Factorization of birational maps on steroids

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This is work with Michael Temkin (Jerusalem)

Statement

Theorem (ℵ-Temkin, after Włodarczyk 03, ℵ-Karu-Matsuki-Wł 02)

Let $\phi : X \to Y$ be a projective birational morphism of regular, noetherian qe schemes. Assume either char $= 0$ or strong resolution holds. Then ϕ factors as

$$X = V_0 \overset{\varphi_1}{\dashrightarrow} V_1 \overset{\varphi_2}{\dashrightarrow} \ldots \overset{\varphi_{\ell-1}}{\dashrightarrow} V_{\ell-1} \overset{\varphi_\ell}{\dashrightarrow} V_\ell = Y\ ,$$

with V_i regular, projective over Y, and φ_i or φ_i^{-1} is the blowing up of a regular Z_i $(\subset X_i$ or $X_{i+1})$.
The factorization is functorial for regular surjective $Y_1 \to Y$, namely for $X_1 = X \times_Y Y_1$ get the factorization with $(V_i)_1 = V_i \times_Y Y_1$ etc.

Question

What's a regular morphism? what's a qe scheme?

Regular morphisms and qe schemes

Definition

$f : Y \to X$ is **regular** if

- flat and
- all geometric fibers of $f : X \to Y$ are regular.

Definition

X is a **qe scheme** if:

- locally noetherian,
- for any Y/X of finite type, $Y_{reg} \subset Y$ is open; and
- For any $x \in X$, $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to X$ is a regular morphism.

I'll give examples if you ask.

Lesson: Commutative rings are as bad as you feared.

Nature

Why?

- Qe schemes are the natural world for resolution of singularities.
- (Temkin) they show up in "nature".

IAS nature $= \mathbb{C}$ analytic. Can we factor?
consider $Y = \mathbb{C}^n$, and for X blow up

- $(1, 0)$ once
- $(2, 0)$ twice

  . . .
- $(n, 0)$ n times at infinitely near points.

There is no way to factor this in finitely many steps.

Problem

The local rings are noetherian, but not the Stein patches.

Affinoids and germs

Consider closed polydisc $D \subset \mathbb{C}^r$ ("Stein compact") and sheaf $\mathcal{O}_D := \mathcal{O}_{\mathbb{C}^r}|_D$ (overconvergent functions).

Theorem (Frisch 67, Matsumura)

$A_D := \Gamma(D, \mathcal{O}_D)$ *is an excellent regular noetherian ring.*

Correspondences

- There is an "algebraization" correspondence: closed complex subspaces of D correspond to closed subschemes of $D^{alg} := \operatorname{Spec} A_D$ etc. No weird boundary phenomena.
- There is an analytification functor from schemes of finite type over $D^{alg} \mapsto$ complex spaces over D. It preserves regularity.
- Use these as patches to build complex geometry of "analytic germs".
- There is a similar picture with affinoids in rigid analytic or Berkovich spaces, affine formal schemes, etc.

Analytic factorization

Theorem (ℵ-Temkin, generalizing the compact complex manifold case (ℵKMW))

Let Y be a compact nonsingular analytic germ. Any $X \to Y$ projective bimeromorphic can be factored into blowings up and down as before.

This requires GAGA.

Theorem (GAGA, Serre's Théorème 3)

Analytification induces a cohomology-preserving equivalence

$$Coh(\mathbb{P}^n_{D^{alg}}) \leftrightarrow Coh(\mathbb{P}^n_D).$$

Lemma (correspondence)

For an affinoid Y, analytification induces bijections

- $\{Blowings\ up\ X/Y^{alg}\} \leftrightarrow \{Blowings\ up\ X^{an}/Y\}$,
- $\{Factorizations\ X \dashrightarrow \dashrightarrow Y^{alg}\} \leftrightarrow \{Factorizations\ X^{an} \dashrightarrow \dashrightarrow Y\}$.

