SHEAVES OF t-STRUCTURES AND VALUATIVE CRITERIA FOR STABLE COMPLEXES

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Contents

1. Introduction

We work with varieties over a field k of characteristic 0, though many arguments go through without this assumption. The notation $D(X)$ is reserved for the *bounded* derived category of *coherent* sheaves on a variety X, namely $D(X) := D^b_c(X)$.

The core of this paper is concerned with the construction of a "constant" t-structure on the bounded derived category of coherent sheaves $D(X \times S)$, with X and S smooth varieties, given a nondegenerate t-structure on $D(X)$ with noetherian heart.

While we believe this construction and the methods involved should be generally useful in studying derived categories, we present here but one application: we prove a valuative criterion for separation and properness for the collection $\mathcal{P}(1)$ of stable objects of phase 1 under a numerical, locally finite and noetherian Bridgeland-Douglas stability condition $(Z, \mathcal{P})$ on $D(X)$, where X is a smooth projective variety.

1.1. **The main results.** Let X be a smooth projective variety and, as always, let $D(X) := D^b_c(X)$ be the derived category of bounded complexes of coherent sheaves. Whenever given a variety S we denote by $p : X \times S \to X$ the projection to the first factor.

1.1.1. *Results on sheaves of t-structures.* In section 2 we are given a nondegenerate t-structure $(D(X)^{\leq 0}, D(X)^{\geq 0})$ with heart $\mathcal{C} = D(X)^{\leq 0} \cap D(X)^{\geq 0}$ (see section 1.2 for definitions and a brief introduction). For a smooth projective variety S with ample line bundle L we define

$$D(X \times S)^{[a,b]} \quad = \quad \left\{ \, E \in D(X \times S) \; \middle| \; \mathbf{R}p_* \left(E \otimes L^n\right) \; \in \; D(X)^{[a,b]} \quad \forall n \gg 0 \, \right\}.$$

Assuming that $\mathcal{C}$ is noetherian we prove

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(1) $(D(X\times S)^{\leq 0}, D(X\times S)^{\geq 0})$ is a nondegenerate t-structure on $D(X\times S)$.
(2) It extends to a sheaf of t-structures over S.
(3) It is independent of the choice of L.
(4) The functor $p^*: D(X)\to D(X\times S)$ is t-exact.
(5) For every closed immersion of a smooth variety $i_T: T\hookrightarrow S$ the functor $i_{T*}: D(X\times T)\to D(X\times S)$ is t-exact.
(6) If $S'\subset S$ is open, then the t-structure $(D(X\times S')^{\leq 0}, D(X\times S')^{\geq 0})$ on $D(X\times S')$ is independent of the projective completion $S'\subset S$.
(7) The heart $\mathcal{C}_{S'} = D(X\times S')^{\leq 0}\cap D(X\times S')^{\geq 0}$ is noetherian.

And finally

(8) If $\mathcal{C}$ is bounded with respect to the standard t-structure on $D(X)$, then we can characterize the t-structure by
$E\in D(X\times S)^{\geq 0} \quad \iff$
$$\mathrm{Hom}\big(F, \mathbf{R}p'_*(j^*E)\big) = 0, \text{ whenever } F\in D(X)^{<0} \text{ and } j: S'\hookrightarrow S \text{ open}.$$

This is the subject of section 2. Parts (1)-(5) and (7) are in Theorem 2.6.1. Part (8) is in Theorem 2.7.2, and (6) is in Theorem 3.4.1 (but see also Theorem 2.7.2).

1.1.2. *Results on flat extensions of objects.* In section 3 we investigate further the sheaf of t-structures discussed above. We denote the heart of this t-structure over a smooth quasi-projective scheme S by $\mathcal{C}_S$.

Apart from a proof of part 6 above, we introduce the notion of a *family of objects in $\mathcal{C}$ parametrized by S* (Definition 3.3.1) and show a few basic properties:

(1) **The open heart property:** if the restriction of $E\in D(X\times S)$ over a smooth closed subvariety $T\subset S$ is in $\mathcal{C}_T$ then $E_U\in\mathcal{C}_U$ for some neighborhood U of T (Proposition 3.3.2).
(2) **The total family is in the heart:** if $E\in D(X\times S)$ is a family of objects in $\mathcal{C}$ then $E\in\mathcal{C}_S$ (Corollary 3.3.3).
(3) **Extension across a divisor:** if $D\subset S$ is a smooth divisor with complement U and if $E_0\in\mathcal{C}_U$, then there is an extension $E\in\mathcal{C}_S$ of E_0 such that $\mathbf{L}i_D^*E\in\mathcal{C}_D$ (Proposition 3.2.2).

In section 4 we fix a Bridgeland–Douglas stability condition $(Z,\mathcal{P})$ on $D(X)$ (see section 1.3 for definitions and a brief introduction). *We assume that its heart is noetherian*, and concentrate on a one-parameter family E_0 of objects in $\mathcal{P}(1)$ parametrized by an open curve $U\subset S$. We prove a "valuative criterion for properness and separation":

(1) **valuative properness:** a t-flat extension E of E_0 over S has its fibers in $\mathcal{P}(1)$;
(2) **separation up to S-equivalence:** any two extensions have S-equivalent fibers, and
(3) **polystable replacement:** after a base change there is an extension with polystable fibers on $S\smallsetminus U$.

See Theorem 4.1.1.

1.1.3. *Examples and questions.* For our results to be of some practical relevance, we show that there is a large source of examples where they apply. For this purpose we prove in Proposition 5.0.1 that

every "discrete" stability condition aligned with $\mathbb{R}\subset\mathbb{C}$ has noetherian heart.

In particular, our result always applies in the case when X is a curve, with a proof very different from the classical one. Also, in the case when X is a K3 surface there exists a connected component in the space of stability conditions (this component is described in [B2]) in which "discrete" stability conditions are dense (see Corollary 5.0.5).

In section 6, we state a number of natural questions about moduli spaces, and about possibilities for extending the scope of validity and reach of our results.

1.2. **t-structures.** Let T be a triangulated category. Recall that a *t-structure*

$$(T^{\leq 0}, T^{\geq 0})$$

is determined by a full subcategory $T^{\leq 0} \subset T$ such that

(1) $T^{\leq 0}$ is stable under positive (left) shifts, i.e. if $a \in T^{\leq 0}$ then $a[1] \in T^{\leq 0}$, and
(2) The embedding $T^{\leq 0} \subset T$ has a right adjoint *truncation functor* $\tau_{\leq 0} : T \to T^{\leq 0}$.

Recall that for a full subcategory $U \subset T$ the right orthogonal full subcategory is defined by

$$U^{\perp} \quad := \quad \big\{ F \in T \mid \mathrm{Hom}(E, F) = 0 \quad \forall E \in U \big\}.$$

Using this one defines as usual $T^{\geq 1} = (T^{\leq 0})^{\perp}$, $T^{\leq -i} = T^{\leq 0}[i]$ and similarly for $T^{\geq -i}$.

The heart of the t-structure is the full subcategory $\mathcal{C} = T^{\leq 0} \cap T^{\geq 0}$.

The embedding $T^{\geq 0} \subset T$ has a left adjoint $\tau_{\geq 0} : T \to T^{\geq 0}$, and the composite functor $H^0 = \tau_{\geq 0}\tau_{\leq 0} : T \to \mathcal{C}$ is the *0-th cohomology functor* corresponding to the t-structure. As usual we denote $H^i(F) = H^0(F[i])$.

The t-structure is said to be *nondegenerate*[1] if

(1) $\cap_i T^{\leq i} = \cap_i T^{\geq i} = 0$, and
(2) $\cup_i T^{\leq i} = \cup_i T^{\geq i} = T$.

There are no definite results about when there is an equivalence $T \simeq D^b(\mathcal{C})$. However, when the t-structure is nondegenerate we have, for $a, b \in \mathcal{C}$, the expected equalities

$$\begin{array}{rrcll} & \mathrm{Hom}^i_{\mathcal{C}}(a,b) & = & \mathrm{Hom}^i_T(a,b) = 0 & \quad \text{for } i < 0, \\ \text{and} & \mathrm{Hom}^i_{\mathcal{C}}(a,b) & = & \mathrm{Hom}^i_T(a,b) & \quad \text{for } i = 0, 1. \end{array}$$

Another notion we will use is the following: a class of objects $\mathcal{E} \subset T$ is said to be *bounded with respect to a t-structure* $(T^{\leq i}, T^{\geq i})$ if there is an integer N such that every object $E \in \mathcal{E}$ satisfies

$$E \in T^{[-N,N]}.$$

The prototypical example of a t-structure is the derived category $D(Sh(X))$ of sheaves on a space, with the usual truncated subcategories $\big(D^{\leq 0}(Sh(X)), D^{\geq 0}(Sh(X))\big)$. This t-structure is nondegenerate when restricted to the bounded derived category $D^b(Sh(X))$. The heart is the abelian category of sheaves.

Let $F : T \to S$ be an exact functor of triangulated categories equipped with t-structures. The functor is said to be *left exact* if it sends $T^{\geq 0}$ to $S^{\geq 0}$, and analogously for *right exact* and *exact*.

Given an exact functor of triangulated categories $G : T \to S$ with *fully faithful* left adjoint $F : S \to T$ and a right adjoint $H : S \to T$, consider the subcategory

$$U \quad = \quad \big\{ a \in T \mid G(a) = 0 \big\}.$$

A theorem of Bernstein, Beilinson and Deligne [BBD] says that any two t-structures, $(U^{\leq 0}, U^{\geq 0})$ on U and $(S^{\leq 0}, S^{\geq 0})$ on S, can be glued together to a t-structure on T, such that both the

[1] We use a condition stronger than [BBD]. Our definition agrees with the standard conjunction "bounded and nondegenerate", but we use "bounded" in a slightly different "relative" context below.

embedding $U \subset T$ and the functor $G : T \to S$ are t-exact. This has an inductive generalization (again in [BBD]), which says that if

$$(T_1, T_2, \dots, T_m)$$

is a semiorthogonal decomposition of T, and if t-structures are given on T_i, then a canonical t-structure can be constructed on T with analogous exactness properties.

1.3. **Stability conditions.** Motivated by work from String Theory (see e.g. [D1, AD, D2]), Tom Bridgeland introduced in [B1] the a notion of *stability condition* on a triangulated category T.

First, a collection of full subcategories $\mathcal{P}(t), t \in \mathbb{R}$ is called a *slicing* if

(1) for all $t \in \mathbb{R}$ we have $\mathcal{P}(t+1) = \mathcal{P}(t)[1]$,
(2) if $t_1 > t_2$ and $A_i \in \mathcal{P}(t_i)$, then $\mathrm{Hom}(A_1, A_2) = 0$, and
(3) every $E \in T$ has a *Harder-Narasimhan filtration*, i.e. there is a finite sequence $t_1 > t_2 > \cdots > t_n$ and a diagram of distinguished triangles

$$\begin{tikzcd}
0 \arrow[r, equal] & E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n \arrow[r, equal] \arrow[dl] & E \\
& & A_1 \arrow[ul, "{[1]}"] & & A_2 \arrow[ul, "{[1]}"] & & & & A_n \arrow[ul, "{[1]}"] & &
\end{tikzcd}$$

with $A_i \in \mathcal{P}(t_i)$.

The objects of $\mathcal{P}(t)$ are called *semistable of phase* t.

Given a slicing $\mathcal{P}$, one defines, for each interval $I \subset \mathbb{R}$ (possibly infinite, with either end open or closed), a subcategory $\mathcal{P}(I)$ as the minimal extension-closed subcategory containing all $\mathcal{P}(t), t \in I$. The category $\mathcal{P}(\,(0,\infty)\,) \subset T$ determines a t-structure, with heart $\mathcal{P}(\,(0,1]\,)$. We call $\mathcal{P}(\,(0,1]\,)$ the *heart* of the slicing $\mathcal{P}$. When $a < b < a+1$ the category $\mathcal{P}(\,(a,b)\,)$ is only *quasi-abelian* (see [B1], section 4).

A slicing is called *locally finite* if there exists $\eta > 0$ such that for all $t \in \mathbb{R}$ the quasi-abelian category $\mathcal{P}(\,(t-\eta, t+\eta)\,)$ is of finite length (with respect to strict short exact sequences, see [B1], Definition 4.1).

A *stability condition* $(Z, \mathcal{P})$ on T consists of a slicing $\mathcal{P}$ and a group homomorphism $Z : K_0(T) \to \mathbb{C}$ such if $E \in \mathcal{P}(\phi)$ we have

$$Z(E) \;=\; m(E)\cdot e^{i\pi\phi}$$

with $m(E) > 0$. The value $m(E) = |Z(E)|$ is called "the mass of E" in the literature. The stability condition is *locally finite* if the slicing $\mathcal{P}$ is.

Suppose T is of finite type over a field k. Then a stability condition $(Z, \mathcal{P})$ is said to be *numerical* if the function Z factors through $\mathcal{N}(T) = K(T)/K(T)^{\perp}$, where the perpendicular is taken with respect to the Euler bilinear form

$$\chi(E, F) \;=\; \sum_i (-1)^i \,\dim_k \mathrm{Hom}_T(E, F[i]).$$

In [B1], Theorem 1.2 and Corollary 1.3 it is shown that the collection of all locally finite numerical stability conditions forms a complex manifold.

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2. Constant families of t-structures

Throughout this section we fix a smooth projective variety X over k and a nondegenerate t-structure $(D^{\leq 0}, D^{\geq 0})$ on $D(X)$. Let S be a smooth (quasiprojective) variety over k. We always denote by $p : X \times S \to X$ the natural projection. Our goal is to construct a t-structure on $D(X \times S)$ such that the functor $p^* : D(X) \to D(X \times S)$ would be t-exact, as well as the restriction functor $j^* : D(X \times S) \to D(X \times U)$ for an open embedding $j : U \hookrightarrow S$. For a smooth projective variety S we will construct a canonical such t-structure. Given a smooth projective variety S we can thus "restrict" such a t-structure to t-structures on $D(X \times U)$ for all open subsets $U \subset S$.

An analogous construction in a different framework can be found in [YZ], Theorems 4.11 and 4.14.

2.1. Sheaves of t-structures.

Let S be a smooth variety over k.

Definition 2.1.1. A *sheaf of t-structures (on X) over S* is a collection of nondegenerate t-structures
$$(D(X \times U)^{\leq 0}, D(X \times U)^{\geq 0})$$
on $D(X \times U)$ for all open subsets $U \subset S$, such that the restriction functors $D(X \times U) \to D(X \times U')$ are t-exact, where $U' \subset U$.

Lemma 2.1.2. *Assume that we have a sheaf of t-structures over S and let U_i be an open covering of S. Assume that for some $F \in D(X \times S)$ we have $F|_{X \times U_i} \in D(X \times U_i)^{\leq 0}$ (resp, $F|_{X \times U_i} \in D(X \times U_i)^{\geq 0}$) for all i. Then $F \in D(X \times S)^{\leq 0}$ (resp., $F \in D(X \times S)^{\geq 0}$).*

Proof. Note that, given an object $G \in D(X \times S)$ such that $G|_{X \times U_i} = 0$ for all i, we have $G = 0$, since quasi-isomorphisms are tested on cohomology sheaves with respect to the standard t-structure. By assumption, the restriction functors commute with the cohomology functors with respect to our t-structures, hence the condition $H^i F = 0$ can be checked locally. □

Proposition 2.1.3. *Assume that we have a sheaf ot t-structures over S. Then for every vector bundle V on S the functor*
$$\begin{array}{ccc} D(X \times S) & \to & D(X \times S) \\ F & \mapsto & F \otimes p_S^* V \end{array}$$
of tensoring with the pull-back of V to $X \times S$ is t-exact.

