PRECONCEPTIONS AND MISCONCEPTIONS ON RELATIVE STABLE MAPS IN THE NORMAL CROSSINGS CASE

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Introduction

We describe our work in progress on relative Gromov–Witten Theory and its degeneration counterpart in the case of a smooth variety relative to a normal crossings divisor. Our work follows the algebraic methods of Jun Li, based on expanded degenerations of the target, and has a symplectic counterpart in work in progress by Joshua Davis. We note that Bernd Siebert has announced in a lecture another approach based on endowing the source curves with certain logarithmic structures, which has the potential advantage for wider and more direct applications, though that theory has not been developed yet.

0.1. **Gromov–Witten theory and relative stable maps.** The theory of relative stable maps was first introduced by Ziv Ran in his paper on the degree of the Severi variety under a different name; the subject was developed within Gromov-Witten theory by a number of people, including A.M. Li–Y. Ruan, E. Ionel–T. Parker, and J. Li. Working in algebraic geometry, we must follow the work of Jun Li. Related work appeared through the years, including Harris–Mumford, Alexander-Hirschowitz, Gathmann, Caporaso–Harris, Vakil.

Stable maps were introduced by Kotsevich as a tool in Gromov–Witten theory, which in particular serves as a tool in enumerative geometry. The main goal from the enumerative point of view is to count the number of curves of given genus g and homology class β on a variety X meeting given cycles $\gamma_1, \ldots, \gamma_n$.

The formalism is based on the moduli space of stable maps along with its evaluation maps:

$$\overline{\mathcal{M}} := \overline{\mathcal{M}}_{g,n}(X,\beta) \xrightarrow{ev_i} X.$$

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The moduli space has a virtual fundamental class $[\overline{\mathcal{M}}]^{\mathrm{vir}}$ allowing to define the Gromov–Witten classes

$$\langle \gamma_1, \ldots \gamma_n \rangle \quad = \quad \int_{[\overline{\mathcal{M}}]^{\mathrm{vir}}} \prod ev_i^* \gamma_i \ .$$

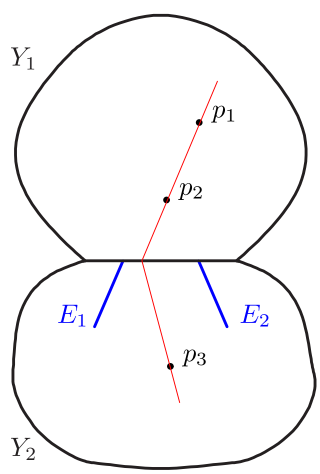
Tools in Gromov–Witten theory include the famous WDVV equation, but much more powerful are the methods of localization and degeneration, see discussion in [**?**, **?**]. We concentrate on the degeneration method.

1. Basic examples

1.1. **Plane sections of a quadric surface.** As a simple example, consider a pencil of quadric surfaces degenerating to two planes meeting along a line Σ, and consider the number of curves of type $(1,1)$ (i.e. conics) on a quadric passing through three points in general position. Of course this number is 1, but it is instructive to see how it is revealed in the degeneration.

Note that the total space of the pencil is singular along the intersection of the base locus with the line Σ, namely two points. Blowing up one of the two planes gives a small resolution of the total space, where one component Y_1 of the singular fiber is a plane, and another Y_2 is the blowing up of the other plane at two points on Σ, with two exceptional curves $E_i \subset Y_2$.

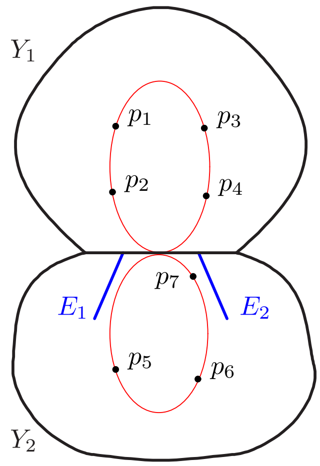
If of the three chosen points p_1, p_2, p_3 we let p_1, p_2 specialize to Y_1 and p_3 to Y_2, the limiting curve through the three points is $L_1 \cup L_2$, where L_1 is the unique line on Y_1 through p_1, p_2, and L_2 the unique line on Y_2 passing through p_3 and the point of intersection $L_1 \cap \Sigma$.



If all three points specialize to Y_1, the resulting curve is the union of E_1, E_2, and the unique conic on Y_1 through p_1, p_2, p_3 and meeting E_1, E_2.

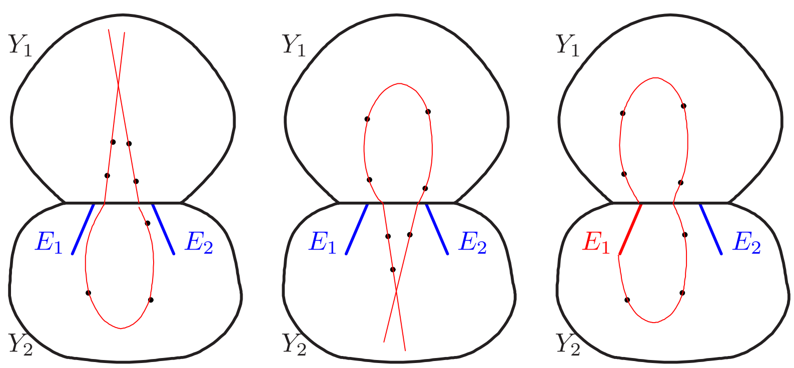
If all three points specialize to Y_2, the resulting curve is the unique proper transform on Y_2 of a conic, passing through p_1, p_2, p_3 and meeting E_1, E_2.