Analytic factorization given correspondence Lemma

- Write $Y = \cup Y_i$ with Y_i affinoids so Y_i^{alg} qe schemes.
- Write $X_i := Bl_I Y_i = X \times_Y Y_i$, so $X_i^{alg} = Bl_{I^{alg}} Y_I^{alg}$ regular.
- Get blowup $\sqcup X_i^{alg} \to \sqcup Y_i^{alg}$.
- Apply algebraic factorization $\sqcup X_i^{alg} \dashrightarrow \dashrightarrow \sqcup Y_i^{alg}$.
- Analytification gives corresponding $\sqcup X_i \dashrightarrow \dashrightarrow \sqcup Y_i$.
- The theorem follows from the claim below:

Claim (Analytic Patching)

Let $Y_* \subset Y_1 \cap Y_2$ be affinoid, and $X_* = Bl_I Y_*$. Then the restrictions of $X_1 \dashrightarrow \dashrightarrow Y_1$ and $X_2 \dashrightarrow \dashrightarrow Y_2$ to $X_* \to Y_*$ coincide.

Claim and Lemma

By the Correspondence Lemma, Analytic Patching follows from

Lemma (Algebraic Patching)

Let $X_*^{alg} = Bl_{I^{alg}} Y_*^{alg}$. Then the restrictions of $X_1^{alg} \dashrightarrow Y_1^{alg}$ and $X_2^{alg} \dashrightarrow Y_2^{alg}$ to $X_*^{alg} \to Y_*^{alg}$ coincide.

Proof.

Let $Z = Y_1^{alg} \sqcup Y_2^{alg}$ and $W = Z \sqcup Y_*^{alg}$. The embeddings $Y_*^{alg} \to Y_i^{alg}$ and the identity $Z \to Z$ give two maps $h_i : W \to Z$. These are regular (Temkin!) and surjective.
Write $X_Z = Bl_{I^{alg}} Z = X_1^{alg} \sqcup X_2^{alg}$. Note that $h_1^* X_Z = h_2^* X_Z$, since they are the blowings up of the same ideal sheaf.
Functoriality for regular surjective morphisms gives the Lemma.

♠

About GAGA

- It is magnificent.
- You can too:

Lemma (Dimension Lemma)

We have $H^i(\mathbb{P}^n_{D^{alg}}, \mathcal{F}) = H^i(\mathbb{P}^n_D, \mathcal{F}^{\mathrm{an}}) = 0$ *for* $i > n$ *and all* $\mathcal{F}$*.*

Lemma (Structure Sheaf Lemma)

We have $H^i(\mathbb{P}^n_{D^{alg}}, \mathcal{O}) = H^i(\mathbb{P}^n_D, \mathcal{O})$ *for all* i*.*

Proof of lemmas

Proof of Dimension Lemma.

Use Čech covers of $\mathbb{P}^n_D = \cup_{i=0}^n D^n[1+\epsilon] \times D$ by closed standard polydisks. ♠

Proof of Structure Sheaf Lemma.

By proper base change for

$$\begin{array}{ccc} \mathbb{P}^n_D & \longrightarrow & \mathbb{P}^n_{\mathbb{CP}^r} \\ \pi \downarrow & & \downarrow \varpi \\ D & \longrightarrow & \mathbb{CP}^r \end{array}$$

get

$$R^i\pi_*\mathcal{O}_{\mathbb{P}^n_D} = R^i\varpi_*\mathcal{O}_{\mathbb{P}^n_{\mathbb{CP}^r}}.$$

♠

More on GAGA in the appendix.

Factorization step 1: birational cobordism

We follow Włodarczyk's original ideas. Much works for schemes.

Claim

There is (functorially) a regular projective $(B \to Y, O_B(1))$, *with* $\mathbb{G}_m$ *action, such that:*

$$B^{ss}_{a_{\min}} /\!\!/ \mathbb{G}_m = X, \quad B^{ss}_{a_{\max}} /\!\!/ \mathbb{G}_m = Y.$$

Proof.

