BIRATIONAL GEOMETRY FOR NUMBER THEORISTS

DAN ABRAMOVICH

Contents

Lecture 0. Introduction: curves

The goal:

GEOMETRY DETERMINES ARITHMETIC.

Glossary:

Geometry: birational geometry of a variety X over a number field k

Arithmetic: rational and integral points, at least after a field extension.

Determines: to be discussed

0.1. **Closed curves.**

Degree of K_C	rational points
$2g-2 \leq 0$	potentially dense
$2g-2 > 0$	finite

In other words,

> **rational points on a curve C of genus g are potentially dense if and only if $g \leq 1$.**

Potentially dense: Zariski-dense after a field extension.

$\boxed{K_C}$: $\mathcal{O}_C(K_C) = \Omega^1_C = \omega_C$

Case $g \leq 1$: explicit construction of points on rational and elliptic curves

Case $g > 1$: Faltings's theorem.

Theorem (Faltings, 1983)**.** *Let C be an algebraic curve of genus > 1 over a number field k. Then $C(k)$ is finite.*

0.2. **Open curves.**

On $\mathbb{A}^1$ one wants to speak of **integral points** $= x \in \mathbb{Z}$.

Only makes sense on the *integral model* $\mathbb{A}^1_{\mathbb{Z}}$.
Otherwise not invariant!

For **open** varieties we use **integral** points on **integral models**.

In general: C an affine curve with completion $\overline{C}$, complement Σ.

Birational invariant: $K_{\overline{C}} + \Sigma$ - "logarithmic differentials".

Ring: $\mathcal{O}_{k,S}$, where S a finite set of primes.

Base scheme: $\operatorname{Spec}\mathcal{O}_{k,S}$
Integral model: $\mathcal{C} = \overline{\mathcal{C}} \smallsetminus \overline{\Sigma}$ over $\operatorname{Spec}\mathcal{O}_{k,S}$

S-integral points on $\mathcal{C}$ = elements of $\mathcal{C}(\mathcal{O}_{k,S})$

= sections of $\mathcal{C} \to \operatorname{Spec}(\mathcal{O}_{k,S})$.

$\overline{\mathcal{C}} \to \operatorname{Spec}\mathcal{O}_{k,S}$ proper: integral points = rational points

degree of $K_{\overline{C}} + \Sigma$	integral points
$2g - 2 + n \leq 0$	potentially dense
$2g - 2 + n > 0$	finite

potentially dense: extend k and S.

$2g - 2 + n \leq 0$: Explicit construction

$2g - 2 + n > 0$: Faltings, plus older

Theorem 0.2.1 (Siegel's Theorem)**.** *If $n \geq 3$, or if $g > 0$ and $n > 0$, then for any integral model $\mathcal{C}$ of C, the set of integral points $\mathcal{C}(\mathcal{O}_{k,S})$ is finite.*

0.3. **Faltings implies Siegel.**

Rational and integral points can be controlled in finite étale covers.

Theorem 0.3.1 (Hermite-Minkowski). *Let k be a number field, S a finite set of finite places, and d a positive integer. Then there are only finitely many extensions k'/k of degree $\leq d$ unramified outside S.*

(So "degree + discriminant" is a good measure of the size of a number field.)

From which one can deduce

Theorem 0.3.2 (Chevalley-Weil). *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a finite étale morphism of schemes over $\mathcal{O}_{k,S}$. Then there is a finite extension k'/k, with S' lying over S, such that $\pi^{-1}\mathcal{Y}(\mathcal{O}_{k,S}) \subset \mathcal{X}(\mathcal{O}_{k',S'})$.*

On the geometric side we have an old topological result

Theorem 0.3.3. *If C is an open curve with $2g-2+n>0$ and $n>0$, defined over k, there is a finite extension k'/k and a finite unramified covering $D\to C_{k'}$, such that $g(D)>1$.*

0.4. **Function field case.** If K is the function field of a complex variety B, then a variety X/K is the generic fiber of a scheme $\mathcal{X}/B$, and a K-rational point $P \in X(K)$ can be thought of as a rational section of $\mathcal{X} \to B$.

If $\dim B = 1$ and $\mathcal{X} \to B$ is proper, then again a K-rational point $P \in X(K)$ is equivalent to a *regular* section $B \to \mathcal{X}$.

The notion of *integral points* is similarly defined using sections, at least when $\dim B = 1$.

Theorem (Manin, Grauert)**.** *Assume $g(C) > 1$. Then the set of nonconstant points $C(K) \smallsetminus C(K)^{const}$ is finite.*

which implies

Theorem (Faltings)**.** *Let C be a curve of genus > 1 over a field k* **finitely generated** *over $\mathbb{Q}$. Then the set of k-rational points $C(k)$ is finite.*

Lecture 1. Kodaira dimension

1.1. **Iitaka dimension.** k: field of characteristic 0

X/k: a smooth, projective, $\dim X = d$.