1.2. **Rational curves of type $(2,2)$.** It is instructive to do a similar analysis of the number of rational curves of type $(2,2)$ through 7 points in general position on the quadric. Again the number is known classically - it is the number of singular curves of genus 1 in a pencil, namely 12. If we let $p_1, \ldots, p_4$ specialize to Y_1 and the rest to Y_2, we get several possible configurations, one of which is the following:



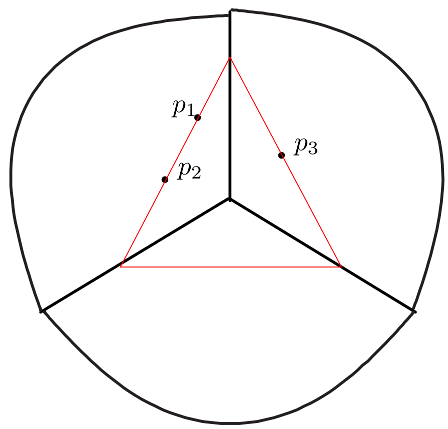
There are exactly two conics on Y_1 through $p_1, \ldots, p_4$ tangent to Σ, and for each there is a unique conic on Y_2 through p_5, p_6, p_7 tangent to Σ *at the same point*. But the correct number is not 2, but 4, since *each such configuration appears with multiplicity two* in the fiber of the degeneration. The verification of this multiplicity 2, and multiplicity 1 otherwise (in general the product of the orders of tangency of the curve with the singular locus) has been a rather painful component in past work.

Other configurations that appear, all with multiplicity 1, are as follows:



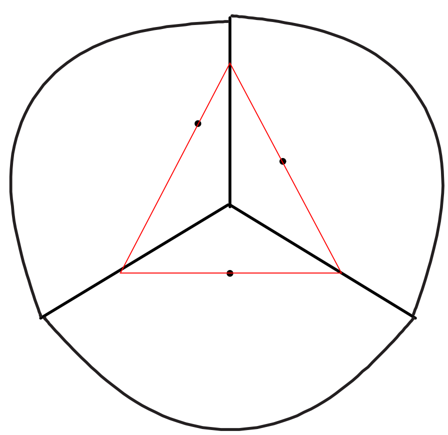
There are three each of the first two, and the last has a brother with E_i switched. The total is indeed 12.

1.3. **Plane section of a cubic.** Consider a cubic surface degenerating to three planes, and consider the unique plane section through three points p_1, p_2, p_3 in general poisition degenerating so that p_1, p_2 are on one plane and p_3 on another. In this case one can easily see the unique configuration of three lines, one on each plane, going through the points.

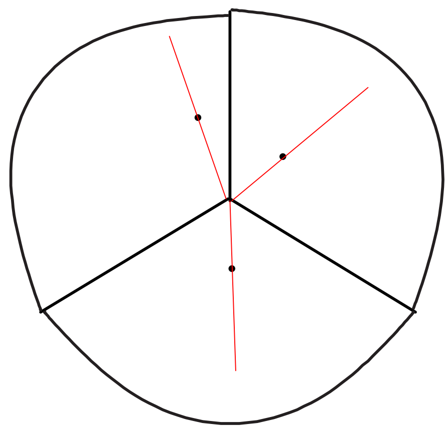


We remark that here as well the formalism requires resolving singularities of the total space, which again can be done using a small modification, but in this simple calculation these do not intervene.

If the points are spread out evenly we again get one configuration

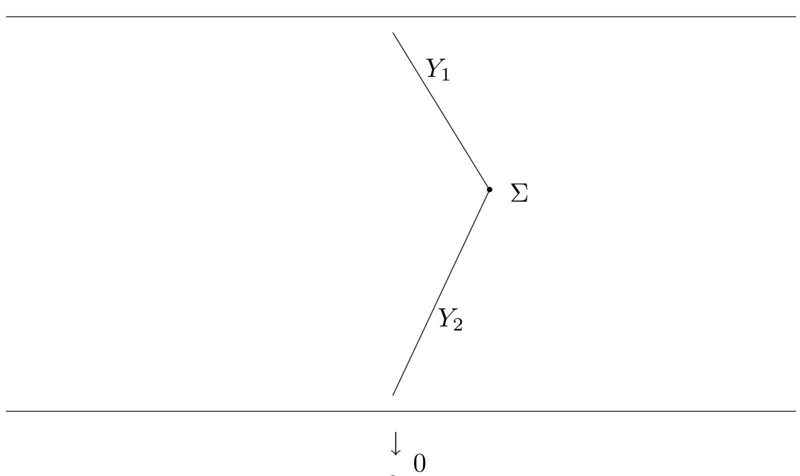


But beware that the naive calculation, in terms of fixed points of the degree 1 map induced by "monodromy" on one of the lines, gives 2. The offending configuration, which does not appear in the limit, is the following:



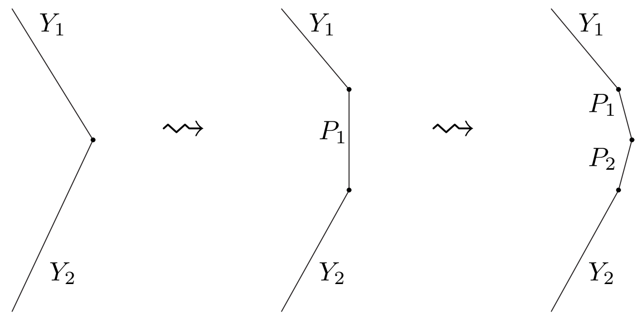
2. The degeneration scheme: old picture

Previous work considered the case of a family of varieties parametrized by a curve, with smooth total space and special fiber consisting of two smooth components meeting transversally along a divisor Σ (thus not encompassing the last example above).



A major issue is that, although a space of stable map fibered over the base exists, the formalism of virtual fundamental classes fails in genus > 0 in case there are components mapping to the singular locus Σ of the fiber.

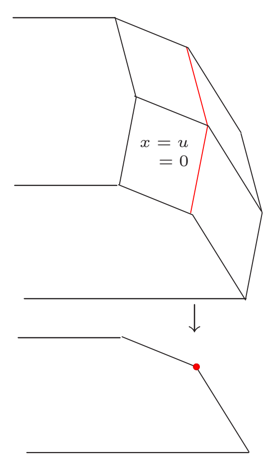
The solution proposed so far involves expanding the fiber by sticking a chain of $\mathbb{P}^1$ bundles over Σ between the two original components of the fiber.



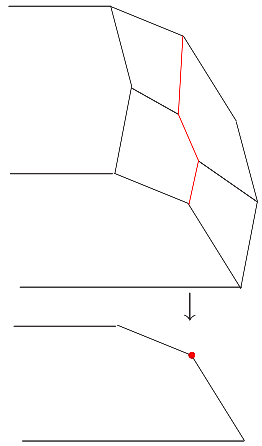
This is introduced in Jun Li's work by replacing the original base curve B by something more involved: suppose B has parameter s such that the origin is defined by $s = 0$. Suppose also that the total space $\mathcal{X}$ of the family has parameters x, y such that $s = xy$, and possibly other "free" parameters we'll ignore.

Consier the smooth surface $B^{(1)}$ with parameters u, v with morphism $B^{(1)} \to B$ defined by $uv = s$. There is a fiberwise action of $\mathbb{C}^*$ on $B^{(1)}$ by $(u, v) \mapsto (t \cdot u, t^{-1} \cdot v)$.

The pullback of $\mathcal{X}$ to $B^{(1)}$ has a threefold A_1 singularity: $xy = uv$.

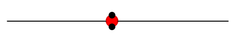


Blowing up the Weil divisor $x = u = 0$ we obtain a small resolution, where the central fiber over $u = v = 0$ has a single $\mathbb{P}^1$-bundle inserted.

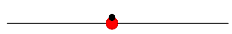


Replacing $B^{(1)}$ by similar $n+1$-folds $B^{(n)}$ with $(\mathbb{C}^*)^n$ action and using a small blowup we obtain a chain of $\mathbb{P}^1$-bundles inserted in the fiber.

Everything is $\mathbb{C}^*$- equivariant, and, in fact, the right base to use is roughly the Artin stack $[B^{(1)}/\mathbb{C}^*]$ (and more generally $[B^{(n)}/(\mathbb{C}^*)^n]$. This stack $[B^{(1)}/\mathbb{C}^*]$ has there "origins" - two regular ones and one a copy of $\mathcal{B}\mathbb{G}_m$, indicated in large red below. The red point is in the closure of each of the back points.



A slightly better choice is described in [5], where it is noted that the two "regular" origins parametrize the same thing and can be merged into one:



One then defines degenerate stable maps to the fibers in this new total space, and similarly relative stable maps to each component Y_i, in which no components of the source curve map entirely into the singular locus. Then one proves a gluing formula for degenerate stable maps in terms of relative stable maps to each of the components. [1] ←1

3. The case of a normal crossings divisor

The problem we set out to solve is:

1. define relative stable maps to (Y, D), where D is a normal crossings divisor on a smooth variety Y,

2. define degenerate stable maps to a variety obtained by gluing such relative (Y_i, D_i) appropriately along the divisors, in such a way that Gromov–Witten invariants are defined and are deformation invariant, and

3. prove a gluing formula comparing the two.

In the following we consider some of the properties of the theory in the case of smooth divisors, and see how they extend to the more general case of normal crossings divisors.

[1](Dan) expand here

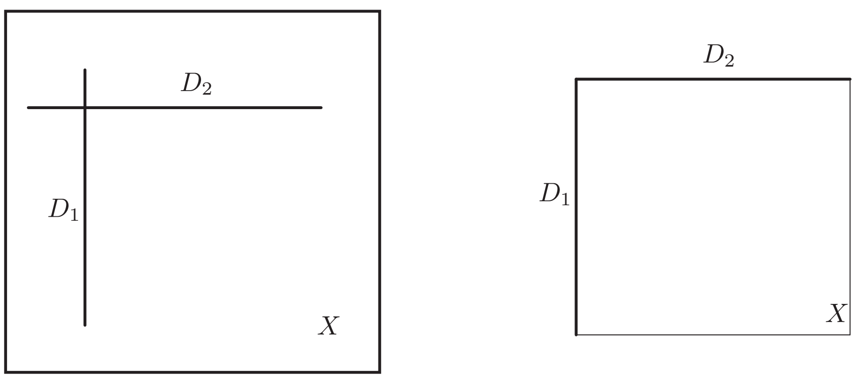
3.1. **Comparing old and new: a normal crossings divisor has already appeared.** Consider stable maps relative to a surface X with $D = D_1 + D_2$ a normal crossings divisor, with D_i smooth.