If $X = Bl_I Y$, then take the deformation of the normal cone of $Z(I)$ and resolve singularities to get B. ♠

Factorization step 2: VGIT

Claim

The quotient $B^{ss}_a \to B^{ss}_a /\!\!/ \mathbb{G}_m$ *is affine,* $a_{\min} \le a \le a_{\max}$,

Claim

There is a functorial factorization of ϕ *into a sequence of*

$$B^{ss}_{a_i-} /\!\!/ \mathbb{G}_m \quad \to \quad B^{ss}_{a_i} /\!\!/ \mathbb{G}_m \quad \leftarrow \quad B^{ss}_{a_i+} /\!\!/ \mathbb{G}_m.$$

For this, study relatively affine actions of diagonalizable groups on locally noetherian schemes. The maps result from

$$B^{ss}_{a_i-} \quad \hookrightarrow \quad B^{ss}_{a_i} \quad \hookleftarrow \quad B^{ss}_{a_i+}.$$

Factorization step 3: torific blowups

Claim

There is (functorially) an invariant ideal J_i on $B^{ss}_{a_i}$ so that $B^{tor}_{a_i} := Bl_{J_i} B^{ss}_{a_i}$, with its exceptional divisor, is toroidal and the $\mathbb{G}_m$ action is a toroidal action.

- Over a field k, a pair (B, E) is toroidal if locally it has a regular morphism to a toric variety (X_σ, D) with its toric divisor $D = X_\sigma - T$. In general there is a criterion by Kato.
- The action is toroidal if the map is equivariant for a subgroup of T.
- The proof requires studying logarithmically regular schemes. The ideal is the torific ideal of ℵ-de Jong and ℵKMW.

Factorization step 4: Luna's fundamental lemma

Definition (Special orbits)

An orbit $\mathbb{G}_m \cdot x \subset X$ is *special* if it is closed in the fiber of $X \to X /\!\!/ \mathbb{G}_m$.

Definition (Inert morphisms)

A $\mathbb{G}_m$-equivariant $X \to Y$ is *inert* if (1) it takes special orbits to special orbits and (2) it preserves inertia groups.

Theorem (Luna's fundamental lemma, [Luna 73,Bardsley-Richardson 85, Alper 10,ℵ-Temkin])

A regular and inert $\mathbb{G}_m$-equivariant $X \to Y$ is strongly regular, *namely (1) $X /\!\!/ \mathbb{G}_m \to Y /\!\!/ \mathbb{G}_m$ is regular and (2) $X = Y \times_{Y /\!\!/ \mathbb{G}_m} X /\!\!/ \mathbb{G}_m$.*

Factorization step 5: torification is torific

Claim

The following diagram is toroidal:

$$\begin{array}{ccccc} B^{tor}_{a_i-} & \longrightarrow & B^{tor}_{a_i} & \longleftarrow & B^{tor}_{a_i+} \\ \downarrow & & \downarrow & & \downarrow \\ B^{tor}_{a_i-} /\!\!/ \mathbb{G}_m & \longrightarrow & B^{tor}_{a_i} /\!\!/ \mathbb{G}_m & \longleftarrow & B^{tor}_{a_i+} /\!\!/ \mathbb{G}_m \end{array}$$

This uses Luna and the properties of torific ideals.

Factorization step 6: resolving and patching

An argument using canonical resolution of $\aleph$KMW allows one to replace $B^{tor}_{a_i-} /\!\!/ \mathbb{G}_m$ by regular toroidal schemes so that $B^{tor}_{a_{i-1}+} /\!\!/ \mathbb{G}_m \dashrightarrow B^{tor}_{a_i-} /\!\!/ \mathbb{G}_m$ is a sequence of blowings down and up of nonsingular centers. Finally we have

Claim (Morelli 96, Wł97, $\aleph$-Matsuki-Rashid, $\aleph$KMW, $\aleph$-Temkin)

There is a toroidal factorization of $B^{tor}_{a_i-} /\!\!/ \mathbb{G}_m \dashrightarrow B^{tor}_{a_i+} /\!\!/ \mathbb{G}_m$, functorial with respect to regular surjective morphisms.