L: line bundle on X, specifically interesting

$\boxed{K_X}$: $\mathcal{O}_X(K_X) = \wedge^d \Omega^1_X = \omega_X$.

Sections of $\mathcal{O}_X(K_X)$ - birational invariant!

Theorem (Iitaka,Moishezon)**.** *$h^0(X, L^n)$ grows polynomially:*
if there is a section, there is a unique integer $\kappa = \kappa(X, L)$ with $0 \leq \kappa \leq d$ such that

$$\limsup_{n\to\infty} \frac{h^0(X, L^n)}{n^\kappa}$$

exists and is nonzero.

$\kappa(X, L) =:$ *the Iitaka dimension of* (X, L).

$\kappa(X) := \kappa(X, K_X) =:$ *the Kodaira dimension of* X.

if $h^0(X, L^n)$ vanishes for all positive integers n, set

$$\kappa(X, L) \quad = \quad -\infty$$

Proposition. *Assume $\kappa(X,L) \geq 0$. Then for sufficiently high and divisible n,*
the image of the rational map $\phi_{L^n} : X \dashrightarrow \mathbb{P}H^0(X,L^n)$ does not depend on n (up to birational equivalence), and
$\dim \phi_{L^n}(X) = \kappa(X,L)$.

- The birational equivalence class of $\phi_{L^{n_0}}(X)$ is denoted $I(X,L)$.
- The rational map $X \to I(X,L)$ is called the *Iitaka fibration* of (X,L).
- In case L is the canonical bundle, this is called the Iitaka fibration of X, written $X \to I(X)$

Definition. The variety X is said to be *of general type* of $\kappa(X) = \dim X$.

$$\boxed{\kappa(X) \text{ is a birational invariant}}$$

$\kappa(\mathbb{P}^n) = -\infty$

$\kappa(A) = 0$ for an abelian variety A.

$$\kappa(C) = \begin{cases} 1 & \text{if } g > 1, \\ 0 & \text{if } g = 1, \text{ and} \\ < 0 & \text{if } g = 0. \end{cases}$$

Easy additivity:
$\kappa(X_1 \times X_2, L_1 \boxtimes L_2) = \kappa(X_1, L_1) + \kappa(X_2, L_2).$

so

$$\kappa(X_1 \times X_2) = \kappa(X_1) + \kappa(X_2).$$

Easy subadditivity:

$X \to B$ dominant morphism with connected fibers. Then

$$\kappa(X) \leq \dim(B) + \kappa(X_{\eta_B})$$

Definition. We say that X is **uniruled** if there is a variety B of dimension $\dim X - 1$ and a dominant rational map $B \times \mathbb{P}^1 \dashrightarrow X$.

So, if X is uniruled, $\kappa(X) = -\infty$.

Converse is important, follows from existence of "good minimal models":

Conjecture.
Assume X is not uniruled. Then $\kappa(X) \geq 0$.

Surfaces:

κ	description
$-\infty$	$\mathbb{P}^2$ or $\mathbb{P}^1 \times C$
0	a. abelian surfaces b. bielliptic surfaces k. K3 surfaces e. Enriques surfaces
1	many elliptic surfaces

Here is a central conjecture of birational geometry:

Conjecture (Iitaka)**.** *Let $X \to B$ be a surjective morphism of smooth projective varieties. Then*
$$\kappa(X) \geq \kappa(B) + \kappa(X_{\eta_B}).$$

Progress: Arakelov, Fujita, Kawamata, Viehweg and Kollár...

Theorem (Kawamata)**.** *Iitaka's conjecture follows from the Minimal Model Program:*

if X_{η_B} has a good minimal model then
$$\kappa(X) \geq \kappa(B) + \kappa(X_{\eta_B}).$$

Theorem (Viehweg)**.** *Iitaka's conjecture holds in case B is of general type, namely:*

Let $X \to B$ be a surjective morphism of smooth projective varieties, and assume $\kappa(B) = \dim B$. Then
$$\kappa(X) = \dim(B) + \kappa(X_{\eta_B}).$$

1.2. **Uniruled varieties and rationally connected fibrations.**

k: algebraically closed field of characteristic 0.

Definition. A smooth projective variety P is said to be *rationally connected* if through any two points $x, y \in P$ there is a morphism from a rational curve $C \to P$ having x and y in its image.

There are various equivalent ways to characterize rationally connected varieties.

Theorem (Campana, Kollár-Miyaoka-Mori)**.** *Let P be a smooth projective variety. The following are equivalent:*

(1) P is rationally connected.
(2) Any two points are connected by a chain of rational curves.
(3) For any finite set of points $S \subset P$, there is a morphism from a rational curve $C \to P$ having S in its image.
(4) There is a "very free" rational curve on P - if $\dim P > 2$ this means there is a rational curve $C \subset P$ such that the normal bundle $N_{C \subset P}$ is ample.