Proof. This follows immediately from the previous lemma. □

The following theorem implies that conversely, if the functors of tensoring with pull-backs of line bundles on S are t-exact, then the t-structure on $D(X \times S)$ extends to a sheaf of t-structures.

Theorem 2.1.4. *Let S be a smooth quasiprojective variety, and let $(D(X \times S)^{\leq 0}, D(X \times S)^{\geq 0})$ be a nondegenerate t-structure on $D(X \times S)$. Assume that for some ample line bundle L on S the functor*
$$\begin{array}{ccc} D(X \times S) & \longrightarrow & D(X \times S) \\ F & \mapsto & F \otimes p_S^* L \end{array}$$
is t-exact. Then there exists a unique extension of the t-structure on $D(X \times S)$ to a sheaf of t-structures over S.

We will need several lemmata for the proof. Let $T \subset S$ be a closed subset. Denote by $D_T(X \times S) \subset D(X \times S)$ the full subcategory of objects supported on $X \times T$. This is the same as the category of objects whose cohomology sheaves are supported in $X \times T$. Note that $D_T(X \times S)$ is a thick subcategory. The first lemma is well known:

Lemma 2.1.5. *Let $f_1, \ldots, f_n$ be sections of some line bundle M on S such that T is the set of common zeroes of $f_1, \ldots, f_n$. Then $F \in D_T(X \times S)$ if and only if the morphisms $f_{i_1} \ldots f_{i_d} : F \to F \otimes M^d$ are zero for all sequences $(i_1, \ldots, i_d)$ of length d for some $d > 0$.*

Proof. The "if" part follows by considering the induced maps on cohomology sheaves. Let us prove the "only if" part. Using the standard t-structure, suppose $F \in D^{[a,b]}$ and apply induction on the length $b - a$, the case $b - a = 0$ being immediate. We take $b \leq c < a$ and consider the distinguished triangle

$$\tau_{\leq c} F \to F \to \tau_{>c} F \to \tau_{\leq c} F[1],$$

along with the associated long exact sequence of $\mathrm{Hom}(F, \bullet)$. By induction there is an integer d which works for the two truncations. We claim that $2d$ will work for F. Indeed, the arrow

$$\tau_{>c} \circ (f_{i_1} \ldots f_{i_d}) : F \to \tau_{>c} F \otimes M^d$$

vanishes since it factors through $f_{i_1} \ldots f_{i_d} : \tau_{>c} F \to \tau_{>c} F \otimes M^d$. Therefore $f_{i_1} \ldots f_{i_d} : F \to F \otimes M^d$ factors through an arrow $g : F \to \tau_{\leq c} F \otimes M^d$. It follows that

$$(f_{i_{d+1}} \ldots f_{i_{2d}}) \circ g : F \to \tau_{\leq c} F \otimes M^{2d}$$

vanishes, and therefore also its image $f_{i_1} \ldots f_{i_{2d}} : F \to F \otimes M^{2d}$ vanishes as well. □

Lemma 2.1.6. *Under the assumptions of Theorem 2.1.4, let H^i, $i \in \mathbb{Z}$ be the cohomology functors with respect to the t-structure on $D(X \times S)$. Then for every closed subset $T \subset S$ one has $F \in D_T(X \times S)$ if and only if $H^i F \in D_T(X \times S)$ for all i.*

Proof. The "if" part follows as in Lemma 2.1.5 since our t-structure is nondegenerate. The "only if" part follows from Lemma 2.1.5 and from the fact that the twisting functor $F \mapsto F \otimes p_S^* L$ is t-exact. Indeed, replacing L by a suitable power we can assume that T is the set of common zeros of sections $f_1, \ldots, f_n$ of L. Now if the morphism $f = f_{i_1} \ldots f_{i_d} : F \to F \otimes L^d$ is zero, then using the defining property of the truncating functor $\tau_{\leq 0}$ (resp., $\tau_{\geq 0}$) we deduce that $f : \tau_{\leq 0} F \to \tau_{\leq 0} F \otimes L^d = \tau_{\leq 0}(F \otimes L^d)$ (respecively, $f : \tau_{\geq 0} F \to \tau_{\geq 0} F \otimes L^d = \tau_{\geq 0}(F \otimes L^d)$) is zero. □

Lemma 2.1.7. *Let $\mathcal{C} \subset D(X \times S)$ be the heart of the t-structure on $D(X \times S)$ considered in Theorem 2.1.4. Then for every closed subset $T \subset S$ the subcategory $D_T(X \times S) \cap \mathcal{C} \subset \mathcal{C}$ is closed under subobjects, quotients and extensions.*

Proof. This follows immediately from Lemma 2.1.5. □

Proof of Theorem 2.1.4. Let $U \subset S$ be an open set with the complement T. Proposition 5.5.4 of [TT] implies that the restriction functor $D(X \times S) \to D(X \times U)$ is essentially surjective. Hence, the category $D(X \times U)$ is the quotient of $D(X \times S)$ by the thick subcategory $D_T(X \times S)$. By the definition, this means that $D(X \times U)$ is the localization of $D(X \times S)$ with respect to the localizing class consisting of morphisms $f : F \to F'$ such that the cone of f belongs to $D_T(X \times S)$. To show that the t-structure on $D(X \times S)$ induces a t-structure on the localized category, it suffices to show that every diagram

$$F \xleftarrow{f} F' \to G$$

such that the cone of f is in $D_T(X \times S)$, $F \in D(X \times S)^{\leq 0}$ and $G \in D(X \times S)^{\geq 1}$, the induced morphism $F \to G$ in the localized category vanishes. Considering the long exact sequence of cohomology associated with the exact triangle $F' \xrightarrow{f} F \to C \to F'[1]$ and using Lemmata 2.1.6

and 2.1.7 we derive that $\tau_{\geq 1}(F') \in D_T(X \times S)$. Therefore, the arrow $\tau_{\leq 0}(F') \to F'$ belongs to our localizing class and the above diagram is equivalent to

$$F \leftarrow \tau_{\leq 0}(F') \to G.$$

But $\mathrm{Hom}(\tau_{\leq 0}(F'), G) = 0$ which proves our claim. $\square$

For later use we record the following related (and probably well known) lemma:

Lemma 2.1.8. *Let $U \subset V$ be an open subset whose complement is a divisor T with defining section $f \in H^0(V, \mathcal{O}_V(T))$. Let F_1 and F_2 be objects in $D(V)$, and let $\phi_U : (F_1)_U \to (F_2)_U$ be a morphism. Then for some k, the morphism $f^k\phi_U$ extends to $\phi' : F_1 \to F_2$.*

Proof. The morphism ϕ_U corresponds to a diagram

$$F_1 \leftarrow H \to F_2$$

where

$$C = Cone(H \to F_1) \in D_T(V).$$

In particular there is an integer k such that the morphism $f^k : C \to C(kT)$ is zero.

In the long exact sequence

$$\cdots \to \mathrm{Hom}(F_1, H(kT)) \to \mathrm{Hom}(F_1, F_1(kT)) \to \mathrm{Hom}(F_1, C(kT)) \to \cdots$$

the image of the element $f^k \in \mathrm{Hom}(F_1, F_1(kT))$ in $\mathrm{Hom}(F_1, C(kT))$ factors through the zero morphism $f^k : C \to C(kT)$, therefore it vanishes. Thus there is $\psi \in \mathrm{Hom}(F_1, H(kT))$ mapping to $f^k \in \mathrm{Hom}(F_1, F_1(kT))$, representing the inverse of $H_U \to (F_1)_U$, as required. $\square$

The next theorem follows from the proof of Theorem 3.2.4 of [BBD] (we only need the case of Zariski sheaves of $\mathcal{O}$-modules, so we leave to the reader to formulate a more general version on arbitrary site similar to the formulation in [BBD]).

Theorem 2.1.9. *Let U_i be a finite open covering of S, and let*

$$\big(\, K_i \in D(X \times U_i)\,, \quad \alpha_{ij} : K_i|_{X \times U_{ij}} \xrightarrow{\sim} K_j|_{X \times U_{ij}} \,\big)$$

be a gluing datum, where $U_{i_1 \dots i_n} := U_{i_1} \cap \ldots \cap U_{i_n}$ (so $\alpha_{jk}\alpha_{ij} = \alpha_{ik}$ over $X \times U_{ijk}$). Assume that $\mathrm{Hom}^j(K_i|_U, K_i|_U) = 0$ for all $j < 0$ and all open sets of the form $U = U_{ii_1 \dots i_n}$. Then there exists $K \in D(X \times S)$ equipped with isomorphisms $K|_{X \times U_i} \simeq K_i$ compatible with (α_{ij}) on double intersections.

For every $F, G \in D(X \times S)$ let us define an object $\mathbf{R}\mathcal{H}om_S(F, G) \in D(S)$ by setting

$$\mathbf{R}\mathcal{H}om_S(F, G) \quad = \quad \mathbf{R}p_{S*}\, \mathbf{R}\mathcal{H}om(F, G).$$

Note that for every open subset $U \subset S$ we have

$$\mathbf{R}\mathrm{Hom}(F_U, G_U) \quad \simeq \quad \mathbf{R}\Gamma(U\,,\ \mathbf{R}\mathcal{H}om_S(F, G)|_U\,),$$

where for $F \in D(X \times S)$ we set $F_U := F|_{X \times U}$.

Lemma 2.1.10. *Let F and G be a pair of objects in $D(X \times S)$ such that*

$$\mathbf{R}\mathcal{H}om_S(F, G) \ \in \ D(S)^{\geq 0}$$

(with respect to the standard t-structure). Then $U \mapsto \mathrm{Hom}(F_U, G_U)$ is a sheaf on S.

Proof. Indeed, for an open set $U \subset S$ we have

$$\begin{aligned} \mathrm{Hom}(F_U, G_U) \ = \ H^0\,\mathbf{R}\mathrm{Hom}(F_U, G_U) \ &= \ H^0\,\mathbf{R}\Gamma\big(U\,,\, \mathbf{R}\mathcal{H}om_S(F, G)|_U\,\big) \\ &= \ \Gamma\big(U\,,\, H^0\,\mathbf{R}\mathcal{H}om_S(F, G)\,\big), \end{aligned}$$

since the functor Γ is left exact. $\square$

Corollary 2.1.11. *Let U_i be a finite open covering of S, and let*

$$\left(K_i \in D(X \times U_i)\,, \quad \alpha_{ij} : K_i|_{X\times U_{ij}} \stackrel{\sim}{\to} K_j|_{X\times U_{ij}}\right)$$

be a gluing datum. Assume that for every i one has $\mathbf{R}\mathcal{H}om_{U_i}(K_i, K_i) \in D(U_i)^{\geq 0}$ (with respect to the standard t-structure). Then there exists a unique $K \in D(X\times S)$ (up to unique isomorphism) equipped with isomorphisms $K|_{X\times U_i} \simeq K_i$ compatible with (α_{ij}) on double intersections.

Proof. Uniqueness follows immediately from Lemma 2.1.10 since our assumptions imply that $\mathbf{R}\mathcal{H}om_S(K, K) \in D(S)^{\geq 0}$. The existence follows from Theorem 2.1.9. Indeed, if $U = U_{ii_1\dots i_k}$ then the condition $\mathbf{R}\mathcal{H}om_U(K_{iU}, K_{iU}) \in D(U)^{\geq 0}$ implies that $\mathrm{Hom}^j(K_{iU}, K_{iU}) = 0$ for $j < 0$. □

Corollary 2.1.12. *Let $U \mapsto (D(X \times U)^{\leq 0}, D(X \times U)^{\geq 0})$ be a sheaf of t-structures over S. Then the hearts $\mathcal{C}_U = (D(X \times U)^{\leq 0} \cap D(X \times U)^{\geq 0})$ form a stack of abelian categories on S.*

Proof. Assume that $F, G \in \mathcal{C}_S$. Then for every open affine subset $U \subset S$ the vanishing of $H^i\mathbf{R}\mathrm{Hom}(F_U, G_U)$ for $i < 0$ implies that $\mathbf{R}\mathcal{H}om_S(F, G)|_U$ belongs to $D(U)^{\geq 0}$ (with respect to the standard t-structure on $D(U)$). Hence, $\mathbf{R}\mathcal{H}om_S(F, G) \in D(S)^{\geq 0}$, so by Lemma 2.1.10 the groups $U \mapsto \mathrm{Hom}(F_U, G_U)$ form a sheaf on S. To glue objects $F_i \in \mathcal{C}_{U_i}$ given on some finite open covering (U_i) of S and equipped with the gluing data we apply Corollary 2.1.11. □

2.2. **Categorification of Hilbert's basis theorem.** In this subsection we will prove a technical result that can be considered as a categorical version of Hilbert's basis theorem (in its graded version). An analogous result can be found in [AZ] in their context (it may be that their result implies ours).

Let $\mathcal{C}$ be an abelian k-linear category.

Definition 2.2.1. Let $A = \oplus_n A_n$ be a $\mathbb{Z}$-graded associative algebra with unit over k such that $\dim_k A_n < \infty$ for every $n \in \mathbb{Z}$.

(a) A *graded A-module in $\mathcal{C}$* is a collection $M = (M_n|\ n \in \mathbb{Z})$ of objects in $\mathcal{C}$ and a collection of morphisms $A_m \otimes M_n \to M_{m+n}$ for $m, n \in \mathbb{Z}$ satisfying the natural associativity condition and such that the composition $k \otimes M_n \to A_0 \otimes M_n \to M_n$ is the identity morphisms for every $n \in \mathbb{Z}$. Graded A-modules in $\mathcal{C}$ form an abelian category in a natural way.

(b) A *free* graded A-module *of finite type* in $\mathcal{C}$ is a finite direct sum of graded A-modules in $\mathcal{C}$ of the form $A \otimes F(i)$, where
$$(A \otimes F(i))_n = A_{n+i} \otimes F,$$
F is an object of $\mathcal{C}$, i is a fixed integer; the morphisms
$$A_m \otimes (A \otimes F(i))_n \to (A \otimes F)_{m+n}$$
are induced by the multiplication in A.

(c) A graded A-module M in $\mathcal{C}$ is called of *finite type* if there exists a surjection $P \to M$ where P is a free graded A-module of finite type in $\mathcal{C}$. Note that if A is generated by A_1 over A_0 and $M_n = 0$ for $n < 0$ then M is of finite type iff the maps $A_1 \otimes M_n \to M_{n+1}$ are surjective for $n \gg 0$.

For example, if $\mathcal{C}$ is the category of finitely generated modules over a noetherian ring R then a graded A-module in $\mathcal{C}$ is the same as a graded $R\otimes_k A$-module with graded components finitely generated over R. A graded A-module in $\mathcal{C}$ is of finite type iff the corresponding $R\otimes_k A$-module is finitely generated in the usual sense.

In the case $A = k[x_1, \dots, x_d]$, we have the following analogue of Hilbert's basis theorem (the proof is also completely analogous).

Theorem 2.2.2. *Assume that $\mathcal{C}$ is noetherian and let $A = k[x_1, \dots, x_d]$ be the algebra of polynomials in d variables with the grading $\deg(x_i) = 1$.*

(i) Let F be an object of $\mathcal{C}$, $A \otimes F$ be the corresponding free graded A-module in $\mathcal{C}$. Then every graded submodule in $A \otimes F$ is of finite type.

(ii) The category of graded A-modules of finite type in $\mathcal{C}$ is abelian and noetherian.

Proof. We use induction in d. When $d = 0$ both (i) and (ii) are clearly true. Let $d > 0$ and assume that the assertion is true for $d - 1$. Let $B = k[x_2, \dots, x_d]$, so that $A = B[x_1]$.