This requires generalized cone complexes of $\aleph$-Caporaso-Payne.

GAGA appendix: Serre's proof - cohomology

Lemma (Twisting Sheaf Lemma)

We have $H^i(\mathbb{P}^r_A, \mathcal{O}(n)) = H^i(\mathbb{P}^r_D, \mathcal{O}(n))$ *for all* i, r, n.

Proof.

Induction on r and $0 \to \mathcal{O}_{\mathbb{P}^r_D}(n-1) \to \mathcal{O}_{\mathbb{P}^r_D}(n) \to \mathcal{O}_{\mathbb{P}^{r-1}_D}(n) \to 0$

$$\begin{array}{ccccccc}
H^{i-1}(\mathbb{P}^{r-1}_A, \mathcal{O}(n)) & \longrightarrow & H^i(\mathbb{P}^r_A, \mathcal{O}(n-1)) & \longrightarrow & H^i(\mathbb{P}^r_A, \mathcal{O}(n)) & \longrightarrow & H^i(\mathbb{P}^{r-1}_A, \mathcal{O}(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{i-1}(\mathbb{P}^{r-1}_D, \mathcal{O}(n)) & \longrightarrow & H^i(\mathbb{P}^r_D, \mathcal{O}(n-1)) & \longrightarrow & H^i(\mathbb{P}^r_D, \mathcal{O}(n)) & \longrightarrow & H^i(\mathbb{P}^{r-1}_D, \mathcal{O}(n)).
\end{array}$$

So the result for n is equivalent to the result for $n-1$.
By the Structure Sheaf Lemma it holds for $n = 0$ so it holds for all n. ♠

GAGA appendix: Serre's proof - cohomology

Proposition (Serre's Théorème 1)

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r_A$. The homomorphism $h^* : H^i(\mathbb{P}^r_A, \mathcal{F}) \to H^i(\mathbb{P}^r_D, h^*\mathcal{F})$ is an isomorphism for all i.

Proof.

Descending induction on i for all coherent $\mathbb{P}^r_A$ modules, the case $i > r$ given by the Dimension Lemma.

Choose a resolution $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$ with $\mathcal{E}$ a sum of twisting sheaves. Flatness of h implies $0 \to h^*\mathcal{G} \to h^*\mathcal{E} \to h^*\mathcal{F} \to 0$ exact.

$$
\begin{array}{ccccccccc}
H^i(\mathbb{P}^r_A, \mathcal{G}) & \longrightarrow & H^i(\mathbb{P}^r_A, \mathcal{E}) & \longrightarrow & H^i(\mathbb{P}^r_A, \mathcal{F}) & \longrightarrow & H^{i+1}(\mathbb{P}^r_A, \mathcal{G}) & \longrightarrow & H^{i+1}(\mathbb{P}^r_A, \mathcal{E}) \\
\downarrow & & \downarrow{=} & & \downarrow & & \downarrow{=} & & \downarrow{=} \\
H^i(\mathbb{P}^r_D, \mathcal{G}) & \longrightarrow & H^i(\mathbb{P}^r_D, \mathcal{E}) & \longrightarrow & H^i(\mathbb{P}^r_D, \mathcal{F}) & \longrightarrow & H^{i+1}(\mathbb{P}^r_D, \mathcal{G}) & \longrightarrow & H^{i+1}(\mathbb{P}^r_D, \mathcal{E})
\end{array}
$$

so the arrow $H^i(\mathbb{P}^r_A, \mathcal{F}) \to H^i(\mathbb{P}^r_D, h^*\mathcal{F})$ surjective, and so also for $\mathcal{G}$, and finish by the 5 lemma. ♠

GAGA appendix: Serre's proof - Homomorphisms

Proposition (Serre's Théorème 2)

For any coherent $\mathbb{P}^r_A$-modules $\mathcal{F}, \mathcal{G}$ the natural homomorphism

$$\underline{\mathrm{Hom}}_{\mathbb{P}^r_A}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathbb{P}^r_D}(h^*\mathcal{F}, h^*\mathcal{G})$$

is an isomorphism. In particular the functor h^* is fully faithful.