Key properties:

Theorem. *Let X and X' be smooth projective varieties, with X rationally connected.*

(1) If $X \dashrightarrow X'$ is a dominant rational map (in particular when X and X' are birationally equivalent) then X' is rationally connected.

(2) If X' is deformation-equivalent to X then X' is rationally connected.

(3) If $X' = X_{k'}$ where k'/k is an algebraically closed field extension, then X' is rationally connected if and only if X is.

Theorem (Kollár-Miyaoka-Mori, Campana)**.** *A Fano variety is rationally connected.*

Now we can break any X up:

Theorem (C, K-M-M, Graber-Harris-Starr)**.**
Let X be a smooth projective variety.
There is a modification $X' \to X$,
a variety $Z(X)$,
and a dominant morphism $X' \to Z(X)$ with connected fibers, such that

(1) The general fiber of $X' \to Z(X)$ is rationally connected, and
(2) $Z(X)$ is not uniruled.

Moreover, $X' \to X$ is an isomorphism in a neighborhood of the general fiber of $X' \to Z(X)$.

The rational map $r_X : X \dashrightarrow Z(X)$ is called the **maximally rationally connected fibration of** X (or MRC fibration of X), and $Z(X)$, which is well defined up to birational equivalence, is called the *MRC quotient* of X.

The MRC fibration has the universal property of being "final" for dominant rational maps $X \to B$ with rationally connected fibers.

The set of rational points on a rational curve is Zariski dense. The following is a natural extension:

Conjecture (Campana)**.** *Let P be a rationally connected variety over a number field k. Then rational points on P are potentially dense.*

This conjecture and its sister below for Kodaira dimension 0 was implicit in works of many, including Bogomolov, Colliot-Thélène, Harris, Hassett, Tschinkel.

1.3. **Geometry and arithmetic of the Iitaka fibration.**

Assume $\kappa(X) \geq 0$. Consider the Iitaka fibration $X \dashrightarrow I(X)$.

Proposition. *Let F be a general fiber of $X \to I(X)$. Then $\kappa(F) = 0$*

Conjecture (Campana)**.** *Let F be a variety over a number field k satisfying $\kappa(F) = 0$. Then rational points on F are potentially dense.*

1.4. **Lang's conjecture.** A highly inspiring set of conjectures in diophantine geometry is the following:

Conjecture (Lang's conjecture, weak form)**.** *Let X/k be a smooth projective variety of general type over finitely generated field. Then $X(k)$ is not Zariski-dense in X.*

Conjecture (Lang's geometric conjecture)**.** *Let X be a smooth projective variety of general type. There is a Zariski closed proper subset $S(X) \subset X$, whose irreducible components are not of general type, and such that every subset $T \subset X$ not of general type is contained in $S(X)$.*

Conjecture (Lang's conjecture, srtong form)**.** *Let X/k be a smooth projective variety of general type over a finitely generated field. Then for any finite extension k'/k, the set $(X \smallsetminus S(X))(k')$ is finite.*

For application:

Proposition. *Assume weak Lang. Let X/k: smooth projective variety over a number field. Assume given a dominant rational map $X \dashrightarrow Z$, with Z a variety of general type,* $\dim Z > 0$.

Then $X(k)$ is not Zariski-dense in X.

1.5. **Uniformity of rational points.** Applying in $\mathcal{M}_{g,n}$ we get

Theorem 1.5.1 (Caporaso, Harris, Mazur)**.**
Assume weak Lang.
Let k = number field, $g > 1$.
There is $N(k, g)$ so that for every genus g curve C/k,

$$\#C(k) \leq N(k, g).$$

Theorem 1.5.2 (Pacelli)**.**
$N(k, g) = N(d, g)$ *with* $d = [k : \mathbb{Q}]$.

Theorem 1.5.3 (Caporaso, Harris, Mazur)**.**
Assume strong Lang. Let $g > 1$.
There is $N(g)$ so that for every finitely generated k

$$\#C(k) \leq N(g)$$

for all but finitely many genus g curves C/k.

More: Hassett, A., Voloch, Matsuki.

1.6. **The search for an arithmetic dichotomy.** Potential density of rational points on curves is determined by geometry.

CAPORASO'S TABLE: RATIONAL POINTS ON SURFACES

Kodaira dim	$X(k)$ potentially dense	$X(k)$ never dense
$\kappa = -\infty$	$\mathbb{P}^2$	$\mathbb{P}^1 \times C$ $(g(C) \geq 2)$
$\kappa = 0$	$E \times E$, many others	**none known**
$\kappa = 1$	many examples	$E \times C$ $(g(C) \geq 2)$
$\kappa = 2$	**none known**	many examples

$\kappa = -\infty$: Mordell (and Campana)

$\kappa = 0$: the subject of Campana's Conjecture.

$\kappa = 2$: the subject of Lang's conjecture,

What to do with $\kappa = 1$?