The first thing we note is that we have indeed already seen such a picture: we have the total space $\mathcal{X}$ of a degeneration over B of a varety with normal crossings degenerate fiber having two smooth components Y_1, Y_2.

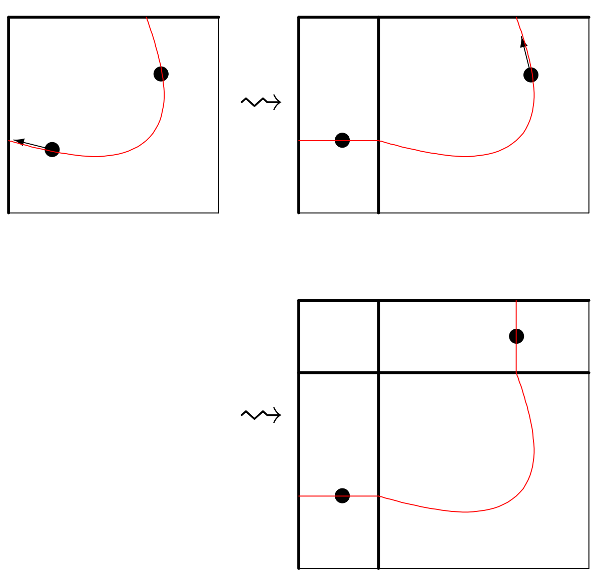
It seems that this is a great misconception - we have no satisfactory way to relate even fiberwise relative stable maps for $(\mathcal{X}, Y_1 + Y_2)$ with the degenerate stable maps. We are informed by Joshua Davis that he is investigating a variant of this problem, but for the time being we must set it aside, and declare that

the new relative picture is different from the old degenerate picture.

3.2. **Old picture: the expansions are normal crossings varieties.** Here is a variety with a divisor, along with a stick-figure (or moment map image) version:



Here is an example of a degeneration of stable configurations of the simplest kind

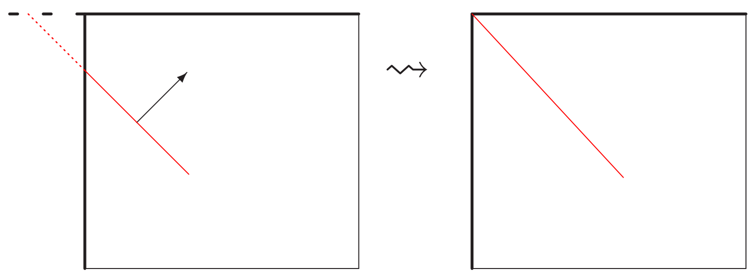


Of course, this last variety is very nice, but it is certainly not a normal crossings variety! A variety with local equation $x_1x_2 = 0, x_3x_4 = 0$ is a product of normal crossings varieties, also known as *semistable*, see [**?**]. We conclude:

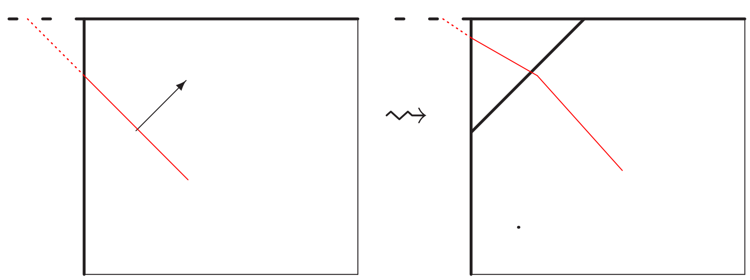
Expanded varieties are semistable, not normal crossings.

3.3. **Old picture: curves meet the strata properly.** As in the old picture, we expect to have some marked points on the curves in the complement of D, which always remain there, and we mark all the points where it meets D, which should again remain in the smooth locus of D.

But what happens when a curve degeneration reaches a point where it passes the intersection $D_1 \cap D_2$?



