MA 205/206 notes: Derived functors and cohomology
Following Hartshorne

Dan Abramovich
Brown University
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Abelian categories

Definition

An abelian category is a category $\mathcal{A}$ with each $\text{Hom}(A, B)$ provided the structure of an abelian group, such that

- Compositions laws $\text{Hom}(B, C) \times \text{Hom}(A, B) \to \text{Hom}(A, C)$ are bilinear (pre-additive category).
- Finite\(^a\) products and coproducts exist and coincide (additive category), in particular an initial and final object 0 exists.
- Every morphism has a kernel and cokernel (pre-abelian).
- Every monomorphism is the kernel of its cokernel.
- Every epimorphism is the cokernel of its kernel.
- Every morphism factors into an epimorphism followed by a monomorphism (image and coimage in some order).

\(^a\)0 and 2 enough
In an additive category, a kernel of $f : X \to Y$ is a fibered product $X \times_Y 0$.

In an additive category, a cokernel of $f : X \to Y$ is a fibered coproduct $Y \oplus^X 0$.

In any category, $f : X \to Y$ is a monomorphism if $g \mapsto f \circ g$ is injective.

In any category, $f : X \to Y$ is an epimorphism if $g \mapsto g \circ f$ is injective.
Examples of abelian categories

- The prototypical example is $\text{Ab}$.
- The typical example is $\text{Mod}(A)$.
- We will use $\text{Ab}(X)$, $\text{Mod}(\mathcal{O}_X)$, $\text{Qcoh}(\mathcal{O}_X)$,
- and, if $X$ noetherian, $\text{Coh}(\mathcal{O}_X)$.

There is an embedding theorem saying any abelian $\mathcal{F}$ is a full subcategory of $\text{Ab}$. 
A complex $A^\bullet$ in an abelian category $\mathcal{A}$ is a sequence with maps $d^i : A^i \rightarrow A^{i+1}$ such that $d^{i+1} \circ d^i = 0$.

Complexes in $\mathcal{A}$ form an abelian category by requiring arrows to commute with $d$ and doing things componentwise.

$h^i(A^\bullet) = \text{Ker}(d^i)/\text{Im}(d^{i-1})$. It is a functor $\mathcal{C}omp(\mathcal{A}) \rightarrow \mathcal{A}$.

If $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ exact, there is $\delta^i : h^i(C^\bullet) \rightarrow h^{i+1}(A^\bullet)$ with a long exact sequence.

A homotopy between $f, g : A^\bullet \rightarrow B^\bullet$ is a collection $k^i : A^i \rightarrow B^{i-1}$ with $f - g = dk + kd$.

If there is a homotopy we say $f$ and $g$ are homotopic, $f \sim g$.

If $f \sim g$ then $h^i(f) = h^i(g)$. 
A covariant $F : \mathcal{A} \to \mathcal{B}$ is additive if\[ \text{Hom}(A, A') \to \text{Hom}(FA, FA') \]is a homomorphism.

Such $F$ is left exact if $0 \to A' \to A \to A'' \to 0$ exact implies $0 \to FA' \to FA \to FA''$ exact.

For contravariant, require $0 \to FA'' \to FA \to FA'$ exact.

Prototypical example: for fixed $A$ the covariant functor $\text{Hom}(A, \bullet) : \mathcal{A} \to \mathcal{Ab}$

\[ A' \mapsto \text{Hom}(A, A') \]

and the contravariant functor $\text{Hom}(\bullet, A) : \mathcal{A} \to \mathcal{Ab}$

\[ A'' \mapsto \text{Hom}(A'', A) \]

are left exact, by the abelian category axioms.
Injectives, resolutions, derived functors

- $I \in \text{Ob}(\mathcal{A})$ is injective if $\text{Hom}(\bullet, I)$ is exact.
- An injective resolution of $A$ is an exact sequence

$$0 \to A \to I^0 \to I^1 \to \cdots$$

with $I^i$ injective.
- $\mathcal{A}$ has enough injectives if every $A$ is a subobject of an injective, so (inductively) has an injective resolution.