Diophantine geometry is not governed by $\kappa(X)$

Example. Take a Lefschetz pencil of cubic curves in $\mathbb{P}^2$, total space S

Take the $S_1 \to S$ base change $t = s^k$, with $k \geq 3$.

- $\kappa(S_1) = 1$
- $S_1(k)$ dense

Existence of maps to varieties of general type not enough:

Example (Colliot-Thélène, Skorobogatov, Swinnerton-Dyer)**.**

C: a hyperelliptic curve with involution $\phi : C \to C$.
E: elliptic with a 2-torsion point a.
$\widetilde{\phi}$: Free action of $\mathbb{Z}/2\mathbb{Z}$ on $Y = E \times C$ given by

$$(x, y) \mapsto (x + a, \phi(y)).$$

$S_2 := Y/\widetilde{\phi}$.

- $\kappa(S_2) = 1$
- $S_2(k)$ not dense by Chevalley-Weil and Faltings.
- $S_2 \to C'$ dominant $\Rightarrow g(C) \leq 1$.

1.7. **Logarithmic Kodaira dimension and Lang-Vojta conjectures.**

$\overline{X}$: smooth projective variety,
D: a reduced normal crossings divisor.
$X := \overline{X} \setminus D$.

$\kappa(X) := \kappa(\overline{X}, K_{\overline{X}} + D) =:$ logarithmic Kodaira dimension of X.

Easy: $\kappa(X)$ independent of completion $X \subset \overline{X}$.

X is of *logarithmic general type* if $\kappa(X) = \dim X$.

$\mathcal{X}$: a model of X over $\mathcal{O}_{k,S}$.

S-integral points $:= \mathcal{X}(\mathcal{O}_{k,S})$

The Lang-Vojta conjecture is the following:

Conjecture 1.7.1. *If X is of logarithmic general type, then S-integral points on a model are not Zariski dense in X.*

The Lang-Vojta conjecture is a consequence of Vojta's conjecture

Lecture 2. Campana's program

THIS SITE IS UNDER CONSTRUCTION
DANGER! HEAVY EQUIPMENT CROSSING

2.1. **One dimensional Campana constellations.**

Had surface examples with $\kappa = 1$:

$S_1 \to \mathbb{P}^1$ and $S_2 \to \mathbb{P}^1$, with $\kappa = 1$.

Arithmetic behavior very different.

Campana's question: is there an underlying structure on the base $\mathbb{P}^1$ from which we can deduce this difference of behavior?

The key point: S_2 has $2g+2$ double fibers over a divisor $D \subset \mathbb{P}^1$.

$S_2 \to \mathbb{P}^1$ can be lifted to $S_2 \to \mathcal{P}$,

$\mathcal{P} := \sqrt{(\mathbb{P}^1, D)}$ =orbifold structure on $\mathbb{P}^1$ obtained by taking the square root of $\mathbf{1}_D$.

Darmon-Granville: consider the canonical $K_{\mathcal{P}}$ of $\mathcal{P}$.

as a $\mathbb{Q}$-divisor, $K_{\mathcal{P}} = K_{\mathbb{P}^1} + (1 - 1/2)D$.

(with m-fold fiber over a divisor D, take D with coefficient $(1 - 1/m)$.)

Theorem (Darmon-Granville, using Chevalley-Weil and Faltings)**.**
An orbifold curve $\mathcal{P}$ has potentially dense set of **integral** *points if and only if the Kodaira dimension $\kappa(\mathcal{P}) = \kappa(\mathcal{P}, K_{\mathcal{P}}) < 1$.*

Key property: extend $\pi_2 : S_2 \to \mathcal{P}$ over $\mathcal{O}_{k,S}$, then

$$\boxed{p \in S_2(k) \quad \Rightarrow \quad \pi(p) \in \mathcal{P}(\mathcal{O}_{k,S})}.$$

This fully explains our example:

Since integral points on $\mathcal{P} = \sqrt{(\mathbb{P}^1, D)}$ are not Zariski dense, and since rational points on S_2 map to integral points on $\mathcal{P}$, rational points on S_2 are not Zariski dense.

Next key point:

What should we declare the structure to be when we have a fiber that looks like $x^2y^3 = 0$, i.e. has two components of multiplicities 2 and 3?

Classical: $\gcd(2, 3) = 1 \quad \Rightarrow$ no new structure.

Campana (Bogomolov sheaves): $\min(2, 3) = 2$.

Definition (Campana)**.**
X, Y: smooth, $f : X \to Y$ dominant, $\boxed{\dim Y = 1}$.

Define $\Delta_f = \sum \delta_p p$: a $\mathbb{Q}$-divisor on Y:

Say $f^*p = \sum m_i C_i$, where C_i distinct integral.

