}(y_j) - b_{i+1\,j})/(1 - b_{i+1\,r}) = \psi^*_{i+1}(y_j) + \epsilon$
  with $\epsilon \in (\Pi^U_{i+1})_*^{-1}(MC(\mathcal{I}), 1)$, as needed.

Order reduction for marked ideals

In other lectures we have seen the inductive proof of functorial order reduction for ideals in dimension $\leq n$, compatible with re-embeddings, assuming functorial order reduction of marked ideals in dimension $< n$

We will show:

Theorem (3.107)

Assume given an order reduction functor $\mathcal{BO}_m$ of ideals in dimension $\leq n$. Then there is an order reduction functor $\mathcal{BMO}$ for marked ideals in dimension n, in such a way that if $\mathcal{I}$ of maximal order m then $\mathcal{BMO}(X, \mathcal{I}, m\emptyset) = \mathcal{BO}_m(X, \mathcal{I}, \emptyset)$.

We will also show its implication: the re-embedding principle:

Corollary

If $Y \subset X$, assume $\mathcal{I} = \mathcal{J}' + \mathcal{I}_Y$ with $\mathcal{J} = \mathcal{J}'\mathcal{O}_Y$ not zero on any component of Y. Then $\mathcal{BMO}(X, \mathcal{I}, 1, \emptyset)$ is the sequence induced by $\mathcal{BMO}(Y, \mathcal{J}, 1, \emptyset)$.

Proof of the re-embedding principle

- The problem is local on X so we may assume Y a complete intersection of codimension r.
- Induction on r says we may assume $Y = V(f)$ a hypersurface.
- $f \in \mathcal{I}$ so $\operatorname{maxord}(\mathcal{I}) = 1$,
- so $\mathcal{BMO}(X, \mathcal{I}, 1, \emptyset) = \mathcal{BO}_1(X, \mathcal{I}, \emptyset)$.
- The latter was constructed to be compatible with re-embeddings, so it is induced by $\mathcal{BMO}(Y, \mathcal{J}, 1, \emptyset)$.

My rule

if you write "Definition-Lemma" you have not thought appropriately about what you want to define and what lemma you want to prove. Don't do it.

Let E be an snc divisor on a regular variety X, and $\mathcal{I} \subset \emptyset_X$ an ideal sheaf.

Definition

$\mathcal{I}$ is said to be monomial if $\mathcal{I} = \prod \mathcal{I}_{E_j}^{a_j}$ where $E_j \subset E$ smooth divisors in X. $\mathcal{I}$ is nowhere monomial if it is not monomial on any neighborhood of $x \in E$.

We note that if $\mathcal{I}$ is monomial (respectively, nowhere monomial) and $Y \to X$ smooth then $\mathcal{I}\mathcal{O}_Y$ is also monomial (respectively, nowhere monomial).

Decomposition of an ideal

Lemma

For any $\mathcal{I}$ there is a unique expression $\mathcal{I} = \mathcal{M}(\mathcal{I})\mathcal{N}(\mathcal{I})$, where $\mathcal{M}(\mathcal{I})$ is monomial and $\mathcal{N}(\mathcal{I})$ is nowhere monomial. If $Y \to X$ smooth then $\mathcal{M}(\mathcal{I}\mathcal{O}_Y) = \mathcal{M}(\mathcal{I})\mathcal{O}_Y$ and $\mathcal{N}(\mathcal{I}\mathcal{O}_Y) = \mathcal{N}(\mathcal{I})\mathcal{O}_Y$

Definition

$\mathcal{M}(\mathcal{I})$ is the monomial part and $\mathcal{N}(\mathcal{I})$ is the nonmonomial part.

I leave the proof for you as an exercise.