Fix an abelian category $\mathcal{A}$ with enough injectives, and for each object $A$ fix an injective resolution $I_A^\bullet$.

**Definition**

For an additive covariant left exact $F : \mathcal{A} \to \mathcal{B}$ define the right derived functors

$$R^i F(A) = h^i(F(I_A^\bullet)).$$
Basic Properties of $R^iF$

**Theorem**

- $R^iF : \mathcal{A} \to \mathcal{B}$ is additive.
- **Independence**: Changing $I_A^*$ results in a naturally isomorphic functor.
- $R^0F \simeq F$
- For exact $E_A : 0 \to A' \to A \to A'' \to 0$ there is $\delta^i : R^iF(A'') \rightarrow R^{i+1}F(A')$ with long exact sequence

$$R^iF(A') \rightarrow R^iF(A) \rightarrow R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \ldots$$

- This is compatible with morphisms of exact sequences $E_A \to E_B$.
- If $I$ injective and $I > 0$ then $R^iF(I) = 0$. 
We say $J$ is $F$-acyclic if $R^i F(J) = 0$ for all $i > 0$. An $F$-acyclic resolution $J^\bullet$ of $A$ is what you think.

**Proposition**

If $A \to J^\bullet$ is an $F$-acyclic resolution then $R^i F(A) \simeq h^i(F(J^\bullet))$.

Try it at home!
One might think that the injective resolution definition is artificial. Read in Hartshorne (or Grothendieck) about $\delta$-functors, universal $\delta$-functors, effaceable $\delta$-functors.

It is shown that that an effaceable $\delta$-functors is universal, and there can be at most one universal $\delta$-functor up to isomorphism.

Then it is shown that if there are enough injectives, the derived functor is effaceable so it is (the unique) universal $\delta$-functor.
Enough injectives in $\mathbb{A}b$ and $\text{Mod}(A)$

- It is not too hard to see that an abelian group is injective if and only if it is divisible.
- Also one can embed any abelian group in a divisible group, so there are enough injectives in $\mathbb{A}b$.
- If $M$ is an $A$-module and $M \to I_\mathbb{Z}$ be an embedding into a divisible group, then the natural embedding $M \to \text{Hom}_\mathbb{Z}(A, I_\mathbb{Z})$ provide an embedding into an injective $A$-module.
- You can find all this explained in a few pages e.g. in http://www.math.leidenuniv.nl/~edix/tag_2009/michiel_2.pdf.
Enough injectives in $\text{Mod}(\mathcal{O}_X)$

**Proposition (Proposition III.2.2)**

*If* $(X, \mathcal{O}_X)$ *a ringed space*¹ *then* $\text{Mod}(\mathcal{O}_X)$ *has enough injectives.*

¹not necessarily lrs

**Embedding:**

- Embed the stalks $\mathcal{F}_x \hookrightarrow I_x$ in injective $\mathcal{O}_{X,x}$-modules.
- Define $\mathcal{J} = \prod_x j_{x*}(I_x)$.
- We have an embedding $\mathcal{F} \hookrightarrow \prod_x j_{x*}(\mathcal{F}_x) \hookrightarrow \prod_x j_{x*}(I_x) = \mathcal{J}$.

**Injectivity:**

- We have
  
  $$\text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{J}) = \prod \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_{x*}(I_x)) = \prod \text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x).$$

- $\mathcal{G} \hookrightarrow \mathcal{G}_x$ and $\mathcal{G}_x \hookrightarrow \text{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$ are exact,

- so $\mathcal{G} \hookrightarrow \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{J})$ is exact, so $\mathcal{J}$ injective.
Since \((X, \mathbb{Z}_X)\) is a ringed space, \(\mathcal{A}b(X)\) has enough injectives.

### Definition

The **sheaf cohomology** \(H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F})\), the right derived functor of

\[
\mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X),
\]

as a functor

\[
\mathcal{A}b(X) \to \mathcal{A}b.
\]

This is clean, but also disturbing, since it ignores any possible structure on \(X\).
We say $\mathcal{F}$ is flasque\(^1\) if for any opens $V \subset U$ the restriction $\mathcal{F}(U \to \mathcal{F}(V))$ is surjective.