Then set

$$\delta_p \;=\; 1 - \frac{1}{m_p}, \quad \text{where} \quad m_p \;=\; \min_i m_i.$$

Definition (Campana)**.**
A Campana constellation **curve** $(Y/\Delta$ is:

a curve Y along with a $\mathbb{Q}$ divisor $\Delta = \sum \delta_p p$,

with each $\delta_p = 1 - 1/m_p$ for some integer m_p.

The Campana constellation base of $f : X \to Y$ is (Y/Δ_f), with Δ_f defined above.

The word used by Campana is *orbifold.* The analogy with classical orbifolds is broken in this very definition.

Campana's definition deliberately does not distinguish between the structure coming from a fiber of type $x^2 = 0$ and one of type $x^2y^3 = 0$.

Definition (Campana)**.**

The **Kodaira dimension** of (Y/Δ) is:

$$\kappa\,(\,(Y/\Delta)\,) := \kappa(Y, K_Y + \Delta).$$

We say that (Y/Δ) is of general type if it has Kodaira dimension 1.

We say that it is *special* if it is not of general type.

We need to speak about *integral points* on an *integral model* of the structure.

$\mathcal{Y}$: an integral model of Y, say proper over $\mathcal{O}_{k,S}$,
$\widetilde{\Delta}$: the closure of Δ.

Definition.
$x \in Y(k)$, considered as an S-integral point $\bar{x}$ of $\mathcal{Y}$,

is said to be a **soft S-integral points on** $(\mathcal{Y}/\widetilde{\Delta})$

if for any nonzero prime $\wp \subset \mathcal{O}_{k,S}$ where the reduction $\bar{x}_\wp$ of $\bar{x}$ coincides with the reduction $\bar{z}_\wp$ of some $\bar{z} \in \tilde{\Delta}$, we have

$$\mathrm{mult}_\wp(\bar{x} \cap \bar{p}) \geq m_p.$$

A key property of this definition is:

Proposition.
Say $f : X \to Y$ extends to a good model $\tilde{f} : \mathcal{X} \to \mathcal{Y}$.

Then the image of a rational point on X is a soft S-integral point on $(\mathcal{Y}/\widetilde{\Delta}_f)$.

So rational points on X can be investigated using integral points on (Y/Δ). This makes the following very much relevant:

Conjecture (Campana)**.** *If (Y/Δ) is of general type then the set of soft S-integral point on any model $\mathcal{Y}$ is not Zariski dense.*

This conjecture does not seem to follow readily from Faltings's theorem.

It does follow from the *abc* conjecture,

in particular we have the following theorem.

Theorem (Campana)**.** *If* (Y/Δ) *is a Campana constellation curve of general type defined over the function field* K *of a curve* B *then for any finite set* $S \subset B$*, the set of non-constant soft* S*-integral point on any model* $\mathcal{Y} \to B \smallsetminus S$ *is not Zariski dense.*

2.2. **Higher dimensional Campana constellations.**

Definition. A *rank 1 discrete valuation* on the function field $\mathcal{K} = \mathcal{K}(Y)$ is a surjective group homomorphism $\nu : \mathcal{K}^\times \to \mathbb{Z}$ satisfying

$$\nu(x+y) \geq \min(\nu(x), \nu(y))$$

with equality unless $\nu(x) = \nu(y)$. We define $\nu(0) = \infty$.

The *valuation ring of* ν is defined as

$$R_\nu \;=\; \left\{x \in \mathcal{K} \;\middle|\; \nu(x) \geq 0\right\}.$$

Denote by $Y_\nu = \operatorname{Spec} R_\nu$, and its unique close point s_ν.

A rank 1 discrete valuation ν is *divisorial* if there is a birational model Y' of Y and an irreducible divisor $D' \subset Y'$ such that for all $x \in \mathcal{K}(X) = \mathcal{K}(X')$ we have

$$\nu(x) \;=\; \operatorname{mult}_{D'} x.$$

In this case we say ν has divisorial center D' in Y'.

Definition.

A b-divisor $\mathbf{\Delta}$ on Y is an expression of the form

$$\mathbf{\Delta} \;=\; \sum_{\nu} c_\nu \cdot \nu,$$

a possibly infinite sum over divisorial valuations of $\mathcal{K}(Y)$ with rational coefficients, which satisfies the following finiteness condition:

- for each birational model Y' there are only finitely many ν with divisorial center on Y' having $c_\nu \neq 0$.

Definition. Let $f : X \to Y$ be a dominant morphism.
For each divisorial valuation ν on $\mathcal{K}(Y)$ consider $f' : X'_\nu \to Y_\nu$,

where X' is a desingularization of the (main component of the) pullback $X \times_Y Y_\nu$.

Write $f^* s_\nu = \sum m_i C_i$. Define

$$\delta_\nu \;=\; 1 - \frac{1}{m_\nu} \quad \text{with} \quad m_\nu = \min_i m_i.$$

The Campana b-divisor on Y associated to a dominant map $f : X \to Y$ is defined to be the b-divisor

$$\boldsymbol{\Delta}_f = \sum \delta_\nu \nu.$$

The definition is independent of the choice of desingularization X'_ν.
This makes the b-divisor $\boldsymbol{\Delta}_f$ a proper birational invariant of f. In particular we can apply it to a dominant rational map f.