Proof of theorem: m-order reduction for $\mathcal{N}(\mathcal{I})$

- If $k = \mathrm{maxord}(\mathcal{N}(\mathcal{I})) \geq m$ consider $\mathcal{BM}_k(X, \mathcal{N}(\mathcal{I}), \mathcal{E})$, reducing the order under k.
- Continuing inductively we obtain a sequence Π_1 of blowings up of order $\geq m$ for $\mathcal{N}(\mathcal{I})$, so in particular of order $\geq m$ for $\mathcal{I}$.
- $\mathcal{N}((\Pi_1)_*^{-1}(\mathcal{I}, m)) = (\Pi_1)_*^{-1}\mathcal{N}(\mathcal{I})$.
- So enough to consider the case $\mathrm{maxord}(\mathcal{N}(\mathcal{I})) < m$.

Separating $V(\mathcal{I}, m)$ from $V(\mathcal{N}(\mathcal{I}))$

Write $s = \mathrm{maxord}\mathcal{N}(\mathcal{I})$.
The following is left as an exercise:

Lemma

$$\mathrm{ord}_Z\, \mathcal{J}_1 \geq s \text{ and } \mathrm{ord}_Z\, \mathcal{J}_2 \geq m \quad \Longleftrightarrow \quad \mathrm{ord}_Z(\mathcal{J}_1^m + \mathcal{J}_2^m) \geq ms$$

Also:

Lemma

A smooth blowup sequence of order ms for $\mathcal{N}(\mathcal{I})^m + \mathcal{I}^s$ is simultaneously a smooth blowup sequence of order s for $\mathcal{N}(\mathcal{I})$ and a smooth blowup sequence of order m for $\mathcal{I}$. It results in $V(\mathcal{I}_r, m) \cap V(\mathcal{N}(\mathcal{I}_r), s) = \emptyset$.

Continuing by induction, we may assume
$V(\mathcal{I}_r, m) \cap V(\mathcal{N}(\mathcal{I}_r)) = \emptyset$.
In other words, we may assume $V(\mathcal{I}_r, m) = V(\mathcal{M}(\mathcal{I}_r), m)$.

Order reduction for $\mathcal{M}(\mathcal{I})$: reducing coefficients below m

- Recall $E = \sum E^j$ with E^j smooth. Write $\mathcal{M} = \prod \mathcal{I}_{E_i^j}^{a_{ij}}$.
- Might as well expand the ordering lexicographically and rewrite $E = \sum E_i^j$ as $E = \sum E^j$ with new indices,
- so that we have $\mathcal{M} = \prod \mathcal{I}_{E^j}^{a_j}$.
- If there is any $a_j > m$ we can blow up E_j. The coefficient in the controlled transform $\pi_*^{-1}\mathcal{M}$ of E_j is reduced by m.

Order reduction for $\mathcal{M}(\mathcal{I})$: *combinatorialinduction*

- Descending induction on the minimum cardinality r of a set I of E^j with $\cap_I E^j \neq \emptyset$ and $\sum_I a_j \geq m$:
- Base: if $\sum a_j < m$ we are done.
- Given r induct on the value of such $\sum_I a_j$ and number of such sets I achieving it.
- Blowing up $\cap_I E^j$ we have a new exceptional with coefficient $\sum_I a_j - m$ in the controlled transform.
- For any new k-tuple intersection we have weight $2\sum_I a_j - m - a_i < \sum_I a_j$, so induction applies.

Compatibility

- Assume $m = \text{maxord}(\mathcal{I})$ and $E = \emptyset$.
- Then $\mathcal{N}(\mathcal{I}) = \mathcal{I}$.
- In the first stem we apply $\mathcal{BO}_m(X, \mathcal{I}, \emptyset)$.
- All blowings up have order m, so $(\Pi_1)_*^{-1}\mathcal{I} = (\Pi_1)_*^{-1}(\mathcal{I}, m)$, with trivial monomial part.
- So $\mathcal{BMO}(X, \mathcal{I}, m, \emptyset) = \mathcal{BO}_m(X, \mathcal{I}, \emptyset)$.