(i) Let $M \subset A \otimes F$ be a graded submodule. This means that for every $n \geq 0$ we have a subobject $M_n \subset A_n \otimes F$ such that for every $i = 1, \dots, d$ the image of M_n under the map $A_n \otimes F \xrightarrow{x_i} A_{n+1} \otimes F$ induced by the multiplication by x_i, is contained in M_{n+1}.

First, for every i we are going to define a graded B-submodule $L^i \subset B \otimes F$ of "leading terms" of M with x_1^i. Consider the filtration $F_0A \subset F_1A \subset \dots$ of A induced by the degree in x_1, so that F_iA are polynomials of degree $\leq i$ in x_1. Let $F_0A_n \subset F_1A_n \subset \dots$ be the induced filtration on A_n. Note that for every i and n we have a natural isomorphism $F_iA_n/F_{i-1}A_{n-1} \simeq B_{n-i}$ (taking the coefficient with x_1^i). Let $p_{n,i} : F_iA_n \otimes F \to B_{n-i} \otimes F$ be the induced projection. Let us set

$$F_iM_n := M_n \cap (F_iA_n \otimes F) \subset M_n,$$
$$L_n^i := p_{n+i,i}(F_iM_{n+i}) \subset B_n \otimes F.$$

Since M is closed under multiplication by B, the collection $(L_n^i |\ n \geq 0)$ is a graded B-submodule L^i in $B \otimes F$. Furthermore, since M is closed under multiplication by x_1, we obtain the inclusion that $L_n^i \subset L_n^{i+1} \subset B_n \otimes F$ for every n. Thus, we have an increasing chain of graded B-submodules $L^0 \subset L^1 \subset \dots$ in $B \otimes F$. Since the category of graded B-modules of finite type in $\mathcal{C}$ is noetherian, this chain necessarily stabilizes. Hence, there exists $N_1 > 0$ such that $L^i = L^{i+1}$ for $i \geq N_1$. By induction assumption the B-modules $L^0, \dots, L^{N_1}$ are of finite type. Hence, we can find $N_2 > 0$ such that the morphisms $B_1 \otimes L_n^i \to L_{n+1}^i$ are surjective for $n \geq N_2$ and $i \leq N_1$.

We claim that in this case the moprhisms $\mu_n : A_1 \otimes M_n \to M_{n+1}$ are surjective for $n \geq N = N_1 + N_2$. It suffices to prove that the induced maps

$$\mu_n^i : \big((x_1) \otimes F_iM_n/F_{i-1}M_n\big) \oplus \big(B_1 \otimes F_{i+1}M_n/F_iM_n\big) \longrightarrow F_{i+1}M_{n+1}/F_iM_{n+1}$$

are surjective for $0 \leq i \leq n$, where $(x_1) \subset A_1$ is the line spanned by x_1. Note that since the kernel of $p_{n,i}$ is $F_{i-1}A_n \otimes F$, we have a natural isomorphism

$$F_iM_n/F_{i-1}M_n \simeq L_{n-i}^i.$$

Moreover, the map μ_n^i can be identified with the natural map

$$L_{n-i}^i \oplus \big(B_1 \otimes L_{n-i-1}^{i+1}\big) \longrightarrow L_{n-i}^{i+1}.$$

If $i \geq N_1$ then the map $L^i = L^{i+1}$, hence μ_n^i is surjective. If $i < N_1$ then $n - i > N_2$ and the result follows from surjectivity of the map $B_1 \otimes L_{n-i-1}^{i+1} \to L_{n-i}^{i+1}$.

(ii) First, let us check that for every object $F \in \mathcal{C}$ the corresponding free graded A-module $A \otimes F$ is a noetherian object. Let $M^1 \subset M^2 \subset \dots$ be an increasing chain of graded A-submodules in $A \otimes F$. For every $n \in \mathbb{Z}$ the corresponding chain $M_n^1 \subset M_n^2 \subset \dots$ of subobjects in $A_n \otimes F$ stabilizes since $\mathcal{C}$ is noetherian. Hence, we can set $M_n := \cup_i M_n^i$. Clearly, $M = (M_n)$ is a graded A-submodule of $A \otimes F$. According to (i) it is of finite type. This easily implies that $M = M^i$ for some i. It follows that every free graded A-module of finite type is a noetherian.

Therefore, every graded A-module of finite type is a noetherian. Conversely, a noetherian graded A-module is of finite type (otherwise, one would get an infinite strictly increasing chain of submodules). □

2.3. **Constant t-structures for projective spaces.**

A. Glued t-structures.

Proposition 2.3.1. *The following is a t-structure on $D(X \times \mathbb{P}^r)$:*

$$D^{[a,b]} \quad = \quad \left\{ F \;\middle|\; \mathbf{R}p_*(F) \in D^{[a,b]},\ \mathbf{R}p_*(F(1)) \in D^{[a,b]},\ \ldots,\ \mathbf{R}p_*(F(r)) \in D^{[a,b]} \right\}.$$

Proof. This t-structure is obtained by gluing of the standard t-structures from the semi-orthogonal decomposition of $D(X \times \mathbb{P}^r)$ into subcategories

$$\big(p^*D(X))(-r),\ \ldots,\ p^*D(X)(-1),\ p^*D(X)\big)$$

□

For every $n \in \mathbb{Z}$ set

$$D^{[a,b]}(X \times \mathbb{P}^r)_n \quad = \quad \left\{ F \;\middle|\; \begin{array}{rcl} \mathbf{R}p_*(F(n)) & \in & D^{[a,b]}, \\ \mathbf{R}p_*(F(n+1)) & \in & D^{[a,b]}, \\ & \vdots & \\ \mathbf{R}p_*(F(n+r)) & \in & D^{[a,b]} \end{array} \right\}. \tag{1}$$

Note that we have $F \in D^{[a,b]}(X \times \mathbb{P}^r)_n$ iff $F(n) \in D^{[a,b]}(X \times \mathbb{P}^r)_0$. Hence, by the previous proposition $(D^{\le 0}(X \times \mathbb{P}^r)_n, D^{\ge 0}(X \times \mathbb{P}^r)_n)$ is a t-structure on $D(X \times \mathbb{P}^r)$. We will denote by $\tau^n_{\ge i}$, $\tau^n_{\le i}$ the truncation functors with respect to the n-th t-structure. It is clear that for every $n \ge 0$ the functor $p^* : D(X) \to D(X \times \mathbb{P}^r)$ is t-exact with respect to the n-th t-structure.

Lemma 2.3.2. *For $m < n$ one has the following inclusions*

$$D^{\le 0}(X \times \mathbb{P}^r)_m \subset D^{\le 0}(X \times \mathbb{P}^r)_n,$$
$$D^{\ge 0}(X \times \mathbb{P}^r)_n \subset D^{\ge 0}(X \times \mathbb{P}^r)_m \subset D^{\ge -r}(X \times \mathbb{P}^r)_n.$$

Proof. For $m < n$ there exists an exact sequence of the form

$$0 \to V_0 \otimes \mathcal{O}_{\mathbb{P}^r}(m) \to V_1 \otimes \mathcal{O}_{\mathbb{P}^r}(m+1) \to \ldots \to V_r \otimes \mathcal{O}_{\mathbb{P}^r}(m+r) \to \mathcal{O}_{\mathbb{P}^r}(n+r) \to 0, \tag{2}$$

where V_r are finite-dimensional k-vector spaces. Let us first apply this in the case $n = m+1$. In this case

$$V_i \simeq \bigwedge^{r+1-i} H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)),$$

so that $V_0 \simeq k$. Therefore, we can view our exact sequence as a resolution for $\mathcal{O}_{\mathbb{P}^r}(m)$ in terms of $\mathcal{O}_{\mathbb{P}^r}(m+1), \ldots, \mathcal{O}_{\mathbb{P}^r}(m+r+1)$ positioned in degrees $[0, \ldots, r]$. Thus, for $F \in D^{\ge 0}(X \times \mathbb{P}^r)_{m+1}$ we conclude that $\mathbf{R}p_*(F(m)) \in D^{\ge 0}$, hence $D^{\ge 0}(X \times \mathbb{P}^r)_{m+1} \subset D^{\ge 0}(X \times \mathbb{P}^r)_m$. Iterating we get one of the required inclusions

$$D^{\ge 0}(X \times \mathbb{P}^r)_n \subset D^{\ge 0}(X \times \mathbb{P}^r)_m$$

for $m < n$. On the other hand, for arbitrary $m < n$ we can view the above exact sequence as a resolution for $\mathcal{O}_{\mathbb{P}^r}(n+r)$ in terms of $\mathcal{O}_{\mathbb{P}^r}(m), \ldots, \mathcal{O}_{\mathbb{P}^r}(m+r)$ positioned in degrees $[-r, \ldots, 0]$. Therefore, for $F \in D^{\ge 0}(X \times \mathbb{P}^r)_m$ we obtain $\mathbf{R}p_*(F(n+r)) \in D^{\ge -r}$ for every $n > m$. This implies that

$$F \in D^{\ge -r}(X \times \mathbb{P}^r)_{n+r} \subset D^{\ge -r}(X \times \mathbb{P}^r)_n.$$

The two other inclusions follow by passing to left orthogonals. □

B. Stabilization of cohomology of the glued t-structures. Assume that we have two t-structures, $(D_1^{\leq 0}, D_1^{\geq 0})$ and $(D_2^{\leq 0}, D_2^{\geq 0})$, on a triangulated category D, such that $D_1^{\leq 0} \subset D_2^{\leq 0}$ (and hence $D_2^{\geq 0} \subset D_1^{\geq 0}$). Let us denote by $\tau^1_{\leq n}$, $\tau^1_{\geq n}$ (respectively, $\tau^2_{\leq n}$, $\tau^2_{\geq n}$) the truncation functors with respect to the first (respectively, second) t-structure. Then for every object $F \in D$ the canonical morphism $\tau^1_{\leq 0}(F) \to F$ factors uniquely through $\tau^2_{\leq 0}(F)$, so we get a morphism of functors $\tau^1_{\leq 0} \to \tau^2_{\leq 0}$. Similarly, the canonical morphism $F \to \tau^2_{\geq 0}(F)$ factors through $\tau^1_{\geq 0}(F)$, so we get a morphism of functors $\tau^1_{\geq 0} \to \tau^2_{\geq 0}$. Let H_1^i (resp. H_2^i) be the i-the cohomology functor with respect to the first (resp. second) t-structure. Then we get a morphism of functors

$$H_1^i(F) = \tau^1_{\geq 0}\tau^1_{\leq 0}(F[i]) \to \tau^2_{\geq 0}\tau^2_{\leq 0}(F[i]) = H_2^i(F).$$

Proposition 2.3.3. *Assume that the heart $D(X)^{\leq 0} \cap D(X)^{\geq 0}$ is noetherian.*

Let H_n^i, $i \in \mathbb{Z}$ be the cohomology functors associated with the n-th t-structure defined by (1). For every $F \in D(X \times \mathbb{P}^r)$ there exists an integer N such that for $n > N$ the morphisms $H_n^i(F) \to H_{n+1}^i(F)$ are isomorphisms for all i.

We need some preparation before giving a proof. Let $\mathcal{C}$ denote the heart of the original t-structure on X. Let us also consider the truncated symmetric algebra

$$A_V := \oplus_0^r S^i V,$$

where $V = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$. We denote by $\mathcal{C}_V$ the full subcategory in the category of graded representations of the algebra A_V in $\mathcal{C}$ formed by representations (M_n) with $M_n = 0$ for $n \notin [0, r]$. We can identify the heart of the 0-th t-structure $D^{[0]}(X \times \mathbb{P}^r)_0$ with $\mathcal{C}_V$ by means of the functor

$$\Phi : F \mapsto \oplus_{i=0}^r \mathbf{R}p_*(F(i)).$$

Let us consider the following collection of functors

$$\begin{array}{rcl} T_i : \mathcal{C}_V & \to & \mathcal{C}_V \\ F & \mapsto & H_0^i(F(1)), \end{array}$$

where we view $\mathcal{C}_V$ as a subcategory in $D(X \times \mathbb{P}^r)$, and where $i \leq 0$ (for $i > 0$ the analogous functors are zero).

Now suppose F is in the heart of the 0-th t-structure and $M = \Phi(F)$. The Koszul complex

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \big(\bigwedge^r V\big) \otimes \mathcal{O}_{\mathbb{P}^r}(1) \to \ldots \to \big(\big(\bigwedge^2 V\big)\big) \otimes \mathcal{O}_{\mathbb{P}^r}(r-1) \to V \otimes \mathcal{O}_{\mathbb{P}^r}(r) \to \mathcal{O}_{\mathbb{P}^r}(r+1) \to 0.$$

induces a quasi-isomorphism

$$Conv(K^\bullet(M)) \to \mathbf{R}p_*(F(r+1))$$

Here we define the complex of objects in $\mathcal{C}$

$$K^\bullet(M) \quad = \quad M_0 \to \big(\bigwedge^r V\big) \otimes M_1 \to \ldots \to \big(\bigwedge^2 V\big) \otimes M_{r-1} \to V \otimes M_r,$$

positioned in degrees $[-r, \ldots, 0]$, and $Conv(K^\bullet(M))$ is its convolution, namely its image under the functor $D^b(\mathcal{C}) \to D(X)$. In particular the cohomology objects of $\mathbf{R}p_*(F(r+1))$ in terms of the t-structure on X are the cohomology objects of the complex $K^\bullet(M)$.

Write

$$M^i_{r+1} = H^i(K^\bullet(M)),$$

in particular we write explicitly

$$M^0_{r+1} \quad = \quad \mathrm{Coker}\Big(\big(\bigwedge^2 V\big) \otimes M_{r-1} \ \to \ V \otimes M_r\Big).$$

Note that this involves only M_{r-1} and M_r. This discussion implies the first part of the following lemma:

Lemma 2.3.4. *(i) For an object $M = (M_0, \ldots, M_r)$ of $\mathcal{C}_V$ one has*

$$\begin{aligned} T_0(M) &\simeq (M_1, \ldots, M_r, M^0_{r+1}), \\ T_i(M) &\simeq (\ 0\ , \ldots,\ 0\ , M^i_{r+1}), \quad i < 0. \end{aligned}$$

(ii) For $M = (M_0, \ldots, M_r) \in \mathcal{C}_V$ and $n > 0$ one has

$$\Phi H^0_0(\Phi^{-1} M(n+r)) = (M'_0, \ldots, M'_r),$$

where

$$M'_i = \operatorname{Coker}\Big(S^{n+i-1}(V) \otimes \big(\bigwedge^2 V\big) \otimes M_{r-1} \quad \to \quad S^{n+i}(V) \otimes M_r\Big)$$

Proof. We need to prove (ii). This follows from the exact sequence (2) for $m = 0$ together with an observation that in this sequence one has $V_r = S^n(V)$,

$$V_{r-1} = \ker\Big(S^n(V) \otimes V \ \to \ S^{n+1}(V)\Big) \ = \ \operatorname{im}\Big(S^{n-1}(V) \otimes \big(\bigwedge^2 V\big) \ \to \ S^n(V) \otimes V\Big).$$

$\square$

Remark 2.3.5. It is easy to see that for every $n \geq 1$ and every $F \in \mathcal{C}_V$ one has

$$H^0_0(F(n)) \simeq T^n_0(F),$$

where $T^n_0 = T_0 \circ \ldots \circ T_0$ is the n-th iteration of T_0. However, we will not use this fact.

Proof of Proposition 2.3.3.