**Proposition**

A flasque sheaf $\mathcal{F}$ of abelian groups is $\Gamma$-acyclic, i.e. $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

**Proposition**

An injective $\mathcal{O}_X$-module is flasque.

**Corollary**

The derived functors of $\Gamma(A, \bullet) : \text{Mod}(\mathcal{O}_X) \to \text{Ab}$ are $H^i(X, \bullet)$.

**Corollary**

if $Y \hookrightarrow X$ closed and $\mathcal{F}$ on $Y$, then $H^k(Y, \mathcal{F}) = H^k(X, i_*\mathcal{F})$.

\(^1\)flabby
An injective $\mathcal{O}_X$-module is flasque

This uses the extension by 0 functor $j_!$ (II.1.19):

- For an open $U \xrightarrow{j_U} X$ consider the $\mathcal{O}_X$-module $\mathcal{O}_U := j_U!(\mathcal{O}_X|_U)$.
- $\text{Hom}(\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$.
- For $V \subset U$ get inclusion $\mathcal{O}_V \to \mathcal{O}_U$.
- Take $\mathcal{I}$ injective, and so $\text{Hom}(\mathcal{O}_U, \mathcal{I}) \to \text{Hom}(\mathcal{O}_V, \mathcal{I})$ surjective.
- So $\mathcal{I}(U) \to \mathcal{I}(V)$ surjective.
If $\mathcal{F}$ flasque then $H^i(X, \mathcal{F}) = 0$ for $i > 0$

This follows from general properties of flasque sheaves (exercise II.1.16), and requires an induction on $i$ for all flasque sheaves:

- Let $\mathcal{F} \to I$ be an embedding in an injective sheaf in $\mathcal{A}b(X)$, with s.e.s
  \[0 \to \mathcal{F} \to I \to \mathcal{G} \to 0.\]

- Since $\mathcal{F}$ and $I$ flasque it follows that $\mathcal{G}$ flasque.

- It also implies $0 \to \mathcal{F}(X) \to I(X) \to \mathcal{G}(X) \to 0$ exact.\(^2\)

- Since $H^i(X, I) = 0$ for $i > 0$, the l.e.s gives $H^1(X, \mathcal{F}) = 0$ and $H^i(X, \mathcal{G}) = H^{i+1}(X, \mathcal{F})$ for $i > 0$.

- Induction gives the result.

\(^2\)we did something similar
The derived functors of $\Gamma(A, \bullet) : \text{Mod}(\mathcal{O}_X) \to \text{Ab}$ coincide with $H^i(X, \bullet)$.

- We compute the derived functor by taking an $\mathcal{O}_X$-injective resolution $\mathcal{F} \to I^\bullet$.
- This is a flasque resolution, which is acyclic, hence computes cohomology.

If $Y \xrightarrow{i} X$ closed then $H^k(Y, \mathcal{F}) = H^k(X, i_*\mathcal{F})$.

If $I^\bullet$ is an injective, hence flasque, resolution of $\mathcal{F}$ then $i_*I^\bullet$ is a flasque resolution of $i_*\mathcal{F}$. 

if $Y \xrightarrow{i} X$ closed then $H^k(Y, \mathcal{F}) = H^k(X, i_*\mathcal{F})$.
Grothendieck’s dimension theorem

**Theorem**

Let $X$ be a noetherian topological space of dimension $n$. Then for $i > n$ and for all sheaf $\mathcal{F}$ on $X$ we have $H^i(X, \mathcal{F}) = 0$.

This is an involved but elegant sequence of reduction steps, with some general input about colimits.

- One uses $i_*$ and $j_!$ and l.e.s to reduce to the case of irreducibles.
- One uses induction on dimension, the case of dimension 0 being trivial.
- One reduces using colimits to sheaves generated by finitely many sections.
- One uses the l.e.s to reduce to $j_! \mathbb{Z}_U$ and subsheaves of such.
- One uses the l.e.s to reduce to $j_! \mathbb{Z}_U$.
- One uses the l.e.s for $j_! \mathbb{Z}_U \subset \mathbb{Z}_X$ and flasqueness of $\mathbb{Z}_X$ to conclude.
Proposition

Let $A$ be a noetherian ring, $I$ an injective $A$-module. Let $\tilde{I}$ be the associated sheaf on $\text{Spec} A$. Then $\tilde{I}$ is flasque.