Definition. A *Campana constellation* $(Y/\mathbf{\Delta})$ consists of a variety Y with a b-divisor $\mathbf{\Delta}$ such that, locally in the étale topology on Y, there is $f : X \to Y$ with $\mathbf{\Delta} = \mathbf{\Delta}_f$.

The trivial constellation on Y is given by the zero b-divisor.

For each birational model Y', define the Y'-divisorial part of $\mathbf{\Delta}$:

$$\Delta_{Y'} = \sum_{\nu \text{ with divisorial support on } Y'} \delta_\nu \nu.$$

Definition. (1) Let $(X/\mathbf{\Delta}_X)$ be a Campana constellation, and $f : X \to Y$ a dominant morphism. The constellation base $(Y, \mathbf{\Delta}_{f,\mathbf{\Delta}_X})$ is defined as follows: for each divisorial valuation ν of Y and each divisorial valuation μ of X with center D dominating the center E of ν, let

$$m_{\mu/\nu} = m_\mu \cdot \mathrm{mult}_D(f^*E).$$

Define

$$m_\nu = \min_{\mu/\nu} m_{\mu/\nu} \quad \text{and} \quad \delta_\nu = 1 - \frac{1}{m_\nu}.$$

Then set as before

$$\mathbf{\Delta}_{f,\mathbf{\Delta}_X} = \sum_\nu \delta_\nu \nu.$$

(2) Let $(X/\mathbf{\Delta}_X)$ and $(Y/\mathbf{\Delta}_Y)$ be Campana constellations and $f : X \to Y$ a dominant morphism. Then f is said to be a *constellation morphism* if $\mathbf{\Delta}_Y \leq \mathbf{\Delta}_{f,\mathbf{\Delta}_X}$, in other words, if for every divisorial valuation ν on Y and any μ/ν we have $m_\nu \leq m_{\mu/\nu}$.

Definition. A rational m-canonical differential ω on Y is said to be regular on $(Y/\mathbf{\Delta})$ if for every divisorial valuation ν on $\mathcal{K}(Y)$, the polar multiplicity satisfies

$$(\omega)_{\infty,\nu} \leq m\delta_\nu.$$

In other words, it is a regular section of $\mathcal{O}_{Y'}(m(K_{Y'} + \Delta_{Y'}))$ on every birational model Y'.

The Kodaira dimension $\kappa(\,(Y/\mathbf{\Delta})\,)$ is defined using regular m-canonical differentials on $(Y/\mathbf{\Delta})$.

This is a birational invariant.

Theorem (Campana)**.** *There is a birational model Y' such that*

$$\kappa(\,(Y/\mathbf{\Delta})\,) = \kappa(Y', K_Y' + \Delta_{Y'}).$$

Definition. A Campana constellation $(Y/\mathbf{\Delta})$ is said to be *of general type* if $\kappa(\,(Y/\mathbf{\Delta})\,) = \dim Y$.

A Campana constellation $(X/\mathbf{\Delta})$ is said to be *special* if there is no dominant morphism $(X/\mathbf{\Delta}) \to (Y/\mathbf{\Delta}')$ where $(Y/\mathbf{\Delta}')$ is of general type.

Definition. (1) A morphism $f : (X/\mathbf{\Delta}_X) \to (Y/\mathbf{\Delta}_Y)$ of Campana constellation is *special*, if its generic fiber is special.

(2) Given a Campana constellation $(X/\mathbf{\Delta}_X)$, a morphism $f : X \to Y$ is said to have *general type base* if $(Y/\mathbf{\Delta}_{f,\mathbf{\Delta}_X})$ is of general type.

(2') In particular, considering X with trivial constellation, a morphism $f : X \to Y$ is said to have *general type base* if $(Y/\mathbf{\Delta}_f)$ is of general type.

Here is the main classification theorem of Campana:

Theorem (Campana)**.** *Let* $(X/\mathbf{\Delta}_X)$ *be a Campana constellation. There exists a dominant rational map* $c : X \dashrightarrow C(X)$*, unique up to birational equivalence, such that*

(1) it has special general fibers, and
(2) it has Campana constellation base of general type.

This map is final for (1) and initial for (2).

This is the Campana core map of $(X/\mathbf{\Delta}_X)$, the constellation $(C(X)/\mathbf{\Delta}_{c,\mathbf{\Delta}_X})$ being the core of $(X/\mathbf{\Delta}_X)$. The key case is when X has the trivial constellation, and then $c : X \dashrightarrow (C(X)/\mathbf{\Delta}_c)$ is the Campana core map of X and $(C(X)/\mathbf{\Delta}_c)$ the core of X.