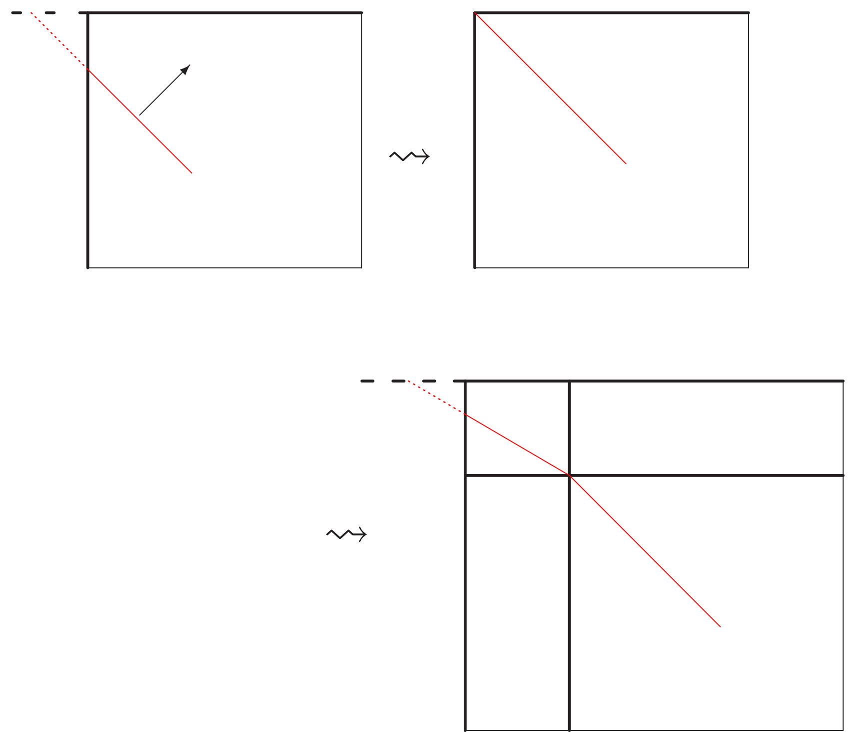
Since we consider the points where the curve meets D_1 and D_2 as a kind of marked points, they may not coincide, therefore something must break. A natural attempt to understand this is to blow up the intersection and get a picture of the following nature:



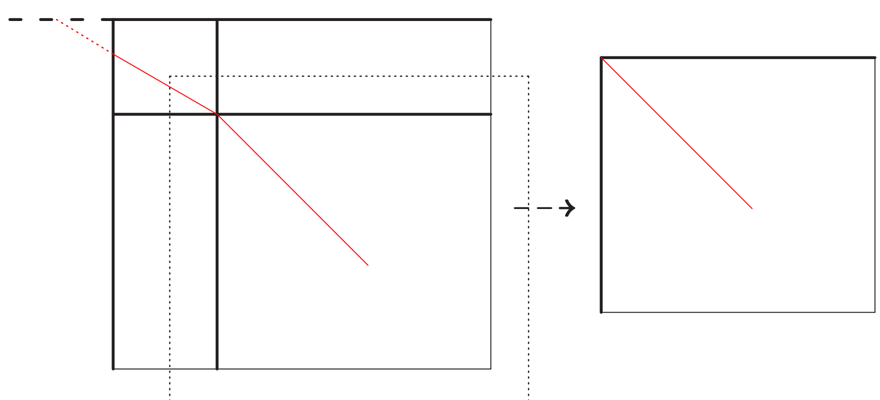
(The triangle stands for the exceptional $\mathbb{P}^2$).

If you think about it, once you start blowing up once, you need to blow up again, and again, with no end in sight to the type of pictures you get. We started looking into this but abandoned it when it looked unwieldy. It appears that Davis's original approach in his thesis followed this path in the case of rational curves, though he too abandoned this when dealing with curves of higher genus.

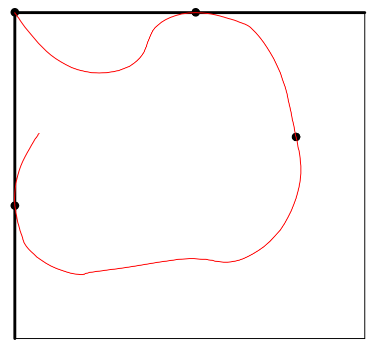
Instead, we keep the degenerations of the target simple, and allow the degenerate curve to pass through a more complicated point in an organized manner:



There is a principle in moduli spaces - whatever you see in a degeneration, you should allow its pieces to appear in the relative picture:



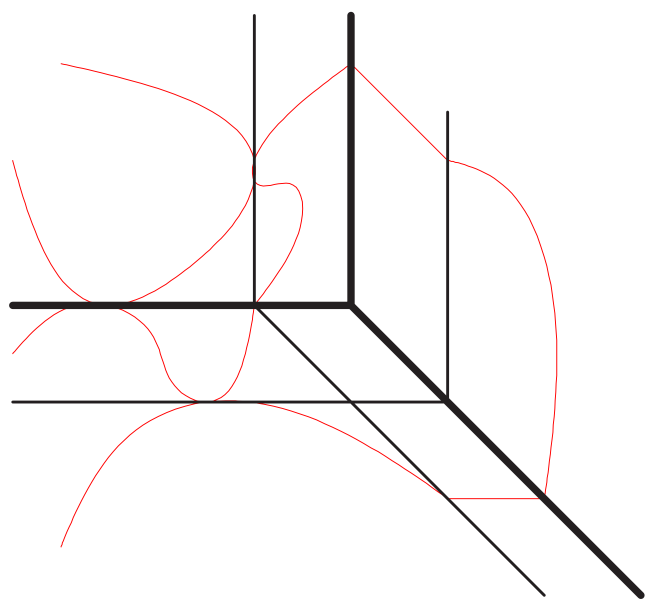
This gives us a more or less general picture of a relative stable map



We conclude:

$\boxed{\textbf{Curves are allowed to meet any stratum, at marked points.}}$

We can now define relative stable maps:[2] ←2
And degenerate stable maps: [3] ←3



3.4. **Old picture: relative stable maps can be glued.** Recall that in the old picture we have a morphism

