Preconceptions and misconceptions on relative stable maps in the normal crossings case

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(joint work with Barbara Fantechi)

I report on joint work in progress with Barbara Fantechi; a closely related work in symplectic geometry is being developed by Joshua Davis of Duke University.

The theory of relative stable maps was first introduced by Ziv Ran in his paper on the degree of the Severi variety under a different name; the subject was developed within Gromov-Witten theory by a number of people, including A.M. Li–Y. Ruan, E. Ionel–T. Parker, and J. Li. Working in algebraic geometry, we must follow the work of Jun Li. Related work appeared through the years, including Harris–Mumford, Alexander-Hirschowitz, Gathmann, Caporaso–Harris, Vakil.

Stable maps were introduced by Kotsevich as a tool in Gromov–Witten theory, which from the point of view of this workshop, serves as a tool in enumerative geometry. The main goal is to count the number of curves of given genus g and homology class β on a variety X meeting given cycles $\gamma_1, \dots \gamma_n$.

Tools in Gromov–Witten theory include the famous WDVV equation, but much more powerful are the methods of localization and degeneration. I concentrate on the degeneration method.

Previous work concentrated on the case of a family of varieties parametrized by a curve, with smooth total space and special fiber consisting of two smooth components meeting transversally along a divisor Σ. The issue is that, although a space of stable map fibered over the base exists, the formalism of virtual fundamental classes fails in genus > 0 in case there are components mapping to the singular locus Σ of the fiber. The solution involves expanding the fiber by sticking a chain of $\mathbb{P}^1$ bundles over Σ between the two original components of the fiber.

One then defines degenerate stable maps to the fibers, and similarly relative stable maps to each component, and one proves a gluing formula for degenerate stable maps in terms of relative stable maps to each of the components.

The problem we set out to solve is:

1. define relative stable maps to (Y, D), where D is a normal crossings divisor on a smooth variety Y,

2. define degenerate stable maps to a variety obtained by gluing such relative (Y_i, D_i) appropriately along the divisors, in such a way that Gromov–Witten invariants are defined and are deformation invariant, and

3. prove a gluing formula comparing the two.

The aim of the talk was to describe a good number of places where one finds welcome and unwelcome surprises in the project. Due to ill planning, most of the talk ended up in explaining the earlier work.

A welcome surprise mentioned in the talk is the following: much grief was brought on previous writers in analyzing the so called "predeformability condition", a closed condition on relative and degenerate stable maps which is unpleasant to work out. Techniques of stacks and twisted stable maps of Olsson and Abramovich–Vistoli enable one to transform this into an open condition on a modified space, thus avoiding much of the grief.

An unwelcome surprise mentioned in the talk is in describing the gluing formula: whereas in the previously studied case the Gromov–Witten numbers of a two-component variety was described in terms of its decomposition to exactly two components, in our case, where the degenerate variety has at least three intersecting components, our formalism requires summing over further decompositions, where each component of the original variety is further "expanded".

References

[1] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. 15 (2002), no. 1, 27–75

[2] James Alexander and André Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geom. 4 (1995), no. 2, 201–222.

[3] Lucia Caporaso and Joe Harris, *Counting plane curves of any genus*, Invent. Math. 131 (1998), no. 2, 345–392.

[4] Andreas Gathmann, *Absolute and relative Gromov-Witten invariants of very ample hypersurfaces*, Duke Math. J. 115 (2002), no. 2, 171–203.

[5] Joe Harris and David Mumford *On the Kodaira dimension of the moduli space of curves*, With an appendix by William Fulton. Invent. Math. 67 (1982), no. 1, 23–88.

[6] André Hirschowitz, *La méthode d'Horace pour l'interpolation à plusieurs variables*, Manuscripta Math. 50 (1985), 337–388.

[7] Eleny-Nicoleta Ionel and Thomas H. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. (2) 157 (2003), no. 1, 45–96.

[8] Eleny-Nicoleta Ionel and Thomas H. Parker, *The symplectic sum formula for Gromov-Witten invariants*, Ann. of Math. (2) 159 (2004), no. 3, 935–1025.

[9] An-Min Li and Yongbin Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. 145 (2001), no. 1, 151–218.

[10] Jun Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. 57 (2001), no. 3, 509–578.

[11] Jun Li, *A degeneration formula of GW-invariants*, J. Differential Geom. 60 (2002), no. 2, 199–293.

[12] martin Olsson, *On (log) twisted curves,* preprint `http://www.ma.utexas.edu/ molsson/Logcurves.pdf`

[13] Ziv Ran, *The degree of a Severi variety*, Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 1, 125–128.

[14] Ravi Vakil, *The enumerative geometry of rational and elliptic curves in projective space*, J. Reine Angew. Math. 529 (2000), 101–153.