The proof is non trivial and goes by way of showing that the sheaves of sections with support along an ideal $\alpha$ is also injective.

Theorem

Let $A$ be a noetherian ring, $X = \text{Spec} A$, and $\mathcal{F}$ quasicoherent. Then for all $i > 0$ we have $H^i(X, \mathcal{F}) = 0$.

Set $M = \Gamma(X, \mathcal{F})$, and let $0 \to M \to I^\bullet$ injective resolution. We have learned that $0 \to \mathcal{F} \to \tilde{I}^\bullet$ exact, flasque resolution! and taking sections $0 \to M \to \Gamma(X, \tilde{I}^\bullet)$ also exact.

So $H^i(X, \mathcal{F}) = h^i(\Gamma(X, \tilde{I}^\bullet)) = 0$.
Corollary

If $X$ is noetherian, $\mathcal{F}$ quasicoherent, then there is an embedding $\mathcal{F} \hookrightarrow \mathcal{G}$ with $\mathcal{G}$ quasicoherent and flasque.

Let $X = \bigcup_{\text{finite}} U_i$

with $U_i = \text{Spec } A_i \xrightarrow{j_i} X$,

write $\mathcal{F}(U_i) = M_i$ and choose $M_i \subset I_i$, with $I_i$ injective $A_i$ modules.

Then $\mathcal{F} \hookrightarrow \bigoplus j_i^! \tilde{I}_i$ injective,

and $\tilde{I}_i$ flasque hence $\bigoplus j_i^! \tilde{I}_i$ flasque.
Serre’s criterion

**Theorem**

Suppose $X$ either noetherian or separated and quasicompact. Then the following are equivalent:

(i) $X$ affine.

(ii) $H^p(X, \mathcal{F})$ for every quasicoherent $\mathcal{F}$ and $p > 0$.

(iii) $H^1(X, \mathcal{F})$ for every quasicoherent $\mathcal{F}$.

(iv) $H^1(X, \mathcal{I})$ for every quasicoherent ideal $\mathcal{I}$.

- Let $A = \mathcal{O}(X)$. Need $\phi : X \to \text{Spec } A$ an isomorphism.
- For $f \in A$ we have $X_f = \phi^{-1}D(f)$ and by an old result $\mathcal{O}_X(X_f) = A[f^{-1}]$.
- If $X_f$ affine then $\phi_{X_f} : X_f \to D(f)$ an isomorphism,
- so it suffices to show (1) each $x \in X$ lies in an affine $X_f$, and (2) $\phi$ surjective.
Serre’s criterion, (1) each \( x \in X \) lies in an affine \( X_f \)

- The closure \( \{x\} \) is quasicompact, hence has a closed point;
- might as well assume \( x \) closed.
- Let \( \mathcal{M} = \mathcal{I}_{\{x\}} \). Let \( U \ni x \) be an affine neighborhood. Let \( J = \mathcal{I}_{X \setminus U} \).
- \( 0 \to \mathcal{M}J \to J \to J/MJ \to 0 \) is exact.
- The latter is a skyscraper with fiber \( k(x) \) at \( x \).
- By assumption \( H^1(X, \mathcal{M}J) = 0 \),
- and by the long exact sequence there is \( f \in J \) such that \( f(x) \neq 0 \).
- Note that \( X_f = D_U(f) \) is an affine neighborhood of \( x \). ♦
Take finitely many \( f_i \) so that \( X = \bigcup X_{f_i} \).

Need to show \( A = \bigcup X_{f_i} \), namely \((f_1, \ldots, f_m) = (1)\).

Consider \( \psi : \mathcal{O}^n_X \to \mathcal{O}_X \), where \( \psi(a_1, \ldots, a_n) = \sum a_i f_i \).