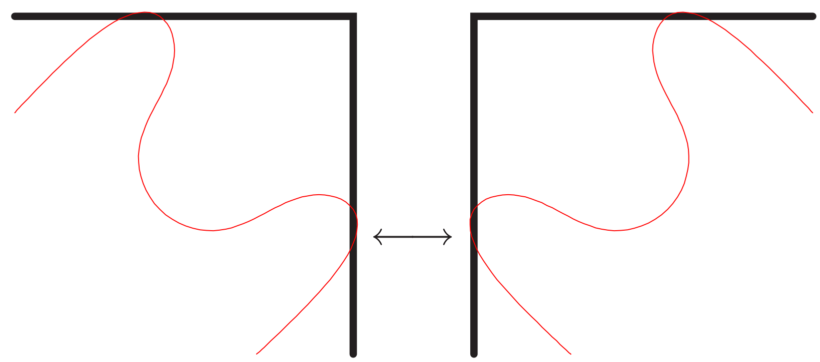
$$\overline{\mathcal{M}}(Y_1, D) \times_{D^m} \overline{\mathcal{M}}(Y_2, D) \longrightarrow \overline{\mathcal{M}}^{\deg}(Y_1 \sqcup^D Y_2),$$

on which the gluing formula for Gromov–Witten invariants relies.

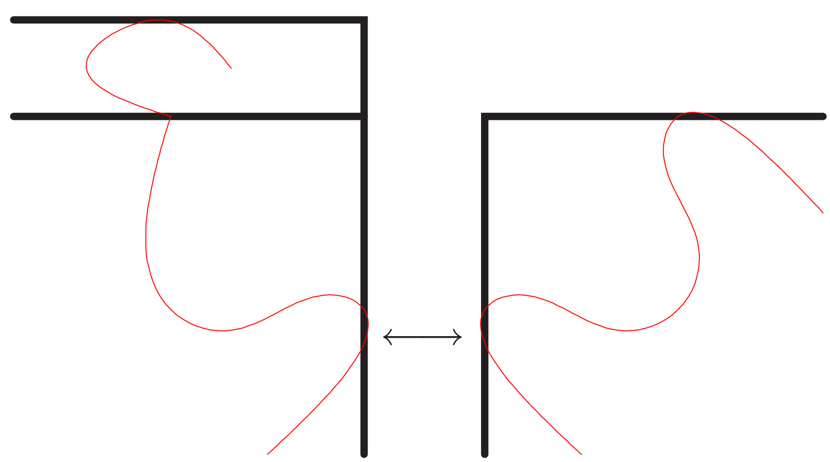
To describe the new situation, it is convenient to use the following slightly simpler "toy model", which is of interest in its own right, and is covered by our work: consider a one parameter degeneration $(\mathcal{X} \to S, \mathcal{E})$ with smooth total space, with

[2](Dan) finish this
[3](Dan) finish this too

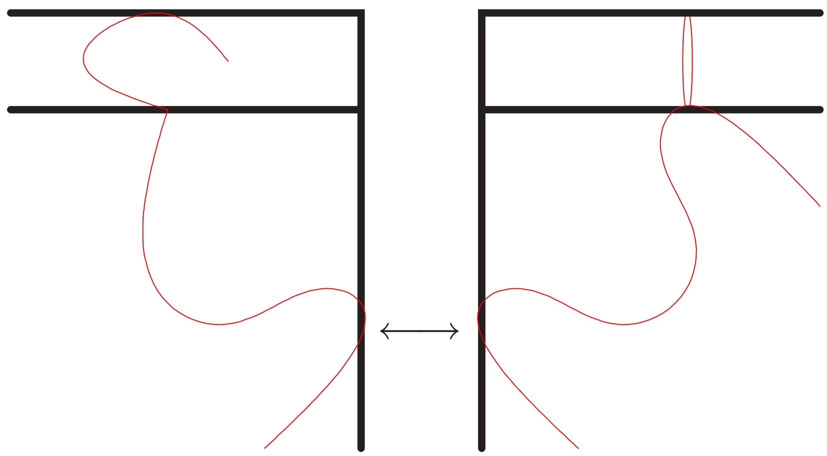
generic fiber (X, E) a smooth variety with a smooth divisor, and special fiber consisting of two smooth components Y_i glued along a smooth divisor D, such that the reduction of E on each Y_i is a smooth divisor E_i meeting D transversally. We then seek a gluing formula for relative Gromov–Witten invariants of $(Y_i, D + E_i)$ giving relative invariants of (X, E). Underlying this we should have a gluing formula for maps:



Let us consider the following degeneration of the above picture:

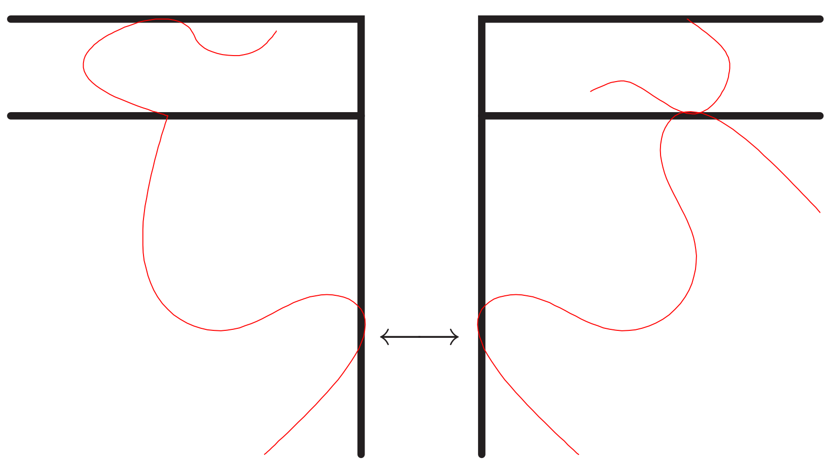


Of course we know how to glue this - the non-degenerate side needs to spawn a matching $\mathbb{P}^1$ bundle, and the curve needs to spawn the appropriate maps of $\mathbb{P}^1$ with matching multiplicities:

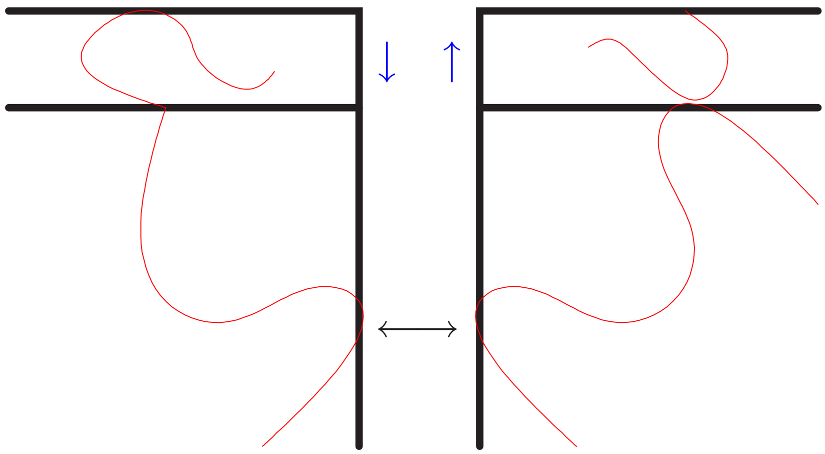


The ellipse on the top right should signify a $\mathbb{P}^1$-multiple cover totally branched at the end-points.

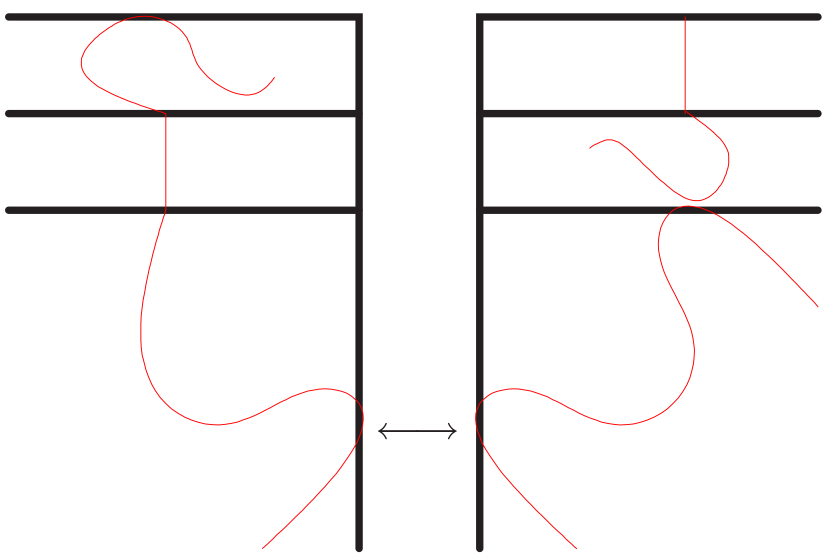
But consider now a case where both sides are degenerate, but the curves on the $\mathbb{P}^1$ bundle need no gluing. Should we glue this way?



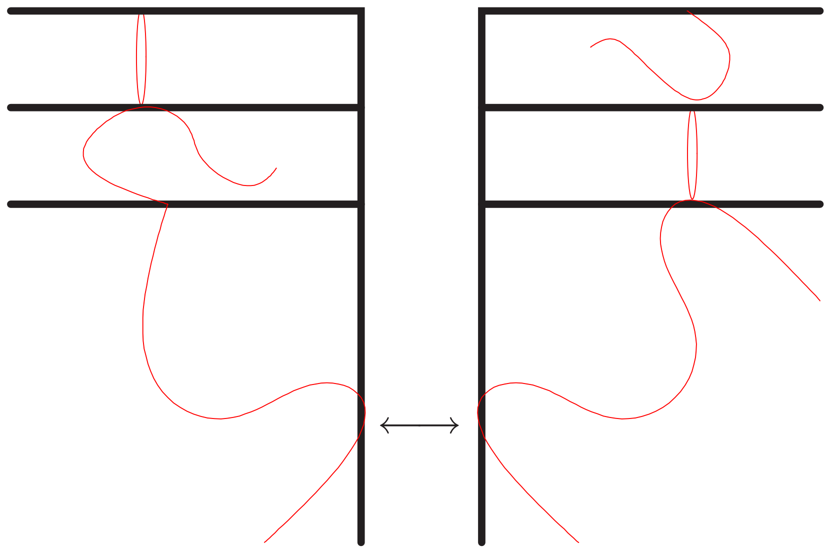
Or, after acting by $\mathbb{G}_m$ on either side, this way?