Proof.

By Serre's Théorème 1, suffices to show that $h^*\mathcal{H}om_{\mathbb{P}^r_A}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathbb{P}^r_D}(h^*\mathcal{F}, h^*\mathcal{G})$ is an isomorphism.

$$\begin{aligned}\left(h^*\mathcal{H}om_{\mathbb{P}^r_A}(\mathcal{F}, \mathcal{G})\right)_x &= Hom_{\mathcal{O}_{x'}}(\mathcal{F}_{x'}, \mathcal{G}_{x'}) \otimes_{\mathcal{O}_{x'}} \mathcal{O}_x \\ &= Hom_{\mathcal{O}_x}(\mathcal{F}_{x'} \otimes_{\mathcal{O}_{x'}} \mathcal{O}_x, \mathcal{G}_{x'} \otimes_{\mathcal{O}_{x'}} \mathcal{O}_x) \\ &= \mathcal{H}om_{\mathbb{P}^r_X}(h^*\mathcal{F}, h^*\mathcal{G})_x.\end{aligned}$$

by flatness. ♠

GAGA appendix: Serre's proof - generation of twisted sheaves

Proposition (Cartan's Théorème A)

For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_D$ there is n_0 so that $\mathcal{F}(n)$ is globally generated whenever $n > n_0$.

Proof.

Induction on r.

Suffices to generate stalk at x. Choose $H \ni x$, and get an exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0$. This gives $\mathcal{F}(-1) \stackrel{\varphi_1}{\to} \mathcal{F} \stackrel{\varphi_0}{\to} \mathcal{F}_H \to 0$ which breaks into

$$0 \to \mathcal{G} \to \mathcal{F}(-1) \to \mathcal{P} \to 0 \qquad \text{and} \qquad 0 \to \mathcal{P} \to \mathcal{F} \to \mathcal{F}_H \to 0,$$

where $\mathcal{G}$ and $\mathcal{F}_H$ are coherent sheaves on H,

Serre's proof - generation of twisted sheaves (continued)

Proof of Cartan's Théorème A, continued.

$$0 \to \mathcal{G} \to \mathcal{F}(-1) \to \mathcal{P} \to 0 \qquad \text{and} \qquad 0 \to \mathcal{P} \to \mathcal{F} \to \mathcal{F}_H \to 0,$$

so right terms in

$$H^1(\mathbb{P}^r_D, \mathcal{F}(n-1)) \to H^1(\mathbb{P}^r_D, \mathcal{P}(n)) \to H^2(H, \mathcal{G}(n))$$

and

$$H^1(\mathbb{P}^r_D, \mathcal{P}(n)) \to H^1(\mathbb{P}^r_D, \mathcal{F}(n)) \to H^1(H, \mathcal{F}_H(n))$$

vanish for large n. So $h^1(\mathbb{P}^r_D, \mathcal{F}(n))$ is descending, and when it stabilizes $H^1(\mathbb{P}^r_D, \mathcal{P}(n)) \to H^1(\mathbb{P}^r_D, \mathcal{F}(n))$ is bijective so $H^0(\mathbb{P}^r_X, \mathcal{F}(n)) \to H^0(H, \mathcal{F}_H(n))$ is surjective.
Sections in $H^0(H, \mathcal{F}_H(n))$ generate $\mathcal{F}_H(n)$ by dimension induction, and by Nakayama the result at $x \in H$ follows.

♠

GAGA appendix: Serre's proof - the equivalence

Peoof of Serre's Théorème 3.

Choose a resolution $\mathcal{O}(-n_1)^{k_1} \xrightarrow{\psi} \mathcal{O}(-n_0)^{k_0} \to \mathcal{F} \to 0$.
By Serre's Théorème 2 the homomorphism ψ is the analytification of an algebraic sheaf homomorphism ψ', so the cokernel $\mathcal{F}$ of ψ is also the analytification of the cokernel of ψ'. ♠