Step 1: reduction to the case $i = 0$ and $F \in D^{\leq 0}(X \times \mathbb{P}^r)_0$. According to Lemma 2.3.2 for $m < n$ we have $D^{[a,b]}(X \times \mathbb{P}^r)_m \subset D^{[a-r,b]}(X \times \mathbb{P}^r)_n$. Hence, for every $F \in D(X \times \mathbb{P}^r)$ there exists an interval $[a, b]$ such that $H^i_n(F) = 0$ for $i \notin [a, b]$ and $n \geq 0$. Let us show using induction in $b - a$ that we can reduce ourselves to the particular case $i = b$ (and hence to the case $i = b = 0$). The case $b - a = 0$ is clear. If $b - a > 0$ then by the assumption we can find $N > 0$ such that for $n \geq N$ the morphisms $H^b_n F \to H^b_{n+1} F$ are isomorphisms. Replacing F by $F(N)$ we can assume that these maps are isomorphisms for $n \geq 0$. This implies that the natural morphisms $\tau^n_{<b} F \to \tau^{n+1}_{<b} F$ are also isomorphisms for $n \geq 0$. Hence, we have $H^i_n(\tau^0_{<b} F) \simeq H^i_n(\tau^n_{<b} F) = 0$ for $n \geq 0$, $i \notin [a, b-1]$. It remains to apply the induction assumption to $\tau_{<b} F$.

Step 2: restatement in terms of 0-th t-structure. The inclusion $D^{\leq -1}(X \times \mathbb{P}^r)_n \subset D^{\leq -1}(X \times \mathbb{P}^r)_{n+1}$ implies that the morphism $H^0_n(F) \to H^0_{n+1}(F)$ can be identified with the canonical morphism $H^0_n(F) \to \tau^{n+1}_{\geq 0}(H^0_n(F))$, Therefore, it is enough to show that $H^0_n(F)$ lies in the heart of the $(n+1)$-st t-structure for all sufficiently large n. Using the isomorphism $H^0_n(F)(n) \simeq H^0_0(F(n))$ we can restate this as the assertion that $H^0_0(F(n))(1)$ lies in the heart of the 0-th t-structure for $n \gg 0$.

Step 3: restatement in terms of M_{r-1} and M_r. In terms of the functors (T_i) introduced above this means that $T_i(\Phi H^0_0(F(n))) = 0$ for $i < 0$ and $n \gg 0$. Let $\Phi H^0_0(F) = (M_0, \ldots, M_r) \in \mathcal{C}_V$. Then by Lemma 2.3.4(ii) we have $\Phi H^0_0(F) = M^n = (M^n_0, \ldots, M^n_r)$, where

$$M^n_i = \operatorname{Coker}\Big(S^{n+i-1}(V) \otimes \big(\bigwedge^2 V\big) \otimes M_{r-1} \quad \to \quad S^{n+i}(V) \otimes M_r\Big).$$

By Lemma 2.3.4(i) we have to check that $H^{<0}(K^\bullet(M^n)) = 0$ for $n \gg 0$.

STEP 4: REDUCTION IN TERMS OF MODULES OF $S(V)$. Let us consider the natural truncation functor $M \to M_{[0,r]}$ from the category of graded $S(V)$-modules in $\mathcal{C}$ to $\mathcal{C}(V)$ that leaves only graded components of M in the range $[0, r]$. For a graded $S(V)$-module M in $\mathcal{C}$ set $K^\bullet(M) = K^\bullet(M_{[0,r]})$. Then we have $K^\bullet(M^n) = K^\bullet(C(n))$, where C is the following graded $S(V)$-module in $\mathcal{C}$:

$$C = \operatorname{Coker}(S(V) \otimes \big(\bigwedge^2 V\big) \otimes M_{r-1}(-1) \to S(V) \otimes M_r).$$

Now for every graded $S(V)$-module N in $\mathcal{C}$ let us denote by $\widetilde{K}^\bullet(N)$ the following complex (placed in degrees $[0, r+1]$):

$$0 \to N_0 \to \big(\bigwedge^r V\big) \otimes N_1 \to \ldots \to \big(\bigwedge^2 V\big) \otimes N_{r-1} \to V \otimes N_r \to N_{r+1} \to 0.$$

It is enough to prove that $\widetilde{K}^\bullet(C(n))$ is exact for $n \gg 0$.

We claim that actually for every graded $S(V)$-module N of finite type in $\mathcal{C}$ the complex $\widetilde{K}^\bullet(N(n))$ is exact for $n \gg 0$. This is clear for *free* modules of finite type by the exactness of the Koszul complex (in all degrees but one).

STEP 5: PROOF OF EXACTNESS. We now prove by descending induction in i, that for a module N of finite type one has $H^i(\widetilde{K}^\bullet(N(n))$ for $n \gg 0$. For $i > r+1$ this is trivial. Assume that the statement is true for $i+1$ and let us consider an exact sequence in the category of graded $S(V)$-modules

$$0 \to N' \to P \to N \to 0,$$

where P is a free module of finite type. Then it induces an exact sequence of complexes

$$0 \to \widetilde{K}^\bullet(N'(n)) \to \widetilde{K}^\bullet(P(n)) \to \widetilde{K}^\bullet(N(n))$$

and hence a long exact sequence of cohomology

$$\ldots \to H^i(\widetilde{K}^\bullet(P(n)) \to H^i(\widetilde{K}^\bullet(N(n)) \to H^{i+1}(\widetilde{K}^\bullet(N'(n)) \to \ldots$$

For $n \gg 0$ we have $H^i(\widetilde{K}^\bullet(P(n)) = 0$. Also, by part (i) of Theorem 2.2.2 the module N' is of finite type. Hence, by the induction assumption we have $H^{i+1}(\widetilde{K}^\bullet(N'(n)) = 0$ for $n \gg 0$ and from the above exact sequence we get that $H^i(\widetilde{K}^\bullet(N(n)) = 0$ for $n \gg 0$. □

C. The constant sheaf of t-structures on $\mathbb{P}^r$.

Theorem 2.3.6. *Assume that the heart $\mathcal{C} = D(X)^{\leq 0} \cap D(X)^{\geq 0}$ is noetherian.*

Then

(1) the subcategories

$$D(X \times \mathbb{P}^r)^{\leq 0} = \{F : \mathbf{R}p_*(F(n)) \in D^{\leq 0} \text{ for all } n \gg 0\},$$
$$D(X \times \mathbb{P}^r)^{\geq 0} = \{F : \mathbf{R}p_*(F(n)) \in D^{\geq 0} \text{ for all } n \gg 0\},$$

define a nondegenerate t-structure on $D(X \times \mathbb{P}^r)$, which extends to a sheaf of t-structures over $\mathbb{P}^r$. Moreover,

(2) the heart of this t-structure on $D(X \times \mathbb{P}^r)$ and on each $D(X \times U)$ is noetherian.

Proof.

PROOF OF (1). Lemma 2.3.2 easily implies that

$$D^{\leq 0}(X \times \mathbb{P}^r) = \cup_n D^{\leq 0}(X \times \mathbb{P}^r)_n,$$
$$D^{\geq 0}(X \times \mathbb{P}^r) = \cap_n D^{\geq 0}(X \times \mathbb{P}^r)_n.$$

This implies that $D^{\geq 1}(X \times \mathbb{P}^r)$ is the right orthogonal of $D^{\leq 0}(X \times \mathbb{P}^r)$. Let $F \in D(X \times \mathbb{P}^r)$ be any object. By Proposition 2.3.3 there exists n such that all objects $(H^i_n(F), i \in \mathbb{Z})$ belong to the hearts of all the t-structures corresponding to $m > n$. It follows that $\tau^n_{\leq 0}(F) \in D^{\leq 0}(X \times \mathbb{P}^r)$ and $\tau^n_{\geq 1}(F) \in D^{\geq 1}(X \times \mathbb{P}^r)$ for $n \gg 0$. This allows us to define a right adjoint $\tau_{\leq 0} : D(X \times \mathbb{P}^r) \to D^{\leq 0}(X \times \mathbb{P}^r)$ by $\tau_{\leq 0}(F) = \tau^n_{\leq 0}(F)$ for $n \gg 0$. This shows that this is a t-structure.

By the definition, this t-structure on $D(X \times \mathbb{P}^r)$ is preserved by the functor of tensoring with $\mathcal{O}_{\mathbb{P}^r}(1)$. Therefore, by Theorem 2.1.4 we obtain a sheaf of t-structures over $\mathbb{P}^r$. This proves (1).

Proof of (2) for $\mathbb{P}^r$.

Let us denote by $\mathcal{C}$ the heart of the original t-structure on $D(X)$ and by $\mathcal{C}_{\mathbb{P}^r}$ the heart of the constructed t-structure on $D(X \times \mathbb{P}^r)$. With every $F \in \mathcal{C}_{\mathbb{P}^r}$ we can associate a graded $S(V)$-module $M(F)$ in $\mathcal{C}$ defined by $M(F)_n = H^0\mathbf{R}p_*(F(n))$ for $n > 0$ and $M(F)_n = 0$ for $n \leq 0$, where $H^0 : D(X) \to \mathcal{C}$ is the cohomology functor associated with the t-structure on $D(X)$.

Claim. *For every $F \in \mathcal{C}_{\mathbb{P}^r}$ the $S(V)$-module $M(F)$ is of finite type.*

Proof of claim. By the definition we have $\mathbf{R}p_*(F(n)) \in \mathcal{C}$ for $n \gg 0$, so for such n we have $M(F)_n = \mathbf{R}p_*(F(n))$. It suffices to show that $V \otimes M(F)_n \to M(F)_{n+1}$ is surjective. Thus, the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^r}(n) \to V \otimes \mathcal{O}_{\mathbb{P}^r}(n-1) \to \mathcal{O}_{\mathbb{P}^r}(n) \to 0$$

shows that it is enough to check that $\mathbf{R}p_*(F \otimes \Omega^1_{\mathbb{P}^r}(n)) \in D^{\leq 0}$.

Assume now that n is large enough so that $\mathbf{R}p_*(F(m)) \in \mathcal{C}$ for $m \geq n - r - 1$. Using the exact sequences

$$0 \to \Omega^i_{\mathbb{P}^r}(n) \to \bigwedge^i V \otimes \mathcal{O}_{\mathbb{P}^r}(n-i) \to \Omega^{i-1}_{\mathbb{P}^r}(n) \to 0$$

for $i = 2, \ldots, r$ we see that it is sufficient to show that $\mathbf{R}p_*(F \otimes \Omega^r_{\mathbb{P}^r}(n)) = \mathbf{R}p_*(F(n-r-1))$ is in $D^{\leq 0}$. But this follows from the assumption that $\mathbf{R}p_*(F(n-r-1))$ is in $\mathcal{C}$. So the claim is proven.

We proceed to prove that the category $\mathcal{C}_{\mathbb{P}^r}$ is noetherian. First note that for any exact sequence

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

in $\mathcal{C}_{\mathbb{P}^r}$ the induced sequence

$$0 \to M(F_1)_n \to M(F_2)_n \to M(F_3)_n \to 0$$

in $\mathcal{C}$ is exact for $n \gg 0$. Next, we claim that if for some $F \in \mathcal{C}_{\mathbb{P}^r}$ we have $M(F)_n = 0$ for $n \gg 0$ then $F = 0$. Indeed, this would mean that $\mathbf{R}p_*(F(n)) = 0$ for $n \gg 0$ which clearly implies $F = 0$ by considering cohomology sheaves with respect to the standard t-structure. Therefore, if for some subobject $F' \subset F$ of an object in $\mathcal{C}_{\mathbb{P}^r}$ we have $M(F')_n = M(F)_n$ for $n \gg 0$ then $F' = F$.

Recall that by Theorem 2.2.2 the category of graded $S(V)$-modules of finite type in $\mathcal{C}$ is noetherian. Since as we have shown above for every $F \in \mathcal{C}_{\mathbb{P}^r}$ the $S(V)$-module $M(F)$ is of finite type, we immediately conclude that every object of $\mathcal{C}_{\mathbb{P}^r}$ is noetherian.

Proof of (2) for open $U \subset \mathbb{P}^r$.

We want to prove that the heart $\mathcal{C}_U$ of the t structure on $X \times U$ is noetherian. Fix an increasing sequence $F_1 \subset F_2 \cdots \subset F$ of objects in $\mathcal{C}_U$. By the sheaf axiom it suffices to show that the sequence stabilizes on a covering family of open subsets of U, so we may assume U affine. The divisor $\mathbb{P}^r \smallsetminus U = D$ is defined by a section f of $\mathcal{O}_{\mathbb{P}^r}(d)$ for some d.

By [TT] there is an extension $\bar{F} \in D(X \times \mathbb{P}^r)$ of F. Note that the restriction $(H^0\bar{F})_U = F$ by the definition of a sheaf of t-structures, so replacing $\bar{F}$ by $H^0\bar{F}$ we may assume $\bar{F} \in \mathcal{C}_{\mathbb{P}^r}$. We similarly construct $\bar{F}_i$. Replacing $\bar{F}_i$ by $\bar{F}_i(-k_iD)$ we may assume that the injection $F_i \hookrightarrow F$ extends to a morphism $\phi_i : \bar{F}_i \to \bar{F}$ (see Lemma 2.1.8). Replacing $\bar{F}_i$ by

$$\sum_{j=1}^{i} \phi_j(\bar{F}_j) \subset \bar{F}$$

we may assume

$$\bar{F}_1 \subset \bar{F}_2 \subset \cdots \subset \bar{F}$$

is increasing. This sequence stabilizes, therefore the original sequence stabilizes, as required. □

The above proof also leads to the following.

Lemma 2.3.7. *In the above notations for every object $F \in \mathcal{C}_{\mathbb{P}^r}$ there exists a surjection of the form $p^*G(n) \to F$ in $\mathcal{C}_{\mathbb{P}^r}$, where $G \in \mathcal{C}$, $n \in \mathbb{Z}$.*

Proof. Without loss of generality we can assume that $\mathbf{R}p_*(F(n)) \in \mathcal{C}$ for $n > 0$. As we have seen in the proof of the above theorem, the graded $S(V)$-module $M(F)$ is of finite type. Hence, there exists $N > 0$ such that the natural map

$$\bigoplus_{n=1}^{N} S(V) \otimes M(F)_n(-n) \quad \longrightarrow \quad M(F)$$

of graded $S(V)$-modules in $\mathcal{C}$ is surjective. This easily implies that the canonical map

$$\bigoplus_{n=1}^{N} \Big(p^* \mathbf{R}p_*(F(n))\Big)(-n) \quad \longrightarrow \quad F$$

is surjective in $\mathcal{C}_{\mathbb{P}^r}$. Therefore, we have a surjection

$$\bigoplus_{n=1}^{N} S^{N-n}(V) \otimes \Big(p^* \mathbf{R}p_*(F(n))\Big)(-N) \quad \longrightarrow \quad F.$$

□

Lemma 2.3.8. *The functor $\mathbf{R}p_* : D(X \times \mathbb{P}^r) \to D(X)$ is left t-exact.*

Proof. This follows from the fact that $\mathbf{R}p_*$ is right adjoint to the t-exact functor p^*. □

D. The constant sheaf of t-structures on $\mathbb{P}^r \times \mathbb{P}^s$.

Proposition 2.3.9. *Consider the t-structure on $D(X \times \mathbb{P}^r \times \mathbb{P}^s)$ obtained by applying the construction of Theorem 2.3.6 twice: first to get a t-structure on $D(X \times \mathbb{P}^s)$ and then to get a t-structure on $D(X \times \mathbb{P}^r \times \mathbb{P}^s)$. Then this t-structure can be described as follows: $F \in D^{\leq 0}$ (resp., $F \in D^{\geq 0}$) iff there exists an integer N such that for all $m, n > N$ one has $\mathbf{R}p_*(F(m,n)) \in D^{\leq 0}$ (resp., $\mathbf{R}p_*(F(m,n)) \in D^{\geq 0}$).*

Proof. It suffices to check that for every object F in the heart of our t-structure on the product $D(X \times \mathbb{P}^r \times \mathbb{P}^s)$ there exists N such that for all $m, n > N$ one has

$$\mathbf{R}p_*(F(m,n)) \quad \in \quad \mathcal{C} \;=\; D(X)^{\leq 0} \cap D(X)^{\geq 0}.$$

Let $p_1 : \;\; X \times \mathbb{P}^r \times \mathbb{P}^s \;\; \to \;\; X \times \mathbb{P}^r$ be the projection. By the definition, there exists an integer N' such that for all $n \geq N'$ one has $\mathbf{R}p_{1*}(F(0,n)) \in \mathcal{C}_{\mathbb{P}^r}$. By Lemma 2.3.8 this implies that $\mathbf{R}p_*(F(m,n)) \in D(X)^{\geq 0}$ for all $n \geq N'$ and all m. On the other hand, for every

$n \geq N'$ there exists an integer M_n such that for all $m \geq M_n$ one has $\mathbf{R}p_*(F(m,n)) \in \mathcal{C}$. Set $M = \max(M_{N'}, M_{N'+1}, \dots, M_{N'+s})$.