2.3. **Firmaments supporting constellations and integral points.** Need: combinatorial computational tool; approach to nondominant morphisms; reduction; integral points.

Use toroidal geometry (more general: logarithmic),

Definition. (1) A toroidal embedding

$$U \subset X$$

is

a variety X and

a dense open set U with complement a Weil divisor $D = X \smallsetminus U$,

such that locally near every point, $U \subset X$ admits an isomorphism with $T \subset V$, with T a torus and V a toric variety.

(2) Let $U_X \subset X$ and $U_Y \subset Y$ be toroidal embeddings,

then a dominant morphism $f : X \to Y$ is said to be toroidal if locally near every point of X there is a toric chart for X near x and for Y near $f(x)$ which is a torus-equivariant morphism of toric varieties.

To a toroidal embedding $U \subset X$ we can attach an integral polyhedral cone complex Σ_X,

consisting of strictly convex cones, attached to each other along faces, and in each cone σ a finitely generated, unit free integral saturated monoid $N_\sigma \subset \sigma$ generating σ as a real cone.

The complex Σ_X can be pieced together using the toric charts. for a toric variety V, cones correspont to toric affine opens V_σ, and the lattice N_σ is the monoid of one-parameter subgroups having a limit point in V_σ; it is dual to the lattice of effective toric Cartier divisors M_σ, which is the quotient of the lattice of regular monomials $\tilde{M}_\sigma$ by the unit monomials.

Relation with discrete valuations:

let R be a discrete valuation ring with valuation ν, special point s_R and generic point η_R; let $\phi : \operatorname{Spec} R \to X$ be a morphism such that $\phi(\eta_R) \subset U$ and $\phi(s_R)$ lying in a stratum having chart $V = \operatorname{Spec} k[\tilde{M}_\sigma]$.

One associates to ϕ the point n_ϕ in N_σ given by the rule:

$$n(m) = \nu(\phi^* m) \quad \forall m \in M.$$

In case $R = R_\nu$ is a valuation ring of Y, I'll call this point n_ν.

Suppose given toridal embeddings $U_X \subset X$ and $U_Y \subset Y$ and a morphism $f : X \to Y$ carrying U_X into U_Y (but not necessarily toroidal).

the description above functorially associates a polyhedral morphism $f_\Sigma : \Sigma_X \to \Sigma_Y$ which is integral, that is, $f_\Sigma(N_\sigma) \subset N_\tau$ whenever $f_\Sigma(\sigma) \subset \tau$.

2.3.1. *Toroidalizing a morphism.* While most morphisns are not toroidal, we have the following:

Theorem (Abramovich-Karu)**.** *Let $f : X \to Y$ be a dominant morphism of varieties. Then there exist modifications $X' \to X$ and $Y' \to Y$ and toroidal structures $U_{X'} \subset X'$, $U_{Y'} \subset Y'$ such that the resulting rational map $f' : X' \to Y'$ is a toroidal morphism:*

$$\begin{array}{ccccc} U_{X'} & \hookrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow f' & & \downarrow f \\ U_{Y'} & \hookrightarrow & Y' & \longrightarrow & Y \end{array}$$

Furthermore, f' can be chosen flat.

We now define firmaments:

Definition. A **toroidal firmament** on a toroidal embedding $U \subset X$ with complex Σ is a finite collection $\mathbf{\Gamma} = \{\Gamma^i_\sigma \subset N_\sigma\}$, where

- each $\Gamma^i_\sigma \subset N_\sigma$ is a finitely generate submonoid, not-necessarily saturated.
- each Γ^i_σ generates the corresponding σ as a cone,
- the collection is closed under restrictions to faces $\tau \prec \sigma$, i.e. $\Gamma^i_\sigma \cap \tau = \Gamma^j_\tau$ for some j, and
- it is irredundant, in the sense that $\Gamma^i_\sigma \not\subset \Gamma^j_\sigma$ for different i, j.

A morphism from a toridal firmament $\mathbf{\Gamma}_X$ on a toroidal embedding $U_X \subset X$ to $\mathbf{\Gamma}_Y$ on $U_Y \subset Y$ is a morphism $f : X \to Y$ with $f(U_X) \subset U_Y$ such that for each σ and i, we have $f_\Sigma(\Gamma^i_\sigma) \subset \Gamma^j_\tau$ for some j.

We say that the toroidal firmament $\mathbf{\Gamma}_X$ is *induced* by $f : X \to Y$ from $\mathbf{\Gamma}_Y$ if for each $\sigma \in \Sigma_X$ such that $f_\Sigma(\sigma) \subset \tau$, we have $\Gamma^i_\sigma = f_\Sigma^{-1}\Gamma^i_\tau \cap N_\sigma$.