0 \to \text{Ker}\psi \to \mathcal{O}^n \to \mathcal{O} \to 0 \) is an exact sequence of quasicoherent sheaves.

Enough to show \( H^1(X, \text{Ker}\psi) = 0 \), since then we have \( A^n \to A \) surjective, as needed.

Write \( K_i = \text{Ker}\psi \cap \mathcal{O}_X^i \) and \( Q_i = K_i / K_{i-1} \).

Note: \( Q_i \subset \mathcal{O}_X \) a sheaf of ideals.

So \( H^1(X, Q_i) = 0 \), and by l.e.s and induction \( H^1(X, \text{Ker}\psi) = 0 \), as needed.
Čech cohomology of sheaves

- Given a covering $\mathcal{U} := \{U_i\}$ of $X$ one defines a complex
  
  $0 \to \mathcal{F}(X) \to C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d_0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d_1} \cdots$,

- where $C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0}, \ldots, U_{i_p})$.

- For $f \in C^p(\mathcal{U}, \mathcal{F})$ one defines
  
  \[ df = \sum_{0}^{p+1} (-1)^k f_{i_0, \ldots, \hat{i}_k, \ldots, i_{p+1}}|_{U_{i_0}, \ldots, i_{p+1}}. \]

- Exercise: $d^2 = 0$.

- Define $\check{H}^p(\mathcal{U}, \mathcal{F}) = h^i(C^\bullet(\mathcal{U}, \mathcal{F}))$. 
Proposition

\[ \check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X). \]

Indeed for a collection of sections \( s_i \in \mathcal{F}(U_i) \) do have \( d_0((s_i)) = 0 \) we need precisely \( s_i|_{U_{ij}} - s_j|_{U_{ij}} = 0 \), and the sheaf axiom gives \( s \in \mathcal{F}(X) \).

Observation

If \( \mathcal{U} \) contains \( n \) opens then \( \check{H}^p(\mathcal{U}, \mathcal{F}) = 0 \) for all \( p \geq n \).

Instead of \( C(\mathcal{U}, \mathcal{F}) \) one can work with alternating chains \( C'(\mathcal{U}, \mathcal{F}) \). We have \( \check{H}(\mathcal{U}, \mathcal{F}) = h^i(C'(\mathcal{U}, \mathcal{F})) \).
Example: Consider $X = \mathbb{P}^1_A$ with the open sets $U_i = D_+(T_i)$.

The Čech complex $C(\mathcal{U}, \mathcal{O}_X)$ is

$$0 \to A[t] \oplus A[t^{-1}] \xrightarrow{d_0} A[t, t^{-1}] \to 0 \cdots .$$

- $\check{H}(\mathcal{U}, \mathcal{O}_X) = \text{Ker}(d_0) = A$,
- $\check{H}^1(\mathcal{U}, \mathcal{O}_X) = \text{Coker}(d_0) = 0$,
- and the rest is 0.

The Čech complex $C(\mathcal{U}, \Omega_X)$ is

$$0 \to A[t]dt \oplus A[s]ds \xrightarrow{d_0} A[t, t^{-1}]dt \to 0 \cdots .$$

- Here $ds$ maps to $-dt/t^2$.
- $\check{H}(\mathcal{U}, \Omega_X) = \text{Ker}(d_0) = 0$,
- $\check{H}^1(\mathcal{U}, \Omega_X) = \text{Coker}(d_0) = A dt/t$,
- (and the rest is 0).
Čech complex of sheaves

**Theorem**

If $X$ noetherian separated, $\mathcal{U}$ affine covering, $\mathcal{F}$ quasicoherent, there is a functorial isom $\check{H}^p(\mathcal{U}, \mathcal{F}) \sim H^p(X, \mathcal{F})$.

- Set $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \ldots < i_p} (j_{i_0}, \ldots, i_p)^* \mathcal{F}$.
- Set $d : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$ as before.
- Note: $\Gamma(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})) = C^p(\mathcal{U}, \mathcal{F})$.

**Lemma**

For any $\mathcal{F}$ we have a resolution $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$.