We claim that for every $n \geq N'$ and every $m \geq M$ one has $\mathbf{R}p_*(F(m,n)) \in \mathcal{C}$. Indeed, it suffices to check that in this range of (m,n) one has $\mathbf{R}p_*(F(m,n)) \in D(X)^{\leq 0}$. We can argue by induction in n. For $N' \leq n \leq N'+s$ the assertion is true by the choice of M. Now let $n > N'+s$ and assume that the assertion is true for all n' such that $N' \leq n' < n$. Then the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^s}(n-s-1) \to \dots \mathcal{O}_{\mathbb{P}^s}(n-1)^{s+1} \to \mathcal{O}_{\mathbb{P}^s}(n) \to 0$$

implies that $\mathbf{R}p_*(F(m,n)) \in D(X)^{\leq 0}$. □

Corollary 2.3.10. *The two t-structures on $D(X \times \mathbb{P}^r \times \mathbb{P}^s)$ obtained from the t-structure on $D(X)$ by iterating the construction of Theorem 2.3.6 coincide. This t-structure extends to a sheaf of t-structures over $\mathbb{P}^r \times \mathbb{P}^s$.*

Proof. The second assertion follows from Theorem 2.1.4 since our t-structure on $D(X \times \mathbb{P}^r \times \mathbb{P}^s)$ is stable under tensoring with the pull-back of any line bundle on $\mathbb{P}^r \times \mathbb{P}^s$. □

We also record the following lemma for later use.

Lemma 2.3.11. *Let $Y \subset \mathbb{P}^r \times \mathbb{P}^s$ be a closed subset such that the natural projection $\pi : Y \to \mathbb{P}^r$ is a finite morphism and let $F \in D(X \times \mathbb{P}^r \times \mathbb{P}^s)$ be an object supported on $X \times Y$. Then $F \in D(X \times \mathbb{P}^r \times \mathbb{P}^s)^{\leq 0}$ (resp., $F \in D(X \times \mathbb{P}^r \times \mathbb{P}^s)^{\geq 0}$) if and only if $\mathbf{R}p_{1*}F \in D(X \times \mathbb{P}^r)^{\leq 0}$ (resp., $\mathbf{R}p_{1*}F \in D(X \times \mathbb{P}^r)^{\geq 0}$), where $p_1 : X \times \mathbb{P}^r \times \mathbb{P}^s \to X \times \mathbb{P}^r$ is the projection.*

Proof. The condition that $F \in \mathcal{C}_{\mathbb{P}^r \times \mathbb{P}^s}$ is equivalent to $\mathbf{R}p_{1*}(F(0,n)) \in \mathcal{C}_{\mathbb{P}^r}$ for all $n \gg 0$. Replacing Y by its finite thickening we can assume that F is a push-forward of an object in $D(X \times Y)$. Let (U_i) be an open covering of $\mathbb{P}^r$ such that the line bundle $\mathcal{O}_Y(0,1)$ is trivial on $\pi^{-1}(U_i)$. Then $\mathbf{R}p_{1*}(F(0,n))|_{X \times U_i} \simeq \mathbf{R}p_{1*}(F)|_{X \times U_i}$ for all $n \in \mathbb{Z}$. It remains to use the fact that we have a sheaf of t-structures over $\mathbb{P}^r$. □

2.4. **The essential image of push-forward along a closed embedding.**

Lemma 2.4.1. *Let $i : Z \to Y$ be the zero locus of a non-zero function f on an affine variety Y. Let also Z' be the zero locus of f^2, $j : Z \to Z'$ and $i' : Z' \to Y$ be the natural embeddings. Then for every $F, G \in D(Z)$ and every morphism $\alpha : i_*F \to i_*G$ in $D(Y)$ there exists a morphism $\beta : j_*F \to j_*G$ in $D(Z')$, such that $\alpha = i'_*\beta$.*

Proof. Let $Y = \operatorname{Spec} A$ and let $F^\bullet$, $G^\bullet$ be (bounded) complexes of finitely generated $A/(f)$-modules representing F and G. We can choose a bounded above complex of finitely generated projective A-modules $P^\bullet$ and a surjective morphism of complexes of A-modules $q : P^\bullet \to F^\bullet$ which is a quasi-isomorphism. Then we have $\alpha = pq^{-1}$ for some chain map $p : P^\bullet \to G^\bullet$. Replacing $P^\bullet$ by $\tau_{\geq n}P^\bullet$ for some $n \ll 0$ we can assume that $P^\bullet$ is a bounded complex equipped with the same two morphisms. This truncation has the effect that not all terms of this complex are projective A-modules, but each of them is a submodule in a projective A-module. Since A is integral these modules are torsion-free. Hence, if $K^\bullet = \operatorname{Ker}(q)$ then $K^\bullet$ is an acyclic complex and the multiplication by f is injective on each K^i. We have $fP^\bullet \subset K^\bullet \subset P^\bullet$. Set $\overline{P}^\bullet = P^\bullet/fK^\bullet$. Then both maps q and p factor through maps $\overline{q} : \overline{P}^\bullet \to F^\bullet$ and $\overline{p} : \overline{P}^\bullet \to G^\bullet$. Moreover, $\overline{q}$ is still a quasi-isomorphism, so $\alpha = \overline{pq}^{-1}$. It remains to note that $\overline{P}^\bullet$ is a complex of $A/(f^2)$-modules. □

Theorem 2.4.2. *Let Y be a smooth affine variety, f be a function on Y such that its divisor of zeros $Z = Z(f)$ is smooth. Let $i : Z \to Y$ denote the natural embedding and let $F \in D(Y)$.*

Assume that the morphism $F \to F$ given by the multiplication with f is zero. Then $F \simeq i_* F'$ for some $F' \in D(Z)$.

Proof. By the projection formula we have $i_* \mathbf{L} i^* F \simeq i_* \mathcal{O}_Z \otimes F$. Now the exact sequence

$$0 \to \mathcal{O}_Y \xrightarrow{f} \mathcal{O}_Y \to i_* \mathcal{O}_Z \to 0$$

gives rise to an exact triangle

$$F \xrightarrow{f} F \to i_* \mathcal{O}_Z \otimes F \to F[1].$$

Therefore, if the map $F \xrightarrow{f} F$ is zero then F is a direct summand in $i_* \mathbf{L} i^* F$. Let us denote by $\pi : i_* \mathbf{L} i^* F \to F$ the corresponding morphism, so that $\pi \circ \mathrm{can}_F = \mathrm{id}_F$, where $\mathrm{can}_F : F \to i_* \mathbf{L} i^* F$ is the canonical morphism.

Since the cohomology sheaves $H^i F$ (with respect to the standard t-structure) are push-forwards of some sheaves on Z there exists a finite order thickening Z' of Z in Y such that $F \simeq i'_* G$ for some $G \in D(Z')$, where $i' : Z' \to Y$ is the natural embedding. Let us denote also by $j : Z \to Z'$ the natural embedding, so that $i = i' \circ j$. Then we have $i_* \mathbf{L} i^* F \simeq i'_* j_* \mathbf{L} i^* F$, so π (resp., can_F) is an element of $\mathrm{Hom}(i'_* j_* \mathbf{L} i^* F, i'_* G)$ (resp., $\mathrm{Hom}(i'_* G, i'_* j_* \mathbf{L} i^* F)$).

By Lemma 2.4.1, replacing Z' by a thickening we may assume that $\pi = i'_* \phi$, $\mathrm{can}_F = i'_* \psi$ for some morphisms $\phi : j_* \mathbf{L} i^* F \to G$ and $\psi : G \to j_* \mathbf{L} i^* F$ in $D(Z')$. Moreover, since $i'_*(\phi \circ \psi) = \mathrm{id}_F$, it follows that $\phi \circ \psi : G \to G$ induces an isomorphism on cohomology sheaves, hence it is an isomorphism.

Since Z is smooth there exists a morphism $p : Z' \to Z$ such that $p \circ j = \mathrm{id}$. Then we can consider the morphism $p_* \phi : \mathbf{L} i^* F \simeq p_* j_* \mathbf{L} i^* F \to p_* G$ in $D(Z)$. Finally, we define the morphism $\alpha : G \to j_* p_* G$ in $D(Z')$ by setting $\alpha := (j_* p_* \phi) \circ \psi$:

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[dr, "\alpha"'] & j_* \mathbf{L} i^* F \arrow[d, "j_* p_* \phi"] \\ & j_* p_* G. \end{tikzcd}$$

We claim that α is an isomorphism. Since the morphism p is finite, it suffices to check that $p_* \alpha$ is an isomorphism. But $p_* \alpha = p_* \phi \circ p_* \psi = p_*(\phi \circ \psi)$ and $\phi \circ \psi$ is an isomorphism. Setting $F' = p_* G$ we obtain

$$i_* F' \simeq i_* p_* G \simeq i'_* j_* p_* G \simeq i'_* G \simeq F,$$

as required. □

2.5. **Restricting a sheaf of t-structures to a closed subset.**

Lemma 2.5.1. *Let $i : Z \to Y$ be a regular embedding of codimension r, where $Z = Z(f_1, \ldots, f_r)$ is the zero locus of an r-tuple of functions $f_1, \ldots, f_r$ on Y. Let $F, F' \in D(Z)$ be two objects such that $\mathrm{Hom}^n(i_* F, i_* F') = 0$ whenever $n < 0$. Then $\mathrm{Hom}^n(F, F') = 0$ whenever $n < 0$, and the natural map $\mathrm{Hom}(F, F') \to \mathrm{Hom}(i_* F, i_* F')$ is an isomorphism.*

Proof. It suffices to consider the case when Z is a divisor in Y: $Z = Z(f)$ for a function f on Y. Note that $\mathrm{Hom}^n(F, F') = 0$ for $n \ll 0$, so we can prove the first assertion by induction. Assume that we know that $\mathrm{Hom}^{n'}(F, F') = 0$ for $n' < n$, where $n < 0$. The canonical distinguished triangle

$$F[1] \to \mathbf{L} i^* i_* F \to F \to F[2]$$

induces a long exact sequence

$$\ldots \to \mathrm{Hom}^{n-2}(F,F') \to \mathrm{Hom}^n(F,F') \to \mathrm{Hom}^n(i_*F,i_*F') \to \mathrm{Hom}^{n-1}(F,F') \to \ldots$$

Since $\mathrm{Hom}^{n-2}(F,F') = \mathrm{Hom}^{n-1}(F,F') = 0$ by the induction assumption, we derive that $\mathrm{Hom}^n(F,F') \simeq \mathrm{Hom}^n(i_*F,i_*F') = 0$ which finishes the proof of the first assertion. Now using the same exact sequence for $n = 0$ we derive the second assertion. □

Theorem 2.5.2. *Let S be a smooth variety, $T \subset S$ be a smooth closed subvariety. Then for every sheaf of t-structures over S there exists a unique sheaf of t-structures over T such that the push-forward functor $D(X \times T) \to D(X \times S)$ is t-exact.*

Proof. Let $i : T \to S$ denote the embedding. We claim that

$$D(X \times T)^{\leq 0} = \{F :\ i_*F \in D^{\leq 0}\},$$

$$D(X \times T)^{\geq 0} = \{F :\ i_*F \in D^{\geq 0}\},$$

is a nondegenerate t-structure on $D(X \times T)$. Let us first check that for $A \in D(X \times T)^{\leq 0}$, $B \in D(X \times T)^{\geq 1}$ one has $\mathrm{Hom}(A,B) = 0$. It suffices to prove that $\mathbf{R}\mathcal{H}om_T(A,B) \in D(T)^{\geq 0}$ (we use the notation of 2.1). This assertion is local on T, so we can assume that S is affine and that T is the zero locus of an r-tuple of functions $(f_1,\ldots,f_r)$ on S (where $r = \dim S - \dim T$). Since T is affine it suffices to check that $\mathrm{Hom}^i(A,B) = 0$ for $i < 0$ in this case. But this follows immediately from Lemma 2.5.1.

It remains to check that $D(X \times T)$ is generated as a triangulated category by $\mathcal{C}_T := D(X \times T)^{\leq 0} \cap D(X \times T)^{\geq 0}$. Let $F \in D(X \times T)$ be an arbitrary object. Without loss of generality we can assume that $i_*F \in D(X \times S)^{\geq 0}$ but $i_*F \notin D(X \times S)^{\geq 1}$. Let us consider the canonical morphism $\phi : \tau_{\leq 0} i_*F \to i_*F$. It suffices to prove that there exists an object $F^0 \in D(X \times T)$ and a morphism $\overline{\phi} : F^0 \to F$ such that

$$\tau_{\leq 0} i_*F \simeq i_*F^0 \tag{3}$$

and $\phi = i_*\overline{\phi}$. Indeed, then we would consider the cone F' of the morphism $\overline{\phi}$. Since i_*F' has smaller cohomological length than i_*F with respect to our t-structure, we would be able to finish the proof using induction.