Given a proper birational equivalence $\phi : X_1 \dashrightarrow X_2$, then two toroidal firmaments $\mathbf{\Gamma}_{X_1}$ and $\mathbf{\Gamma}_{X_2}$ are said to be *equivalent* if there is a toroidal X_3, and a commutative diagram

$$\begin{array}{ccc} & X_3 & \\ {\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_2} \\ X_1 & \overset{\phi}{\dashrightarrow} & X_2, \end{array}$$

where f_i are modifications, such that the two firmaments on X_3 induced by f_i from $\mathbf{\Gamma}_{X_i}$ are identical.

A firmament on an arbitrary X is an equivalence class represented by a modification $X' \to X$ with a toroidal embedding $U' \subset X'$ and a toroidal firmament $\mathbf{\Gamma}$ on $\Sigma_{X'}$.

The trivial firmament is defined by $\Gamma_\sigma = N_\sigma$ for all σ in Σ.

Definition. (1) Let $f : X \to Y$ be a flat toroidal morphism of toroidal embeddings. The *base firmament* $\mathbf{\Gamma}_f$ associated to $X \to Y$ is defined by the images $\Gamma_\sigma^\tau = f_\Sigma(N_\tau)$ for each cone $\tau \in \Sigma_X$ over $\sigma \in \Sigma_Y$.

(2) Let $f : X \to Y$ be a dominant morphism of varieties. The base firmament of f is represented by any $\mathbf{\Gamma}_{f'}$, where $f' : X' \to Y'$ is a flat toroidal birational model of f.

(3) If X is reducible, decomposed as $X = \cup X_i$, but $f : X_i \to Y$ is dominant for all i, we define the base firmament by the (maximal elements of) the union of all the firmaments associated to $X_i \to Y$.

Definition. Let $\mathbf{\Gamma}$ be a firmament on Y. Define the Campana constellation $(Y/\mathbf{\Delta})$ hanging from $\mathbf{\Gamma}$ (or supported by $\mathbf{\Gamma}$) as follows: say $\mathbf{\Gamma}$ is a toroidal formament on some birational model Y'. Let ν be a divisorial valuation. We have associated to it a point $n_\nu \in \sigma$ for the cone σ associated to the stratum in which s_ν lies. Define

$$m_\nu \;=\; \min\{k \,|\, k \cdot n_\nu \;\in\; \Gamma_\sigma^i \;\text{ for some }\; i\,\}.$$

An absolutely important result is:

Proposition. *This is independent of the choice of representative in the equivalence class* $\mathbf{\Gamma}$*, and is a constellation, i.e. always induced, locally in the étale topology, from a morphism* $X \to Y$*.*

Also, the Campana constellation supported by the base firmament of a dominant morphism $X \to Y$ *is the same as the base constellation associated to* $X \to Y$*.*

We need to talk about integral points on integral models. I'll restrict to the toroidal case.

Definition. An S-integral model of a toroidal firmament $\mathbf{\Gamma}$ on Y consists of an integral toroidal model $\mathcal{Y}'$ of Y'.

Definition. Consider a toroidal firmament $\mathbf{\Gamma}$ on Y/k, and a rational point y such that the firmament is trivial in a neighborhood of y. Let $\mathcal{Y}$ be a toroidal S-integral model.

Then y is a *firm integral point of* $\mathcal{Y}$ with respect to $\mathbf{\Gamma}$ if the section $\operatorname{Spec} \mathcal{O}_{k,S} \to \mathcal{Y}$ is a morphism of firmaments, when $\operatorname{Spec} \mathcal{O}_{k,S}$ is endowed with the trivial firmament.

Explicitly, at each prime $\wp \in \operatorname{Spec} \mathcal{O}_{k,S}$ where y reduces to a stratum with cone σ, consider the associated point $n_{y_\wp} \in N_\sigma$. Then y is firmly S-integral if for every $\wp$ we have $n_{y_\wp} \in \Gamma_\sigma^i$ for some i.

Theorem. *Let $f : X \to Y$ be a proper dominant morphism of varieties over k. There exists a toroidal birational model $X' \to Y'$ and an integral model $\mathcal{Y}'$,*
and the image of a rational point on X' is a firm S-integral point on $\mathcal{Y}'$ with respect to $\mathbf{\Gamma}_f$.

Conjecture (Campana)**.** *Let $(Y/\mathbf{\Delta})$ be a smooth projective Campana constellation supported by firmament $\mathbf{\Gamma}$. Then integral points are potentially dense if and only if $(Y/\mathbf{\Delta})$ is special.*

Proposition (Campana)**.** *Assume the conjecture holds true. Let X be a smooth projective variety. Then rational points are potentially dense if and only if X is special.*

Lecture 3. The minimal model program

3.1. **Cone of curves.**

3.2. **Bend and break.**

3.3. **Cone theorem.**

3.4. **The minimal model program.**

Lecture 4. Vojta, Campana and *abc*

4.1. **Heights: local and global.**

4.2. **Vojta's conjecture.**

4.3. **Vojta and *abc*.**

4.4. **Campana and *abc*.**

4.5. **Vojta and Campana.**