① recall: we were taught that

if I is D-balanced $\star$, H M.C. hypersurf

then a BUS for $I|H$ of order $\geq m$

gives a BUS for I of order ~~$\geq$~~ m

and vice versa.

$\star$ D-balanced: $(D^i I)^{\text{sat}} \subset \cancel{D}(I)^{\text{sat}-i}$

(2) Usually we only find MaCi hypersurfaces *locally* (still looking for an example).

So if we have a BUS for I_H on $\mathcal{U}$, and for $I_{H'}$ on $\mathcal{U}'$ what happens on $\mathcal{U} \cap \mathcal{U}'$??

Włodarczyk's idea: (X, I, E, H) is "locally isomorphic" to (X, I, E, H'), if I is "tuned".

So a BUS *Functor*, necessarily compatible with "local isomorphisms", will automatically "coincide" on $\mathcal{U} \cap \mathcal{U}'$.

③ for this purpose:

I is "toned" $\iff$ "MC invariant"

$\overset{\text{def}}{\iff}$ $MC(I) \cdot D(I) \subset I$.

<u>note</u>: different from D-balanced, but will be able to "ensure" both.

(4) local isomorphism: formal version

$p \in X$ $\hat{X}$ = formal completion = spec $\hat{\mathcal{O}}_{X,x}$.

$\hat{\mathcal{O}}_{X,x} = \hat{\mathcal{O}}$

$\phi : \hat{X} \longrightarrow \hat{X}$ s.t.

(1) $\phi(\hat{H}) = \hat{H}'$

(2) $\phi^*(\hat{I}) = \hat{I}$

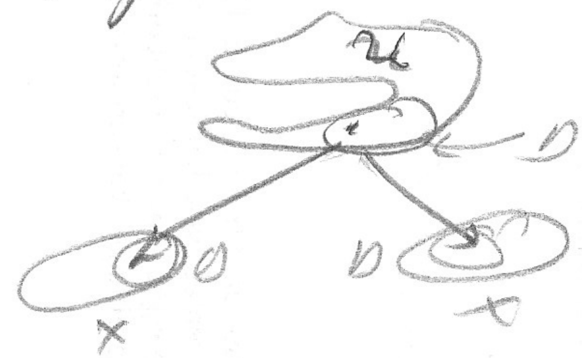
(3) $\phi(\hat{E}^i) = \hat{E}^i$ (if a NCD specified)

(4) $\forall h \in \hat{\mathcal{O}}$ $h - \phi^* h \in MC(\hat{I})$.

condition 4 guarantees that ϕ is the identity on $V(MC(I))$
$= V(I, m)$

(5) formal completion is not a "finite" "algebraic" operation. But can be approximated:

$$\psi, \psi' : \underset{q}{\mathcal{U}} \to \underset{p}{X}$$ an étale surjection



gives isom on completion.

(1) $\psi^{-1}(H) = \psi'^{-1}(H)$

(2) $\psi^* I = \psi'^* I$

(3) $\psi^{-1}(E') = \psi'^{-1}(E')$ if NCD specified

(4) $\psi^* h - \psi'^* h \in MC(\psi^*(I))$ OK.

$= MC(\psi'^*(I))$

(6) invariance theorem

<u>Thm</u> if X smooth, I MC-invariant,

$m = \max \operatorname{ord} I$, E snc, H, H' MC hypersurfaces

at $*$ with nc w/E.

then H, H' <u>étale</u> equiv w.r.t. (X, I, E)

(first we prove <u>formal</u> equiv, then approx)

⑦ we work w/ $R = k[[x_1, \dots, x_n]]$ $\underline{m} = (x_1, \dots, x_n)$

Nakayama: consider the homomorphism

ϕ: $x_i \longmapsto g_i$ $g_i \in \underline{m}$

then ϕ is an automorphism

$\Longleftrightarrow$ $\overline{x_i} \mapsto \overline{g_i}$ the linear map on m/m^2

is an automorphism.

$\Longleftrightarrow$ $\overline{g_i}$ lin indep.

(8) $B \subset m$ an ideal $b_i \in B$

then for generic $\lambda_j \in K$

the map $\bar{x}_i \mapsto \bar{x}_i + \lambda \overline{b_i}$

is invertible (invertible on $\lambda = 0$ $\Rightarrow$ open set)

So the map $x_i \mapsto x_i + \lambda b_i$

is an autom.