Note that since $F' \in D(X \times T)^{\geq 1}$, using orthogonality of $D(X \times T)^{\leq 0}$ and $D(X \times T)^{\geq 1}$ one can immediately check that the pair $(F^0, \overline{\phi})$ (if exists) is uniquely determined up to a unique isomorphism. To prove existence let us first consider a similar problem for the restrictions $F_i = F|_{X \times (T \cap U_i)}$, where (U_i) is sufficiently fine open affine covering of S. Indeed, locally we can assume that T is given in S by r equations $f_1,\ldots,f_r$ (such that all intersections of divisors (f_i) are smooth). Furthermore, we can reduce to the case when $T = Z(f)$ is a divisor in S. In this case using the universal property of $\tau_{\leq 0}$ one immediately checks that multiplication by f gives a zero endomorphism of $\tau_{\leq 0} i_*F$, so we can apply Theorem 2.4.2 to find $F^0 \in D(X \times T)$ such that (3) holds. By Lemma 2.5.1 there exists a unique morphism $\overline{\phi} : F^0 \to F$ such that $\phi = i_*\overline{\phi}$. Applying this argument to all restrictions $F_i \in D(X \times (T \cap U_i))$ we find a collection of objects $F_i^0 \in D(X \times (T \cap U_i))$ and morphisms $\overline{\phi}_i : F_i^0 \to F_i$ with required property. By uniqueness, we get the gluing data for these pairs. It remains to note that since $\mathrm{Hom}^j(F_i^0, F_i^0) = 0$ for $j < 0$ we have $\mathbf{R}\mathcal{H}om_{U_i}(F_i^0, F_i^0) \in D(U_i)^{\geq 0}$. Therefore, we can apply Corollary 2.1.11 to glue objects (F_i^0) into an object $F^0 \in D(X \times T)$. Similarly, we have $\mathbf{R}\mathcal{H}om_{U_i}(F_i^0, F_i) \in D(U_i)^{\geq 0}$. Hence, by Lemma 2.1.10 we can glue morphisms $\overline{\phi}_i$ into a morphism $\overline{\phi} : F^0 \to F$. □

Lemma 2.5.3. *In the situation of the above theorem the functor $\mathbf{L}i^* : D(X \times S) \to D(X \times T)$ is left t-exact and for every $F \in \mathcal{C}_S := D^{\leq 0}(X \times S) \cap D^{\geq 0}(X \times S)$ one has $H^0 \mathbf{L}i^* i_*F \simeq F$.*

Proof. The functor $\mathbf{L}i^*$ is left t-exact since it is left adjoint to the t-exact functor i_*, so it remains to check the second assertion of the lemma. Since we are dealing we sheaves of t-structures, the assertion is local. Also, if i decomposes into a composition $i_1 \circ i_2$ of closed embeddings (and the intermediate variety is smooth) then it suffices to prove the lemma for i_1 and i_2. Therefore, it is enough to consider the case when T is a divisor in S defined as the zero locus of some function $f \in \mathcal{O}(S)$. But in this case the assertion follows immediately from the exact triangle

$$F[1] \to \mathbf{L}i^* i_* F \to F \to F[2].$$

□

2.6. **The constant sheaf of t-structures over a smooth projective variety.**

Theorem 2.6.1. *Assume that the heart $D(X)^{\leq 0} \cap D(X)^{\geq 0}$ is noetherian.*

Let S be a smooth projective variety. Let L be an ample line bundle on S. Then the subcategories

$$D(X \times S)_L^{\leq 0} = \{F : \ \mathbf{R}p_*(F \otimes L^n) \in D^{\leq 0} \text{ for all } n \gg 0\},$$
$$D(X \times S)_L^{\geq 0} = \{F : \ \mathbf{R}p_*(F \otimes L^n) \in D^{\geq 0} \text{ for all } n \gg 0\},$$

define a nondegenerate t-structure on $D(X \times S)$ with a noetherian heart. Furthermore, this t-structure does not depend on a choice of L and extends uniquely to a sheaf of t-structures over S.

Proof. Theorems 2.3.6 and 2.1.4 give the required sheaf of t-structures over every projective space. Now let L be an ample line bundle on S, $i : S \to \mathbb{P}^r$ be the closed embedding given by some power L^n of L. Applying Theorem 2.5.2 we get a sheaf of t-structures over S, extending the t-structure $(D(X \times S)_{L^n}^{\leq 0}, D(X \times S)_{L^n}^{\geq 0})$ on $D(X \times S)$. Moreover, since the functor i_* is t-exact and $i_*(F) = 0$ only for $F = 0$, we immediately obtain that the heart of the t-structure on $D(X \times S)$ is Noetherian. By Theorem 2.1.4 the functor of tensoring with L is t-exact. Therefore, for every object F in $D(X \times S)_{L^n}^{\leq 0}$ we have $F \otimes L, \ldots, F \otimes L^{n-1} \in D(X \times S)_{L^n}^{\leq 0}$, hence, $F \in D(X \times S)_L^{\leq 0}$. Thus, $D(X \times S)_L^{\leq 0} = D(X \times S)_{L^n}^{\leq 0}$. Similarly, $D(X \times S)_L^{\geq 0} = D(X \times S)_{L^n}^{\geq 0}$. It remains to show that this t-structure does not depend on a choice of projective embedding. Assume that we have two projective embeddings $i_1 : S \to \mathbb{P}^r$ and $i_2 : S \to \mathbb{P}^s$, corresponding to very ample line bundles L_1 and L_2 on S. Let $i : S \to \mathbb{P}^r \times \mathbb{P}^s$ be the diagonal embedding. Recall that by Corollary 2.3.10 the two natural t-structures on $D(X \times \mathbb{P}^r \times \mathbb{P}^s)$ coincide. Now applying Lemma 2.3.11 we see that $F \in D(X \times S)_{L_1}^{\leq 0}$ (resp., $F \in D(X \times S)_{L_1}^{\geq 0}$) iff $i_* F \in D(X \times \mathbb{P}^r \times \mathbb{P}^s)^{\leq 0}$ (resp., $i_* F \in D(X \times \mathbb{P}^r \times \mathbb{P}^s)^{\geq 0}$). Since a similar assertion holds for L_2, the t-structures associated with L_1 and L_2 coincide. The obtained t-structure on $D(X \times S)$ is invariant with respect to tensoring by any ample line bundle. Hence, by Theorem 2.1.4 it extends to a sheaf of t-structures. □

Let $\mathcal{C}$ be the heart of the t-structure on $D(X)$ and $\mathcal{C}_S$ be the heart of the t-structure on $D(X \times S)$ constructed above.

Lemma 2.6.2. *In the situation of Theorem 2.6.1 every object F of $\mathcal{C}_S$ has a left resolution in $\mathcal{C}_S$ of the form*

$$\ldots \to p^* G_2 \otimes L^{n_2} \to p^* G_1 \otimes L^{n_1} \to p^* G_0 \otimes L^{n_0} \to F \to 0$$

for some integers $n_0 > n_1 > n_2 > \ldots$ and some objects $G_i \in \mathcal{C}$.

Proof. It suffices to check that for every F there exists a surjection of the form $p^* G \otimes L^n \to F$, where $G \in \mathcal{C}$, $n \in \mathbb{Z}$. In the case when S is a projective space this follows from Lemma 2.3.7. In the general case we can assume that L is very ample and consider the projective embedding $i : S \to \mathbb{P}^r$ given by L. There exists a surjection in $\mathcal{C}_{\mathbb{P}^r}$ of the form $\pi^* G(n) \to i_* F$, where $G \in \mathcal{C}$,

$n \in \mathbb{Z}$, $\pi : X \times \mathbb{P}^r \to X$ is the natural projection. Since the functor $\mathbf{L}i^* : D(X \times \mathbb{P}^r) \to D(X \times S)$ is left t-exact the morphism

$$H^0\big(\mathbf{L}i^*(\pi^* G(n))\big) \to H^0\big(\mathbf{L}i^* i_* F\big)$$

is surjective in $\mathcal{C}_S$. But $H^0 \mathbf{L}i^* i_* F \simeq F$ by Lemma 2.5.3 and

$$H^0\big(\mathbf{L}i^*(\pi^* G(n))\big) \quad \simeq \quad H^0 p^* G \otimes L^n \quad \simeq \quad p^* G \otimes L^n.$$

□

2.7. **The constant sheaf of t-structures over a smooth quasi-projective variety.** In this section we use freely the fact that the category $D(V)$ is small for any variety V. We also use the notion of a pre-aisle, namely a suspended subcategory $\mathcal{S} \subset \mathcal{T}$ in a triangulated category, where the shift in $\mathcal{S}$ is induced from $\mathcal{S}$ and the triangles in $\mathcal{S}$ come from $\mathcal{T}$. See, e.g., [AJS]. A pre-aisle gives a t-structure if and only if there is a "truncation functor" $\tau : \mathcal{T} \to \mathcal{S}$ which is right adjoint to the inclusion $\mathcal{S} \subset \mathcal{T}$. It is said to be *cocomplete* if it is stable under coproducts.

Let us denote by $D_{qc}(X)$ the unbounded derived category of quasicoherent sheaves on X. The following lemma shows that a t-structure $(D^{\leq 0}, D^{\geq 0})$ on $D^b(X)$ extends to a t-structure on $D_{qc}(X)$.

Lemma 2.7.1. *Let $D_{qc}^{\leq 0}(X)$ be the cocomplete pre-aisle in $D_{qc}(X)$ generated by $D^{\leq 0} \subset D^b(X)$. Let also $D_{qc}^{\geq 0}(X)$ be the right orthogonal of $D^{\leq -1} \subset D^b(X)$ in $D_{qc}(X)$. Then $(D_{qc}^{\leq 0}(X), D_{qc}^{\geq 0}(X))$ is a t-structure on D_{qc} and one has $D_{qc}^{[a,b]}(X) \cap D^b(X) = D^{[a,b]}$.*

Proof. It is easy to check that $D_{qc}^{\geq 0}(X)$ is exactly the right orthogonal of $D_{qc}^{\leq -1}(X)$ (see Lemma 3.1 of [AJS]). Furthermore, by Theorem A.1 of loc. cit. these subcategories form a t-structure. The equality

$$D_{qc}^{\geq 0}(X) \cap D^b(X) = D^{\geq 0}$$

is clear since both sides coincide with the right orthogonal to $D^{\leq -1}$ in $D^b(X)$. We have the inclusion $D^{\leq 0} \subset D_{qc}^{\leq 0}(X) \cap D^b(X)$ by the definition. The inverse inclusion follows from the fact that $D_{qc}^{\leq 0}(X) \cap D^b(X)$ is left orthogonal to $D^{\geq 1}$. □

We use this extension of a t-structure to quasicoherent complexes in the following theorem.

Theorem 2.7.2. *Assume that the heart $\mathcal{C} = D(X)^{\leq 0} \cap D(X)^{\geq 0}$ is noetherian and* bounded with respect to the standard t-structure on $D^b(X)$. *Let S be a smooth quasiprojective variety. For an open subset $U \in S$ set*

$$D^{\geq 0}(X \times U) = \{F \in D^b(X \times U) |\ \mathbf{R}p_*(F|_{X \times U'}) \in D_{qc}^{\geq 0}(X) \textit{ for every open } U' \subset U\}$$

and define $D^{\leq 0}(U)$ to be the left orthogonal to $D^{\geq 1}(X \times U)$. Then

(1) these subcategories define a sheaf of nondegenerate t-structures over S.
(2) When S is projective this sheaf of t-structures coincides with the one defined in Theorem 2.6.1.
(3) When S is quasi-projective, the t-structure on S coincides with the one induced from the sheaf of t-structures on any projective completion of S.

Proof. Without loss of generality we can assume S to be projective. Let L be an ample line bundle on S. Then for every open subset $U \subset S$ we have a t-structure $(D^{\leq 0}(X \times U), D^{\geq 0}(X \times U))$ on $D^b(X \times U)$ coming from Theorem 2.6.1. Using Lemma 2.7.1 we can extend this t-structure to the unbounded derived category of quasicoherent sheaves on $X \times U$. Let $P_L(U) \subset D_{qc}(X \times U)$ be the cocomplete pre-aisle generated by $(p^* D^{\leq 0} \otimes L^n, n \in \mathbb{Z})$.

Claim. *The pre-aisle $P_L(U)$ coincides with $D^{\leq 0}_{qc}(X\times U)$ of Lemma 2.7.1.*

Proof of claim.

Note that the inclusion $P_L(U)\subset D^{\leq 0}_{qc}(X\times U)$ is clear. To check the inverse inclusion let us first consider the case $U=S$.

The case $U=S$. Since $D^{\leq 0}_{qc}(X\times S)$ is the minimal pre-aisle containing $D^{\leq 0}(X\times S)$, it suffices to check that $D^{\leq 0}(X\times S)\subset P_L(S)$. Moreover, it is enough to check that $\mathcal{C}_S\subset P_L(S)$. Using Lemma 2.6.2 we can find a resolution of an arbitrary object $F\in\mathcal{C}_S$ of the form

$$\ldots\to P_2\stackrel{d_2}{\to}P_1\stackrel{d_1}{\to}P_0\stackrel{d_0}{\to}F\to 0$$

with $P_i\in\mathcal{C}_S\cap P_L(S)$.

Note that our assumption that $\mathcal{C}$ is bounded with respect to the standard t-structure on $D^b(X)$ immediately implies (by taking the pushforward) that $\mathcal{C}_S$ is bounded with respect to the standard t-structure on $D^b(X\times S)$. In particular, there exists $N>0$ such that $\mathrm{Hom}^{>N}_{D(X\times S)}(\mathcal{C}_S,\mathcal{C}_S)=0$. Set $K_n=\ker(d_n)\subset P_n$. Then we have a sequence of morphisms in $D^b(X\times S)$

$$F\to K_0[1]\to K_1[2]\to\ldots$$

Note that, by the definition of N, the composed map $F\to K_n[n+1]$ is zero for $n>N$. Let $C_n\subset D^b(X\times S)$ be the convolution of the complex

$$P_n\to P_{n-1}\to\ldots\to P_0$$

(this convolution exists since P_i lie in the heart of some t-structure). Then we have an exact triangle

$$K_n[n]\to C_n\to F\to K_n[n+1].$$

Therefore, F is a direct summand of C_n for $n>N$. But C_n belongs to $P_L(S)$. Recall that cocomplete pre-aisles are closed under taking direct summands - see [AJS], Corollary 1.4: if the complement of F is $F^\perp$ then

$$C_n^{\oplus\infty}\;\simeq\;\bigoplus_{i=1}^{\infty}(F\oplus F^\perp)\;\simeq\;\bigoplus_{i=1}^{\infty}(F^\perp\oplus F)\;\simeq\;F^\perp\oplus\bigoplus_{i=2}^{\infty}(F\oplus F^\perp),$$

and the cone of the canonical embedding

$$C_n^{\oplus\infty}\;\simeq\;F^\perp\oplus\bigoplus_{i=2}^{\infty}(F\oplus F^\perp)\quad\longrightarrow\quad C_n^{\oplus\infty}\;\simeq\;\bigoplus_{i=1}^{\infty}(F\oplus F^\perp)$$

is isomorphic to F. Hence, F lies in $P_L(S)$.

The general case. Now let us consider the case of general open subset $j:U\hookrightarrow S$. Note that the set of $F\in D_{qc}(X\times S)$ such that $j^*F\in P_L(U)$ forms a pre-aisle in $D_{qc}(X\times S)$. Hence, we have $j^*P_L(S)\subset P_L(U)$. Therefore,

$$D^{\leq 0}(X\times U)=j^*D^{\leq 0}(X\times S)\subset j^*P_L(S)\subset P_L(U)$$

and as above this implies the inclusion $D^{\leq 0}_{qc}(X\times U)\subset P_L(U)$. This finishes the proof of our claim.

We continue with the proof of the theorem. The subcategory $D^{\geq 1}_{qc}(X\times U)$ is defined as the right orthogonal to $D^{\leq 0}_{qc}(X\times U)=P_L(U)$. Therefore, by Lemma 3.1 of [AJS] we have

$$D^{\geq 0}_{qc}(X\times U)=\{F\in D_{qc}(X\times U)|\ \mathrm{Hom}^{<0}(p^*G\otimes L^n,F)=0\text{ for all }G\in D^{\leq 0},n\in\mathbb{Z}\}.$$

Using the isomorphisms

$$\mathrm{Hom}^i(p^*G\otimes L^n, F)\simeq \mathrm{Hom}^i(G, \mathbf{R}p_*(F\otimes L^{-n})),$$

we deduce that

$$D_{qc}^{\geq 0}(X\times U)=\{F\in D_{qc}(X\times U)|\ \mathbf{R}p_*(F\otimes L^n)\in D_{qc}^{\geq 0} \text{ for all } n\in\mathbb{Z}\}.$$

Note that if U is sufficiently small then the restriction of L to U is trivial, hence for such U we will get

$$D_{qc}^{\geq 0}(X\times U)=\{F\in D_{qc}(X\times U)|\ \mathbf{R}p_*F\in D^{\geq 0}\}.$$

It remains to restrict this equality to $D^b(X\times U)$ and to use the fact that we have a sheaf of t-structures to finish the proof. □

3. Families of objects of the heart and their extensions

3.1. **Torsion subobjects.** As before, let $(D(X)^{\leq 0}, D(X)^{\geq 0})$ be a nondegenerate t-structure with noetherian heart $\mathcal{C}$. We fix a smooth projective variety S. We denote the heart of the lifted t-structure $(D(X\times S)^{\leq 0}, D(X\times S)^{\geq 0})$ by $\mathcal{C}_S$. For an open subset $S'\subset S$ we denote the heart coming from the sheaf of t-structures $\mathcal{C}_{S'\subset S}$, or $\mathcal{C}_{S'}$ if there is no possibility of confusion. Recall that we have already shown that $\mathcal{C}_{S'}$ is independent of the compactification S *if $\mathcal{C}$ is bounded with respect to the standard t-structure*, with an explicit description of this heart. We will soon prove the independence of $\mathcal{C}$ without the boundedness assumption.

