EXERCISES FOR LECTURE 1

MARTIN OLSSON

1. Show that the inclusions

\[(\text{integral monoids}) \subset (\text{monoids})\]

and

\[(\text{integral and saturated monoids}) \subset (\text{integral monoids})\]

have left adjoints.

2. Let \( f : X \to Y \) be a morphism of schemes, \( P \) a monoid, and \( M \) the log structure on \( Y \) associated to a morphism \( \beta : P \to \Gamma(Y, \mathcal{O}_Y) \). Show that \( f^*M \) is the log structure associated to the composite map

\[ P \to \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X). \]

3. Let \( f : X \to Y \) be a morphism of schemes. Show that the functor

\[ f^* : (\text{log structures on } Y) \to (\text{log structures on } X) \]

has a right adjoint \( f^\log_* \). Give an example to show that \( f^\log_* \) does not in general take fine log structures to fine log structures.

4. Let \( P \) be a fs monoid, \( k \) a field, and let \( X \) be the scheme \( \text{Spec}(k[P]) \). Let \( j : U \hookrightarrow X \) be the open subset

\[ \text{Spec}(k[P_{gp}]) \hookrightarrow \text{Spec}(k[P]). \]

Show that \( j^\log_* \mathcal{O}_U^* \) is the log structure associated to the natural map \( P \to k[P] \).

5. Let \((E, e)\) be an elliptic curve over an algebraically closed field \( k \) of characteristic \( \neq 3 \). Let \( j : E \hookrightarrow \mathbb{P}^2 \) be an embedding given by choosing an isomorphism \( \Gamma(E, \mathcal{O}_E(3e)) \simeq k^3 \). Let \( x_1, \ldots, x_9 \in E(k) \) be 9 points, and let \( \pi : P \to \mathbb{P}^2 \) be the blowup of \( \mathbb{P}^2 \) at the nine points \( j(x_i) \). Show the following:

(i) The strict transform of \( E \) in \( P \) projects isomorphically onto \( E \), so we get a lifting \( \tilde{j} : E \hookrightarrow P \) of \( j \).

(ii) \( E \in | - K_P| \).

(iii) Let \( I \) be the ideal of \( E \) in \( \mathcal{O}_P \). Then

\[ \tilde{j}^*I \simeq \mathcal{O}_E(\sum_{i=1}^{9}(e - x_i)). \]