<u>Def</u>: these are autom. of the form $\mathbb{1} + B$.

If I is invariant under $\mathbb{1} + B$ say "B - invariant"

(9) characterization of invariance

Prop 3.94: TFAE

(1) I invariant under $\mathbb{1}+B$ autom.

(2) $B \cdot D(I) \subset I$

(3) $B^j \cdot D^j(I) \subset I \quad \forall j \geq 1$

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(2) ⇒ (3): induction on j.

$B^{j+1} \cdot D^{j+1}(I)$ generated by $b_0 \cdots b_j \cdot \partial g$

$g \in D^j(I)$

$$b_0 \cdots b_j \, \partial g = b_0 \, \partial(b_1 \cdots b_j g) - \sum_{i \geq 1} \overbrace{(b_0 \cdots \hat{b}_i \cdots b_j g)}^{\in B^j} \underline{D(b_i)}$$

$\in B \, D(B^j D^j I) + B^j (D^j I)$

$\subset B \, D(I) + I \subset I.$

10 (3) $\Rightarrow$ (1): $f \in I$ then $\phi^b f = f(x_1+b_1, \dots, x_n+b_n)$

$$\phi^b f = f(x_1, \dots, x_n) + \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum b_i b_{i'} \frac{\partial^2 f}{\partial x_i \partial x_{i'}} + \cdots$$

$$\phi^b f \in I + BDI + \cdots + B^i \cdot D^i(I) + \cdots + m^{s+1} V_s$$

$$\Rightarrow \phi^b f \in I + m^{s+1} V_s$$

$$\Rightarrow \phi^b f \in I$$

(1) $\Rightarrow$ (2): if I is invariant, then for b, λ_i

$$f(x_1 + \lambda b_1, \dots, x_n + \lambda b_n) = f + \sum \frac{(\lambda b)^s}{s!} \frac{\partial^j f}{\partial x_i^s} + m^{s+1}$$

for s different λ_i get van der Monde

So $b \frac{\partial f}{\partial x_i} \in I + m^{s+1} \Rightarrow \in I$

$$\Rightarrow BD(I) \subset I$$

(11) Proof of invariance theorem (P 6): formal step

choose: $H = \{x_1 = 0\}$, $H' = \{x_1' = 0\}$, $x_2, \ldots, x_{s+1}$ for E'

$x_{s+2}, \ldots, x_n$ remaining params

So that both $(x_1, x_2, \ldots, x_n)$ and $(x_1', x_2, \ldots, x_n)$ are regular systems of parameters.

note: $\delta = x_1 - x_1' \in MC(I)$

so $\phi(x_1', x_2, \ldots, x_n) = (x_1' + \delta, x_2, \ldots x_n) = (x_1, \ldots x_n)$

is $\mathbb{1} + MC(I)$.

By 3.94 (characterization) $\phi^* \hat{I} = \hat{I}$ (2)

clearly (1) $\phi^* \hat{H} = \hat{H}'$, (3) $\phi^* \hat{E}' = \hat{E}'$

and checking on variables, $h - \phi^* h \in MC(I)$

(12) proof of Main thm (3.92) - étale case

on $X \times X$ consider the closed set $\mathcal{U}_p = V(x_{11} - x_{21}', x_{12} - x_{22}, \ldots, x_{1n} - x_{2n})$

$\hookrightarrow$ coord $x_{11}, \ldots x_{1n}$, x_{21}', x_{22}, x_{2n}

<u>note:</u> $\hat{\mathcal{U}}_{(p,p)}$ = graph of $\hat{\Phi}_p$.

get ~~std~~ som. on completions $\hat{\mathcal{U}}_p \xrightarrow{\hat{\psi}_1} \hat{X}_p$, $\hat{\mathcal{U}}_p \xrightarrow{\hat{\psi}_2} \hat{X}_p$

<u>Fact:</u> there is a nbhd $\mathcal{U}_p^0$ of (p,p) s.t. ψ_i are étale.

<u>Fact:</u> equality of ideals after completion $\Rightarrow$ equality on a nbhd

this gives a nbhd for p. take a finite union of $\mathcal{U}_p$'s.