Definition 3.1.1. Define an object $F\in\mathcal{C}_S$ to be *S-torsion* if $F\in D(X\times T)$ for some closed subscheme $T\subset S$. Equivalently, for every divisor $D\subset S$ containing T, with defining equation $f\in\mathcal{O}_S(D)$, there is an integer k such that the morphism $f^k: F\to F\overset{\mathrm{L}}{\otimes}\mathcal{O}_S(kD)$ is zero.

An object $F\in\mathcal{C}_S$ is *S-torsion-free* if it contains no nonzero S-torsion subobject.

An object $E\in\mathcal{C}_S$ is said to be *t-flat* if for every closed point $s\in S$ we have $\mathbf{L}i_s^*E\ \in\ \mathcal{C}$.

The following is immediate:

Proposition 3.1.2. *Let $E\in\mathcal{C}_S$. Then*

(1) there is a maximal S-torsion subobject $F\subset E$; and
(2) for every closed $T\subset S$ there is a maximal S-torsion object supported in T.

Proof. We prove the first statement, the other being similar. Consider an increasing sequence

$$F_1\subset F_2\subset\cdots\subset E$$

of S-torsion subobjects. Since $\mathcal{C}_S$ is noetherian, this sequence stabilizes. □

Corollary 3.1.3. *(1) Let $E\in\mathcal{C}_S$. Then there exists an S-torsion-free object $\bar{E}\in\mathcal{C}_S$, an epimorphism $\phi: E\to\bar{E}$, and a dense open immersion $j: S'\to S$ such that $j^*\phi: j^*E\to j^*\bar{E}$ is an isomorphism.*

(2) If $D\subset S$ is a smooth divisor with defining section $f\in\mathcal{O}_S(D)$, then $\mathbf{L}i_D^*\bar{E}\in\mathcal{C}_D$ and

$$i_{D*}\mathbf{L}i_D^*\bar{E}\quad=\quad Cone(\bar{E}(-D)\xrightarrow{f}\bar{E})\quad=\quad\bar{E}/f(\bar{E}(-D)).$$

(3) If $\dim S=1$ then $\bar{E}$ is t-flat.

Proof. Let $F\subset E$ be the maximal S-torsion subobject, supported over some closed $T\subset S$. Let $S'=S\smallsetminus T$. Let $\bar{E}=E/F$. Then $j^*\phi: j^*E\to j^*\bar{E}$ is an isomorphism. If $\bar{G}\subset\bar{E}$ is S-torsion then its inverse image $G\subset E$ is S-torsion as well, containing F, and thus $G=F$ and $\bar{G}=0$. Thus $\bar{E}$ is S-torsion free.

For the second statement we have $\bar{E} \otimes p_S^* \mathcal{O}_S(-D) \to \bar{E}$ injective. Thus the cone of this morphism is in $\mathcal{C}_S$. But this cone is isomorphic to $i_{D*} \mathbf{L} i_D^* \bar{E}$. Since i_{D*} is t-exact and sends nonzero objects to nonzero objects, we have that $\mathbf{L} i_D^* \bar{E} \in \mathcal{C}$. This also implies the last statement, namely $\bar{E}$ is t-flat when $\dim S = 1$. $\square$.

3.2. **Extensions in the heart across a divisor.** First we note that the sheaf of t-structurees is flasque:

Lemma 3.2.1. *Let $j : S' \to S$ be a dense open immersion, and $E' \in \mathcal{C}_{S'}$. Then there is an object $E \in \mathcal{C}_S$ such that $j^*E = E'$.*

Proof. By [TT] there is an object $F \in D(X \times S)$ such that $j^*F = E'$. Consider $E = H^0(F)$. Now by Definition 2.1.1 we have $j^*E = H^0(j^*F) = H^0(E') = E'$. $\square$

We have the following:

Proposition 3.2.2. *Let $i_D : D \hookrightarrow S$ be the closed immersion of a smooth divisor, with complement let $j : S' = S \smallsetminus D \hookrightarrow S$. Fix $E' \in \mathcal{C}_{S'}$. Then*

(1) there exists an object $E \in \mathcal{C}_S$, with no S-torsion supported in D, such that $j^*E = E'$.
(2) If E' is S'-torsion-free then E is S-torsion free.
(3) If, moreover, $\dim S = 1$ then E is t-flat.

Proof. By Lemma 3.2.1 there is an extension $E_0 \in \mathcal{C}_S$. Let F be the maximal S-torsion object supported in D. Then $j^*(E_0/F) = E'$, and E_0/F has no S-torsion supported in D. The rest is immediate. $\square$

3.3. **Families of objects in $\mathcal{C}$.**

Definition 3.3.1. Let S be a scheme of finite type over the base field k. *A family of objects in $\mathcal{C}$ parametrized by S* is an object

$$E \in D(X \times S)$$

such that for every closed point $s \in S$ we have

$$\mathbf{L} i_s^* E \in \mathcal{C}.$$

We have the following *open heart property:*

Proposition 3.3.2. *Let $E \in D(X \times S)$ and let $T \subset S$ be a smooth closed subscheme such that $\mathbf{L} i_T^* E \in \mathcal{C}_T$. Then there is an open neighborhood $T \subset U \subset S$ such that $E_U \in \mathcal{C}_U$.*

Applying the sheaf property we obtain:

Corollary 3.3.3. *Let S be a smooth quasi-projective variety. Let $E \in D(X \times S)$ be a family of objects in $\mathcal{C}$ parametrized by S. Then $E \in \mathcal{C}_S$.*

We rely on the following lemma:

Lemma 3.3.4. *Let S be a smooth quasi-projective scheme. Let $E \in \mathcal{C}_S$, let $T \subset S$ be a smooth subscheme, and assume $H^0(\mathbf{L} i_T^* E) = 0$. Then there is an open neighborhood $T \subset U \subset S$ such that $E_U = 0$.*

Proof of lemma. We apply induction on the codimension.

Step 1: we consider the case where T is a divisor in S.

We have an exact sequence in $\mathcal{C}_S$:

$$0 \to F \to E \to \bar{E} \to 0$$

where F is the maximal S-torsion subobject supported in T. Since $\mathbf{L}i_T^*$ is t-right-exact, we have a surjection $H^0(\mathbf{L}i_T^*E) \to H^0(\mathbf{L}i_T^*\bar{E})$. It follows that $\mathbf{L}i_T^*\bar{E} = H^0(\mathbf{L}i_T^*\bar{E}) = 0$. Considering cohomology sheaves of the standard t-structure, we have that $\bar{E}$ is supported away from T, and thus it vanishes on some open neighborhood of T.

So we may assume $E = F$ is supported in T. Let $f \in H^0(S, \mathcal{O}_S(T))$ be the defining section of T. We have that E is annihilated by f^k for some minimal integer $k \geq 0$. We apply induction on k. We have that $i_{T*}\mathbf{L}i_T^*E = Cone(E(-T) \xrightarrow{f} E)$, and therefore

$$i_{T*}H^0(\mathbf{L}i_T^*E) \quad = \quad H^0(i_{T*}\mathbf{L}i_T^*E) \quad = \quad E/fE(-T).$$

Since $H^0(\mathbf{L}i_T^*E) = 0$ we have $E = fE(-T)$ and therefore E is annihilated by f^{k-1}, contradicting minimality.

Step 2. Consider the case $\dim S - \dim T > 1$ and assumed the result holds for lower codimensions. Write

$$T \overset{i_{T\subset S_1}}{\hookrightarrow} S_1 \overset{i_{S_1}}{\hookrightarrow} S$$

where $S_1 \subset S$ is a smooth divisor. Then

$$H^0(\mathbf{L}i_T^*E) \quad = \quad H^0\big(\,\mathbf{L}i_{T\subset S_1}^*\, H^0(\mathbf{L}i_{S_1}^*E\,)\,\big).$$

By induction we have that there is an open neighborhood $T \subset U_1 \subset S_1$ such that

$$\Big(H^0(\mathbf{L}i_{S_1}^*E)\Big)_{U_1} = 0.$$

Replacing S by by an open subset containing U_1 as a divisor, we get using Step 1 that $E_U = 0$, which is what we needed. $\square$

Proof of Proposition. We apply induction on the codimension as in the Lemma, and it suffices to consider the case where T is a divisor. Then $\mathbf{L}i_T^*$ is right exact and $\mathbf{L}i_T^*[-1]$ is left exact.

Consider

$$M = \max\{i \,|\, \mathrm{Supp}(\tau_{\geq i}E) \cap T \neq \emptyset\}.$$

Replacing S by an open we may assume $\tau_{>M}E = 0$ and thus $\tau_{\geq M}E = H^M(E)[-M]$. The distinguished triangle $\tau_{<M}E \to E \to \tau_{\geq M}E$ induces $\mathbf{L}i_T^*(\tau_{<M}E) \to \mathbf{L}i_T^*E \to \mathbf{L}i_T^*(\tau_{\geq M}E)$.

Assume by contradiction $M > 0$. Since $\mathbf{L}i_T^*$ is right exact we get that

$$H^0\big(\mathbf{L}i_T^*(H^ME)\big) \quad = \quad H^M\big(\mathbf{L}i_T^*(\tau_{\geq M}E)\big) \quad = \quad H^M(\mathbf{L}i_T^*E) = 0$$

so, by Lemma 3.3.4, $H^M(E)_U = 0$, which contradicts the definition of M.

This implies $M \leq 0$. We now replace S by an open neighborhood of T so that $\tau_{>0}E = 0$.

Taking the long exact sequence of cohomology of the distinguished triangle $\mathbf{L}i_T^*(\tau_{<0}E) \to \mathbf{L}i_T^*E \to \mathbf{L}i_T^*(H^0E)$ we get

$$H^{i-1}\big(\mathbf{L}i_T^*(H^0E)\big) \quad \to \quad H^i\big(\mathbf{L}i_T^*(\tau_{<0}E)\big) \quad \to \quad H^i(\mathbf{L}i_T^*E) \quad \to \quad H^i\big(\mathbf{L}i_T^*(H^0E)\big) \quad \to \cdots$$

Note that, by right exactness of $\mathbf{L}i_T^*$ and left exactness of $\mathbf{L}i_T^*[-1]$, we have that

- $H^i\big(\mathbf{L}i_T^*(H^0E)\big) = 0$ for $i \neq 0, -1$,
- $H^i\big(\mathbf{L}i_T^*(\tau_{<0}E)\big) = 0$ for $i > -1$.

also we have

- $H^i(\mathbf{L}i_T^*E) = 0$ for $i \neq 0$

by assumption.

The long exact sequence implies that $H^i\big(\mathbf{L}i_T^*(\tau_{<0}E)\big) = 0$ for all i, so $\mathbf{L}i_T^*(\tau_{<0}E) = 0$, and $E = H^0(E)$. □

3.4. **Invariance of the heart revisited.** We have proved earlier, that *assuming boundedness of $\mathcal{C}$ with respect to the standard t-structure,* the t-structure over a quasi-projective variety is indepeendent of the projective completion. We can now remove the assumption:

Proposition 3.4.1. *Let $U \subset S_1$ be an open subset of a smooth projective variety. Let $E \subset D(X \times U)$ and assume $E \in \mathcal{C}_{U\subset S_1}$. Then for every other open embedding $U \subset S_2$ in a smooth projective variety, we have $E \in \mathcal{C}_{U\subset S_2}$.*

Proof. We apply induction on the dimension of U, the case of dimension 0 (or 1) being trivial.

Step 1: reductions. By the sheaf property it suffices to prove the result for a neighborhood of every point $s \in U$.

Also, by Hironaka's theorem (or any of its variants) we may choose a common resolution of singularities of S_1, S_2 which is an isomorphism in a neighborhood of any given $s \in U$, so it suffices to consider the case where either $S_1 \dashrightarrow S_2$ or its inverse is a morphism; in either case there are birational morphisms $S_i \to S'$ which are isomorphisms on the same neighborhood.

Finally we take a general divisor D' on S' through s such that its pull-backs to both S_1 and S_2 are nonsingular. We denote its pullback to U by D.

Step 2: separating out the torsion. Consider the maximal subobject $F \subset E$ with support in D - see statement (2) in Proposition 3.1.2. We have an exact sequence

$$0 \to F \to E \to \bar{E} \to 0$$

of objects in $\mathcal{C}_{U\subset S_1}$. It suffices to show that F and $\bar{E}$ are in $\mathcal{C}_{U\subset S_2}$, since the latter category is closed under extensions, being the heart of a t-structure.

Step 3: the non-torsion case. Let $D \subset D_1$; $D \subset D_2$ be the completions inside S_1 and S_2. Since $\bar{E}$ has no torsion in D, we have that $\mathbf{L}i_D^*\bar{E} \in \mathcal{C}_{D\subset D_1}$. The induction hypothesis implies that $\mathbf{L}i_D^*\bar{E} \in \mathcal{C}_{D\subset D_2}$ as well. By the open heart property (Proposition 3.3.2) we have $\bar{E}'_U \in \mathcal{C}_{U'\subset S_2}$ for some neighborhood U' of D, which is in particular a neighborhood of s as required.

Step 4: the torsion case. Denote the defining section of D by $f \in H^0(U, \mathcal{O}_U(D))$. Suppose F is annihilated by f^k. We have an exact sequence

$$0 \to f \cdot F(-D) \to F \to F_0 \to 0$$

of objects in $\mathcal{C}_{U\subset S_1}$, where both $f \cdot F(-D)$ is annihilated by f^{k-1} and F_0 by f. It suffices to prove the result for $f \cdot F(-D)$ and F_0, and by induction on k it suffices to consider $k = 1$. So $F = i_{D*}F'$ where $F' \in \mathcal{C}_{D\subset D_1} = \mathcal{C}_{D\subset D_2}$, so $F \in \mathcal{C}_{U\subset S_2}$ as needed. □

3.5. **The generic flatness problem.** The following problem is fundamental:

Problem 3.5.1. *Let $E \in \mathcal{C}_S$. Is there an open dense set $U \subset S$ such that E_U is t-flat over U?*

At this point we have only partial results.

Lemma 3.5.2. *If S is a curve, and $E \in \mathcal{C}_S$, there is an open dense set $U \subset S$ such that E_U is t-flat over U.*

Proof. Let $F \subset E$ be the maximal S-torsion subobject, and let T be its support. Set $U = S \smallsetminus T$. Then E_U is S-torsion-free and therefore t-flat. $\square$

Proposition 3.5.3. *Let $E \in \mathcal{C}_S$. There is a dense subset $Z \subset S$ such that $\mathbf{L}i_s^*E \in \mathcal{C}$ for all $s \in Z$.*

Proof. We apply induction on the dimension. First, we replace S by an open set where E is S-torsion-free. Shrinking S further we may assume S admits a smooth map $S \to \mathbb{P}^1$. The result holds for the fibers by induction, and therefore it holds for S. $\square$

According to [AZ], the generic flatness problem has an affirmative answer if the category $\mathcal{C}_S$ can be defined and proved noetherian for certain noetherian dedekind domains which are not necessarily of finite type. At this point we know this to be true in very few examples.