Or maybe break it up this way?



Or this way?



The answer is, of course, that all are allowed. The point is the following:

First, in the old picture, the gluing data sitting in D^n plus extra data of multiplicities in which the curves meet D, should be thought of as the space of maps of a set of points to D. The gluing data in the new picture is based on maps of a set of points to the universal family of all expanded degenerations of D. The possible changes by $\mathbb{G}_m$ of gluings are accounted for in the automorphisms of such maps. Let us write $\overline{\mathcal{MP}}(D^{\exp})$ to denote the moduli stack of maps of points (with multiplicity data) to expanded versions of D.

Second, in the old picture we glued stable maps. Now can only glue maps to expanded degenerations of Y_i. It is enough to glue maps which are stable as maps to each expanded fiber, when the automorphisms of the fiber are not taken into account. These automorphisms are crucial in the resulting gluing map. Let us denote these $\overline{\mathcal{M}}(Y_i^{\exp})$. And now we have a map[4] ←4

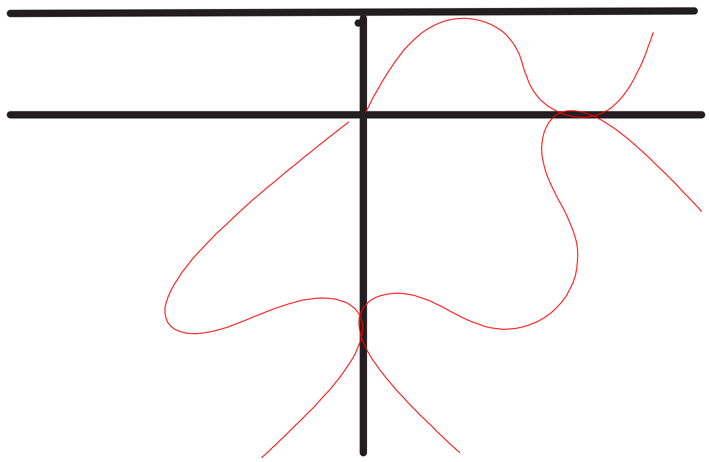
$$\overline{\mathcal{M}}(Y_1^{\exp}) \underset{\overline{\mathcal{MP}}(D^{\exp})}{\times} \overline{\mathcal{M}}(Y_2^{\exp}) \longrightarrow \overline{\mathcal{M}}\left((Y_1 \sqcup^D Y_2)^{\exp}\right).$$

In summary,

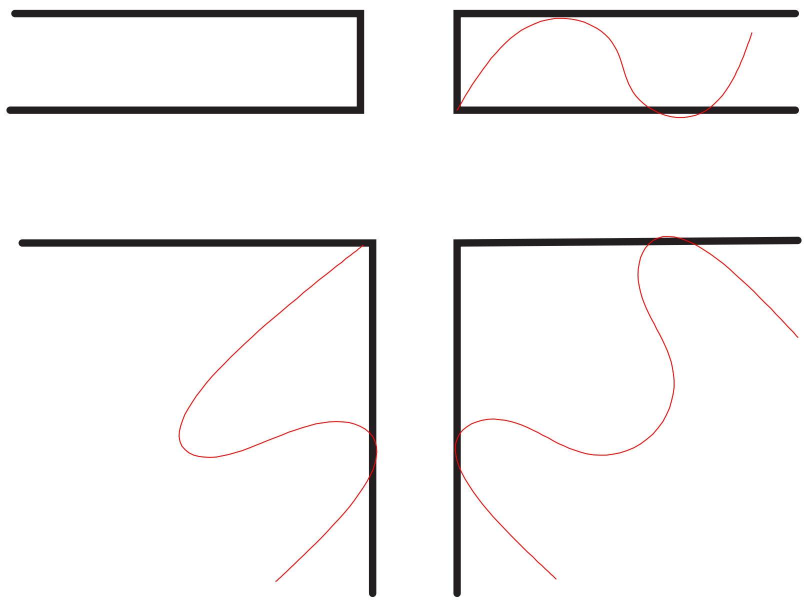
$$\boxed{\textbf{we can only glue “semistable” maps, using “semistable” gluing data.}}$$

3.5. **Old picture: we can glue two curves to get any degenerate picture.** An unwelcome surprise mentioned in the talk is in describing the gluing formula: whereas in the previously studied case the Gromov–Witten numbers of a two-component variety was described in terms of its decomposition to exactly two components, in our case, where the degenerate variety has at least three intersecting components, our formalism requires summing over further decompositions, where each component of the original variety is further “expanded”.

[4](Dan) Notation needs much thought.



which can only decompose to



3.6. **Old picture: the horrors of deformation theory.** A welcome surprise mentioned in the talk is the following: much grief was brought on previous writers in analyzing the so called "predeformability condition", a closed condition on relative and degenerate stable maps which is unpleasant to work out. Techniques of stacks and twisted stable maps of Abramovich–Vistoli and Olsson enable one to transform this into an open condition on a modified space, thus avoiding much of the grief.

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