(B) Etale equivalent BUS are equal:

sos $(X_r, I_r) \to \cdots \to (X_0, I_0) = (X, I)$ $(\mathbb{B})$

$(X_r', I_r') \to \cdots \to (X_0', I_0') = (X, I)$ $(\mathbb{B}')$

are mBUS. suppose $\exists$ $\psi, \psi' : U \to X$ etale,

m = maxord $\underline{I}$

$\psi^* I = \psi'^* I$

$\psi^* h - \psi'^* h \in MC(\psi^* I)$ $\forall h \in \mathcal{O}$

$\psi^* \mathbb{B} = \psi'^* \mathbb{B}'$

then they are "etale equivalent"

$\underline{\text{Thm 3.97}}$ if étale equiv then $\mathbb{B} = \mathbb{B}'$

(14) proof of equality of ~~stable~~ equivalent BUS (3.47):

We show by induction

(1) $(X_i, I_i) = (X_i', I_i')$

(2) ψ, ψ' give $\psi_i, \psi_i' : U_i \to X_i$

s.t. $\psi_i^* h - \psi_i'^* h \in (\Pi_i^U)_*^{-1} \overbrace{\left(MC(I^U, 1)\right)}^{M_i}$

(3) and the centers $Z_{i-1} = Z_{i-1}'$.

$i=0$ trivial, use induction.

Set $W_i = V(M_i)$. By defn. $Z_i^U \subset W_i$
$\subset U$

By induction $\psi_i|_{W_i} = \psi_i'|_{W_i}$ so (3) $Z_i = \psi_i(Z_i^U) = \psi_i'(Z_i^U) = Z_i'$

so (1) $X_{i+1} = X_{i+1}'$.

(15) cont

set coords st $z_i = z_i' + \ell(x_1, \dots, x_k)$ -

<u>induct</u>: $\psi_i'^*(x_i) = \psi_i^*(x_i) - b_{ij}$ $\quad b_{ij} \in M_i$

X_{i+1}: $y_1 = \frac{x_1}{x_r}, \dots, y_{r-1} = \frac{x_{r-1}}{x_r}, y_r = x_r, \dots, y_n = x_n$.

Claim: $\psi_i^{**}(y_j) - \psi_{i+1}'^*(x_j) \in M_{j+1}$

note: ok for $y_r, \dots, y_n$.

$b_{ij} \in M_i \rightarrow \pi_i^{\nu *} b_{ij} = \psi_{r+1}^*(x_r)\, \underline{b_{i+1,j}}$ for $b_{i+1,j} \in M_{j+1}$

$$\psi_{i+1}'^*(y_j) = (\pi_i^{\nu})^* \left(\frac{\psi_i'^*(x_j)}{\psi_i'^*(x_r)} \right) = (\pi_i^{\nu})^* \frac{\psi_i^*(x_j) - b_{ij}}{\psi_i^*(x_r) - b_{ir}}$$

$$= \frac{\psi_{i+1}^*(x_j) - \psi_{i+1}^*(x_r)\, b_{i+1,j}}{\underline{\psi_{i+1}^*(x_r) - \psi_{i+1}^*(x_r)}\, b_{i+1,r}} = \frac{\cdots\cdots}{\psi_{i+1}^*(x_r)(1 - b_{i+1,r})} = \frac{\psi_{i+1}^*(y_j) - b_{i+1,j}}{1 - b_{i+1,r}}$$

<u>16 end</u>

$$\psi'^{*}_{i+1}(y_i) = \frac{1}{1-b_{i+1,r}}\left(\psi^{*}_{i+1}(y_i) - b_{i+1,i}\right)$$

<u>trivially:</u> $$\psi_{i+1}(y_i) = \frac{1}{1-b_{i+1,r}}\left(\psi^{*}_{i+1}(y_i) - \psi^{*}_{r+1}(y_i)\, b_{i+1,r}\right)$$

$$\delta y_i = \underbrace{\frac{1}{1-b_{i+1,r}}}_{\text{unit}}\left(\underset{\underset{M_{i+1}}{\uparrow}}{b_{i+1,i}} - x^{*}_{r+1}(y_i)\, \underset{\underset{M_{i+1}}{\uparrow}}{\underline{b_{i+1,r}}}\right) \in M_{i+1}$$