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We are interested in intrinsic properties (dimension, smoothness . . . )

We are interested in ways to embed a variety in projective space,

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An affine scheme $\text{Spec } A$ is the set of primes in the commutative ring $A$,

which is a departure from varieties, where maximal ideals are taken.

$\text{Spec } A$ is provided a topology by declaring $V(f) = \{p : f \in p\}$ to be closed

(or $D(f) = \{p : f \not\in p\}$ to be open).

$X = \text{Spec } A$ is made a locally ringed space by declaring 

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Schemes are locally ringed spaces which are locally affine schemes.

Arrows are arrows of locally ringed spaces (so $\text{Sch} \subset \text{LRS}$ a full subcategory).

$\text{AffSch} \simeq \text{ComRings}^{op}$.

Then starts a barrage of adjectives: reduced, irreducible, integral, quasicompact, noetherian, regular, ... 

Further adjectives for morphisms (or $S$-schemes).

Important: *separated* and *proper* morphisms.

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We are working with schemes $X$.

The structure is governed by sheaves of abelian groups, such as $\mathcal{O}_X$.

Most important are sheaves of $\mathcal{O}_X$-modules.

Particularly useful are quasi-coherent sheaves of $\mathcal{O}_X$-modules.

We want to understand their sections.

For instance: we classified morphisms $X \to \mathbb{P}^n$ through sections of an invertible sheaf.$^1$

Understanding sections is a fundamental question of varieties.

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Reminder: failure of right-exactness

Recall the sheaf axiom $0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_{ij})$.

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact then $0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X)$ exact…

but right exactness fails in general:

say $Y = \text{two points in } X = \mathbb{P}^1_k$;

then $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$, but $0 \to 0 \to k \to k^2 \to 0$ is not.
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Measuring the failure

- Failure of right exactness is a fact of life\(^2\).
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\(^2\text{Mathematical life} \)
We’ll follow Hartshorne, who follows **Grothendieck**, *Sur quelques points d’algèbre homologique*[^3], to resolve this using derived functors. This works in the context of left-exact additive functors on abelian categories with enough injective objects.

Liu follows **Serre**, *Faisceaux algèbriques cohérents*, to resolve using Čech cohomology. This works for sections of quasi-coherent sheaves, and will be subsumed in Hartshorne’s treatment.

An important modern approach uses derived categories (**Gelfand–Manin**, **Weibel**), still in the additive realm.

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