4. Valuative criteria for semistable objects in $\mathcal{P}(1)$.

4.1. **One parameter families for objects in $\mathcal{P}(1)$.** Consider a numerical locally finite stability condition $(Z, \mathcal{P})$ with $\mathcal{P}(\phi) \subset D(X)$ and $Z : N(X) \to \mathbb{C}$. Assume that its heart $\mathcal{C} = \mathcal{P}((0,1])$ is noetherian. We denote by $(D(X)^{\leq 0}, D(X)^{\geq 0})$ the corresponding t-structure, where $D(X)^{\leq 0} = \mathcal{P}((0,\infty))$ and $D(X)^{>0} = \mathcal{P}((-\infty,0])$.

The main result of this section is the following:

Theorem 4.1.1. *Let S be a smooth curve and $U \subset S$ a dense open subset.*

(1) Every family F_U of objects in $\mathcal{P}(1)$ over U extends to a family F of objects in $\mathcal{P}(1)$ over S.
(2) Let F_1 and F_2 be families of objects in $\mathcal{P}(1)$ over S, and let $\phi_U : (F_1)_U \to (F_2)_U$ be an isomorphism. Then $\mathbf{L}i^*F_1$ and $\mathbf{L}i^*F_2$ are S-equivalent.
(3) Every family F_U of objects in $\mathcal{P}(1)$ over U extends, after a suitable finite surjective base change $g : S' \to S$, to a family F of objects in $\mathcal{P}(1)$ over S' with polystable fibers in $S' \smallsetminus g^{-1}U$.

Part (1) of the theorem follows from Proposition 3.2.2, in conjunction with the following lemma. We note that this lemma is the one point where our arguments restrict to $\mathcal{P}(1)$ and do not extend to an arbitrary $\mathcal{P}(t)$.

Lemma 4.1.2. *Let E be a family of objects in $\mathcal{C}$ with connected base S such that for some $s_0 \in S$ we have*

$$\mathbf{L}i_{s_0}^*E \in \mathcal{P}(1).$$

Then for all $s \in S$ we have $\mathbf{L}i_s^*E \in \mathcal{P}(1)$.

Proof. The objects $\mathbf{L}i_s^*E \in \mathcal{C}$ are numerically equivalent to each other. Therefore $Z(\mathbf{L}i_s^*E) = Z(\mathbf{L}i_{s_0}^*E) \in \mathbf{R}_{<0}$. It follows that $\mathbf{L}i_s^*E \in \mathcal{P}(1)$. $\square$

4.2. **Elementary modifications.** We proceed towards uniqueness in a standard manner. We start with elementary modifications. For simplicity we work over an affine curve S, with a fixed closed point s having defining equation π. We write $i : \{s\} \to S$ and

$$U = S \smallsetminus \{s\}.$$

Let $F \in \mathcal{C}_S$ be a family of objects in $\mathcal{P}(1)$. Consider an exact sequence

$$0 \to E \to \mathbf{L}i^*F \to Q \to 0.$$

Since $\mathbf{L}i^*F \in \mathcal{P}(1)$ we have that $E, Q \in \mathcal{P}(1)$. Since F is t-flat, we have a surjection $F \to i_*\mathbf{L}i^*F$. Since i_* is t-exact we have a surjection $F \to i_*Q \to 0$. Let G be its kernel.

Definition 4.2.1. We call G the *elementary modification of F at Q.*

Lemma 4.2.2. *The object G is a family of objects in $\mathcal{P}(1)$. We have an exact sequence*

$$0 \to Q \to \mathbf{L}i^*G \to E \to 0,$$

in particular $\mathbf{L}i^*G$ is S-equivalent to $\mathbf{L}i^*F$. The elementary modification of G at E is isomorphic to F, and the composition $F \to G \to F$ is the map $\pi : F \to F$.

Proof. Since G is a subobject of F, the kernel of $\pi^n : G \to G$ is a subobject of the kernel of $\pi^n : F \to F$, which is 0. Thus G is t-flat. By Lemma 4.1.2 it is a family of objects in $\mathcal{P}(1)$.

Since i_*Q is annihilated by π, we have that $\pi : F \to F$ factors through the inclusion $G \hookrightarrow F$. The fundamental isomorphism

$$\left(F/\pi F\right)\Big/\left(G/\pi F\right) \quad \simeq \quad F/G$$

shows that $G/\pi F \simeq i_*E$, and exchanging the roles of F and G we find that the kernel of $\mathbf{L}i^*G \to E$ is Q. $\square$

Lemma 4.2.3. *Let $\phi : F_A \to F_\Omega$ be a morphism of families of objects of $\mathcal{P}(1)$ giving an isomorphism $(F_A)_U \to (F_\Omega)_U$. Then there is a sequence of elementary modifications*

$$F_A = F_0 \subset F_1 \subset \cdots \subset F_n = F_\Omega$$

whose composition is ϕ. In particular $\mathbf{L}i^*F_A$ and $\mathbf{L}i^*F_\Omega$ are S-equivalent.

Proof. The object $Q = F_\Omega/F_A$ has support at $\pi = 0$, therefore it is annihilated by π^n for some positive n. Define

$$K_i = \operatorname{Ker}(\pi^{n-i} : Q \to Q), \quad L_i = Q/K_i,$$

and let $F_i = Ker(F_\Omega \to L_i)$. Then $F_{i+1}/F_i \simeq K_{i+1}/K_i$ is annihilated by π. Thus F_i is an elementary modification of F_{i+1}. The claim follows from the previous lemma. $\square$

Using Lemma 2.1.8 we obtain:

Lemma 4.2.4. *Let F_A and F_B be two families of objects in $\mathcal{P}(1)$, and let $\phi : F_{A|U} \to F_{B|U}$ be an isomorphism. Then $\mathbf{L}i^*F_A$ and $\mathbf{L}i^*F_B$ are S-equivalent.*

4.3. **Polystable replacement.** For a polystable replacement we need a finite base-change. For this we have

Lemma 4.3.1. *Let $g : S' \to S$ be a finite flat morphism of nonsingular projective varieties. Then g_* and g^* are t-exact.*

Proof. Let L be ample on S and $L' = g^*L$. Denote by $p' : X \times S' \to X$ the projection. Consider an object $E \in \mathcal{C}_S$, and let $E' = g^*E$. To show that $E' \in \mathcal{C}_{S'}$ we need to show that $\mathbf{R}p'_*(E' \otimes L'^n) \in \mathcal{C}$ for large n, which, by the projection formula is the same as showing that $\mathbf{R}p_*(E \otimes g_*\mathcal{O}_{S'} \otimes L^n) \in \mathcal{C}$, which is equivalent to $E \otimes g_*\mathcal{O}_{S'} \in \mathcal{C}$, which follows from Proposition 2.1.3, since $g_*\mathcal{O}_{S'}$ is a locally free sheaf.

Now let $E' \in \mathcal{C}_{S'}$ and let us show that $g_*E' \in \mathcal{C}_S$. We have by definition $\mathbf{R}p'_*E' \otimes L'^n \in \mathcal{C}$ for large n, so by the projection formula $\mathbf{R}p_*(g_*E' \otimes L^n) \in \mathcal{C}$ for large n, which means $g_*E' \in \mathcal{C}_S$. $\square$

Lemma 4.3.2. *Let F be a family of objects in $P(1)$, with a sequence*

$$0 \to E \to \mathbf{L}i^*F \to Q \to 0.$$

Consider its pullback via $g : S' \to S$ given by $\varpi^2 = \pi$, and the incusion $i' : \{s'\} \hookrightarrow S'$ of the point $\varpi = 0$. Let H be the elementary modification of g^*F at Q. Then the exact sequence

$$0 \to Q \to \mathbf{L}i'^* H \to E \to 0$$

splits.

Proof. Pulling back the exact sequence

$$0 \to p^*Q \stackrel{\pi}{\to} p^*Q \to i_*Q \to 0$$

and using Lemma 4.3.1 we get an exact sequence

$$0 \to g^*p^*Q \stackrel{\varpi^2}{\to} g^*p^*Q \to g^*i_*Q \to 0.$$

We thus have an exact sequence

$$0 \to i'_*Q \stackrel{\varpi}{\to} g^*i_*Q \to i'_*Q \to 0,$$

and H is the pullback of i'_*Q on the left along $g^*F \to g^*i_*Q$. Thus we have surjections $H \to i'_*Q$ and $H \to i'_*E$.

The subobject $\varpi F \subset H$ surjects to i'_*Q and maps to 0 in i'_*E.

Now consider g^*G, where G is the elementary modification of F at Q. We have that g^*G is canonically the elementary modification of H at i'_*Q, in particular it maps to 0 in i'_*Q. At the same time it surjects to g^*i_*E, in particular to i'_*E.

Altogether we have that H surjects to $i'_*Q \oplus i'_*E$, so $i'^*H \to Q \oplus E$ is an isomorphism. □

This completes the proof of Theorem 4.1.1

5. The noetherian property of discrete stability conditions

In order for our results to be of some content, we need examples of stability conditions with noetherian heart.

Proposition 5.0.1. *Let $(Z, \mathcal{P})$ be a stability condition such that both*

$$Image(Z : K_0(X) \to \mathbb{C}) \subset \mathbb{C}$$

and

$$Image(ImZ : K_0(X) \to \mathbb{R}) \subset \mathbb{R}$$

are discrete subgroups. Then the heart $\mathcal{P}((0,1])$ of the stability condition is noetherian.

Corollary 5.0.2. *Let T be a trangulated category of finite type with numerical rank 2. Then every numerical, locally finite stability condition on T such that $\mathcal{P}(0) \neq 0$ is noetherian.*

In particular, this holds in the following cases:

(1) X is a smooth curve, and $T = D(X)$, and
(2) X a smooth projective variety, $\mathcal{C} = Coh^{d,d-2}(X)$ is the category of coherent sheaves supported in dimension d modulo the Serre subcategory of sheaves supported in dimension $d-2$, and $T = D^b(\mathcal{C})$.

Proof of corollary. Let $(Z, \mathcal{P})$ be a numerical, locally finite stability condition on T. If the image of Z lies in a real line, then local finiteness implies that $\mathcal{P}(1) = \mathcal{P}((0,1])$ is noetherian. Otherwise the image of Z is discrete, and since it contains a subgroup $\mathbb{Z} \cdot Z(\mathcal{P}(0))$ of rank 1, the imaginary part is discrete as well. □

Proof of proposition. We consider an increasing sequence

$$F_1 \subset F_2 \subset \cdots \subset E$$

of nonzero objects in the heart $\mathcal{C} = \mathcal{P}((0,1])$.

Lemma 5.0.3. *Suppose $E \in \mathcal{P}(1)$. Then $F_i \in \mathcal{P}(1)$ and the sequence stabilizes.*

Proof. We have

$$Im(Z(F_i)) = -Im Z(E/F_i)$$

and both are nonnegative, therefore $Im(Z(F_i)) = 0$ and thus $E_i \in \mathcal{P}(1)$. Now $Z(F_i)$ and $Z(F_{i+1}/F_i)$ are nonpositive, therefore $Z(F_i)$ is nonincreasing. For the same reason $Z(F_i) > Z(E)$, and by discreteness $Z(F_i)$ stabilizes. But at that point $Z(F_{i+1}/F_i) = 0$ and thus $F_{i+1}/F_i = 0$, so the sequence stabilizes. □

Lemma 5.0.4. *Let $E_0 \subset E$ with $E_0 \in \mathcal{P}(1)$ be the (possibly zero) Harder–Narasimhan subobject of phase 1, and let $\phi_{\max}(E/E_0) = a < 1$ be the phase of the next Harder–Narasimhan piece. Let $T_i \subset F_i$ be the maximal subobject with Harder–Narasimhan constituents of phases $> a$.*

Then $T_i \subset T_{i+1} \subset E_0$ form an increasing sequence and this sequence stabilizes.

Proof. The morphism $T_i \to E/E_0$ vanishes since $\phi_{\min}(T_i) > \phi_{\max}(E/E_0) = a$. Therefore $T_i \hookrightarrow E$ factors through E_0. Similarly $T_i \hookrightarrow F_{i+1}$ factors through T_{i+1}. This sequence stabilizes by the previous lemma. □

Proof of proposition. By the previous lemmas we may assume that $E \notin \mathcal{P}(1)$ and $T_i = T$ are constant. Consider the increasing sequence of subobjects $F_i/T \subset E/T$. By the discreteness assumption we have that the nondecresing sequence $Im\, Z(F_i/T)$ stabilizes, at some $b > 0$. So we may assume it is constant. Then $Re\, Z(F_i/T)$ is nonincreasing, but $\phi_{\max}(F_i/T) \leq a$. Therefore $Z(F_i/T)$ stabilizes, and as before this implies that F_i/T stabilizes and thus F_i does. □

Recall that to every connected component Σ in the space of numerical stability conditions on T Bridgeland associates a subspace $V(\Sigma) \subset (\mathcal{N}(T) \otimes \mathbb{C})^*$, such that the map sending a stability to the corresponding central charge induces a local homeomorphism $\Sigma \to V(\Sigma)$. It seems to be unknown whether the subspace $V(\Sigma)$ is always defined over $\mathbb{Q}$. This is the case for all the components explicitly described in [B1] and [B2] for the cases of curves and K3 surfaces.

Corollary 5.0.5. *Let Σ be a connected component in the space of numerical stability conditions for which $V(\Sigma)$ is defined over $\mathbb{Q}$. The the set of stability conditions satisfying the assumptions of Proposition 5.0.1 is dense in Σ.*

Proof. Indeed, if $V(\Sigma)$ is defined over $\mathbb{Q}$ then the set of homomorphisms $Z : \mathcal{N}(T) \to \mathbb{C}$ with the image contained in $\mathbb{Q} + i\mathbb{Q}$ is dense in $V(\Sigma)$. □

6. Questions

6.1. **Towards a moduli space of semistable objects.** In order to construct proper moduli spaces of S-equivalence classes of objects in $\mathcal{P}(1)$ with fixed numerical class, two problems remain:

(1) The generic flatness problem (Problem 3.5.1).
(2) Boundedness: fixing a class $c \in H^*(X, \mathbb{Q})$ there should be a scheme of finite type S and a family $E \in D(X \times S)$ including all objects in $\mathcal{P}(1)$ having Chern character c.

Generic flatness would imply openness of versality using the open heart property (Proposition 3.3.2). If this holds, then essentially by the work of Inaba [I], versal deformation spaces exist, and the Isom functor is representable. Inaba proves his result only for so called *simple complexes*, but his result should extend readily to our situation. Altogether this would imply the existence of an Artin algebraic stack of semistable objects. The moduli space of S-equivalence classes of objects in $\mathcal{P}(1)$ can be pieced together from this stack - this will be pursued elsewhere.

Boundedness is necessary for proving properness, and we have not begun pursuing the various approaches to address this. It is expected to be a difficult problem.

6.2. **Alternative approaches.** Our construction of the sheaf of t-structures is rather roundabout, and given the rather simple characterization of the resulting t-structure one may wonder if there is no direct way to this construction. It is conceivable that a further study of the quasi-coherent picture as in [AJS], may lead to a direct construction, which may be more general.

Given an abelian category with a number of assumptions, Artin and Zhang [AZ] study various aspects of moduli problems for objects in this category, in particular *Quot* functors. It is interesting to see to what extent our method can be combined with theirs (or maybe even overridden by their approach).

6.3. **Removal of superfluous assumptions.** Various aspects of our discussion requires assumptions which might not be necessary.

(1) It would be nice to extend the results to $\mathcal{P}(t)$ for $t \neq 1$.
(2) The characterization of the t-structure over a quasi-projective variety should be general, and should require neither boundedness of $\mathcal{C}$ with respect to the standard t-structure nor resolution of singularities.
(3) it would be interesting to see to what extent the noetherian assumption may be weakened. After all, Bridgeland's stability conditions are only assumed locally finite in general.

Techniques such as in [BB, K, YZ] may prove useful in pursuing these directions.

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