**Proposition**

If $\mathcal{F}$ is flasque then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

**Lemma**

There are functorial maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$. 
Lemma: we have a resolution $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$

- Check on stalks, say at $x \in U_j$.
- Construct homotopy $k : \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x \to \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})_x$.
- A section $\alpha_x \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x$ lifts to $\alpha \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})(V)$ for some neighborhood $x \in V \subset U_j$.
- Viewing $\alpha$ as an alternating cochain define $k(\alpha)_{i_0,\ldots,i_{p-1}} = \alpha_{j,i_0,\ldots,i_{p-1}}$.

- This is well defined on $V$, hence at $x$, since $V \subset U_j$.
- Check $(dk + kd)(\alpha) = \alpha$, so $id \sim 0$ and $h^p(\mathcal{C}^\bullet_\mathcal{U}) = 0$. 

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Proposition: If $\mathcal{F}$ flasque then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.

- Products and direct images of flasque sheaves are flasque.
- The sheaves $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are thus flasque.
- We have a flasque resolution $0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$,
- hence
  
  \[ H^p(X, \mathcal{F}) = h^p(\Gamma(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}))) = h^p(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})) = \check{H}^p(\mathcal{U}, \mathcal{F}). \]
- But since $\mathcal{F}$ is itself flasque we have
  
  \[ \check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}) = 0 \]

  for $p > 0$. 
Lemma: There are functorial maps $\check{H}^p(U, F) \to H^p(X, F)$.

- Take an injective resolution $0 \to F \to J^\bullet$.
- Since $F \to C^0(F)$ injective there is an extension $C^0(F) \to J^0$.
- The composite $C^0(F) \to J^1$ factors through $C^0(F)/F$.
- Since $C^0(F)/F \to C^1(F)$ injective, there is an extension $C^1(F) \to J^1$.
- Induction provides an arrow $C^\bullet(U, F) \to J^\bullet$.
- Then apply $h^p(\Gamma(X, \bullet))$.
- Functoriality is trickier
X noetherian separated, \( U \) affine, \( \mathcal{F} \) quasicoherent
\[ \Rightarrow \check{H}^p(U, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}). \]

- Induction on \( p \) for all \( \mathcal{F} \) etc.
- Take \( 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0 \) s.e.s with \( \mathcal{G} \) quasicoherent flasque (so \( \mathcal{Q} \) quasicoherent).
- \( X \) separated \( \Rightarrow \) \( U_{i_0}, \ldots, i_p \) affine.
- \( \Rightarrow 0 \to \mathcal{F}(U_{i_0}, \ldots, i_p) \to \mathcal{G}(U_{i_0}, \ldots, i_p) \to \mathcal{Q}(U_{i_0}, \ldots, i_p) \to 0 \) exact.
- \( \Rightarrow 0 \to C^\bullet(U, \mathcal{F}) \to C^\bullet(U, \mathcal{G}) \to C^\bullet(U, \mathcal{Q}) \to 0 \) exact.
- Since \( \mathcal{G} \) flasque \( \check{H}^p(U, \mathcal{G}) = H^p(X, \mathcal{G}) = 0 \) for \( p > 0 \).
- The long exact sequences and functoriality give

\[
\begin{align*}
0 & \Rightarrow \check{H}^0(U, \mathcal{F}) \Rightarrow \check{H}^0(U, \mathcal{G}) \Rightarrow \check{H}^0(U, \mathcal{Q}) \Rightarrow \check{H}^1(U, \mathcal{F}) & \Rightarrow 0 & \text{and} & \check{H}^p(U, \mathcal{Q}) & \Rightarrow \check{H}^{p+1}(U, \mathcal{F}) \\
\downarrow & & \downarrow & & & \downarrow \\
0 & \Rightarrow H^0(X, \mathcal{F}) \Rightarrow H^0(X, \mathcal{G}) \Rightarrow H^0(X, \mathcal{Q}) \Rightarrow H^1(X, \mathcal{F}) & \Rightarrow 0 & & \Rightarrow H^p(X, \mathcal{Q}) & \Rightarrow H^{p+1}(X, \mathcal{F})
\end{align*}
\]

Induction gives the required isomorphisms. ♠