RATIONAL NORMAL CURVES AND HILBERT QUOTIENTS

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Definition 0.1. Let $\mathcal{H}_d$ denote the closure in the Hilbert scheme $\mathtt{Hilb}^{3d+1}(\mathbb{P}^d)$ of the locus parametrizing rational normal curves of degree d. The curves it parametrizes we shall refer to as *generalized* rational normal curves. Denote by $\mathtt{RNC}(n,d)$ the incidence locus inside $\mathcal{H}_d \times (\mathbb{P}^d)^n$.

Remark 0.2. There is a unique rational normal curve of degree 1 in $\mathbb{P}^1$, namely $\mathbb{P}^1$ itself, so $\mathcal{H}_1 = \{\mathtt{pt}\}$, and $\mathcal{H}_2 = \mathbb{P}^5$, thus

$$\mathtt{RNC}(n,1) = (\mathbb{P}^1)^n$$

$$\mathtt{RNC}(n,2) = \mathtt{Con}(n)$$

where the space $\mathtt{Con}(n)$ of n-pointed conics was defined in [GS09].

The group $\mathtt{SL}_{d+1}$ naturally acts on $\mathtt{RNC}(n,d)$, and because a nonsingular rational normal curve is isomorphic to $\mathbb{P}^1$ and the action of $\mathtt{SL}_{d+1}$ stabilizing such a curve is the same as the action of $\mathtt{SL}_2$ on $\mathbb{P}^1$, quotients of $\mathtt{RNC}(n,d)$ by this action are compactifications of the moduli space

$$\mathcal{M}_{0,n} = ((\mathbb{P}^1)^n \setminus \{\mathtt{diagonals}\})/\mathtt{SL}_2$$

of n distinct points on the line. We expect that using GIT to form these quotients yields coarser compactifications than the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{0,n}$, but that Hilbert and Chow quotients yield precisely $\overline{\mathcal{M}}_{0,n}$.

We shall briefly recall the definition of Hilbert and Chow quotients. Let X be a projective variety over $\mathbb{C}$ and G an algebraic group acting on it. For some sufficiently small Zariski open subset $U \subset X$ (which can be taken to be G-invariant) the orbit closures $\overline{G.x}$ all have the same homology class so the correspondence $x \mapsto \overline{G.x}$ defines an embedding of U/X into the appropriate Chow variety. The *Chow quotient* $X/\!\!/G$ is by definition the closure of U/X in this embedding. Similarly, we can find U small enough so that the orbit closures $\overline{G.x}$ form a flat family over U/X and hence induce an embedding $U/X \hookrightarrow \mathtt{Hilb}(X)$. The *Hilbert quotient* $X/\!\!/\!\!/G$ is by definition the closure in this embedding. Kapranov showed in [Kap93] that there is a canonical "Hilbert-Chow" birational morphism $X/\!\!/\!\!/G \to X/\!\!/G$. Moreover, if G is reductive and L is an ample linearized line bundle then there is a canonical birational morphism $X/\!\!/G \to X/\!\!/_L G$ from the Chow quotient to the GIT quotient.

Conjecture 0.3. *There is an isomorphism* $\mathtt{RNC}(n,d)/\!\!/\!\!/\mathtt{SL}_{d+1} \cong \overline{\mathcal{M}}_{0,n}$. *In particular, for any linearization* L *on* $\mathtt{RNC}(n,d)$ *such that the semistable locus is nonempty, there is a morphism* $\overline{\mathcal{M}}_{0,n} \to \mathtt{RNC}(n,d)/\!\!/_L\mathtt{SL}_{d+1}$.

Remark 0.4. This conjecture, if true, would imply the existence of morphisms to all GIT quotients even before we have worked out the stability conditions for those quotients. For $\gamma = 0$ (where γ indicates the weight of the linearization on $\mathcal{H}_d$) stability is extremely simple and so it is easy to describe explicitly what the morphism from $\overline{\mathcal{M}}_{0,n}$ to the GIT quotient does, but proving that such a morphism exists even in this case is a necessary and non-trivial step for working on the conformal blocks project suggested by Angela. Thus proving the conjecture would allow work to proceed on Angela's project as well as being an interesting result in its own regard.

One direction of the conjecture follows readily from earlier work of Kapranov. It was shown in [Kap93] that $\mathtt{RNC}(n,1)/\!/\!/\mathtt{SL}_2 \cong \overline{\mathcal{M}}_{0,n}$ and that this is the inverse limit of all GIT quotients $\mathtt{RNC}(n,1)/\!/_L\mathtt{SL}_2$. For linearizations $L' = (\gamma, L)$ with $\gamma >> 0$ there is an isomorphism $\mathtt{RNC}(n,d)/\!/_{L'}\mathtt{SL}_{d+1} \cong \mathtt{RNC}(n,1)/\!/_L\mathtt{SL}_2$ so the fact that $\mathtt{RNC}(n,d)/\!/\!/\mathtt{SL}_{d+1}$ maps to all the GIT quotients $\mathtt{RNC}(n,d)/\!/_{L'}\mathtt{SL}_{d+1}$ means in particular that it maps to all the GIT quotients $\mathtt{RNC}(n,1)/\!/_L\mathtt{SL}_2$ and hence that there is a morphism $\mathtt{RNC}(n,d)/\!/\!/\mathtt{SL}_{d+1} \to \overline{\mathcal{M}}_{0,n}$ to the inverse limit. This is certainly birational.

We next turn to constructing a birational morphism $\overline{\mathcal{M}}_{0,n} \to \mathtt{RNC}(n,d)/\!/\!/\mathtt{SL}_{d+1}$. For the open $\mathtt{SL}_{d+1}$-invariant subset $U \subset \mathtt{RNC}(n,d)$ in the definition of the Hilbert quotient we can take

$$U := \{(C, p_1, \ldots, p_n) \in \mathtt{RNC}(n,d) \mid C \cong \mathbb{P}^1 \text{ and } p_i \neq p_j \text{ for } i \neq j\}$$

A characterizing property of (nonsingular) rational normal curves of degree d is that any collection of k distinct points, $k \leq d+1$, span a $\mathbb{P}^{k-1}$, so for a pointed curve $(C, p_1, \ldots, p_n) \in U$ the collection of points $(p_1, \ldots, p_n) \in (\mathbb{P}^d)^n$ is *generic* in the sense that $(p_1, \ldots, p_n)$ is in the open $\mathtt{SL}_{d+1}$-invariant subset Kapranov uses to define the Hilbert quotient $(\mathbb{P}^d)^n/\!/\!/\mathtt{SL}_{d+1}$. The orbit closures $\overline{\mathtt{SL}_{d+1}\cdot u}$ for $u \in U$ form a flat family over $U/\mathtt{SL}_{d+1}$ which induces an embedding

$$\mathcal{M}_{0,n} = U/\mathtt{SL}_{d+1} \hookrightarrow \mathtt{Hilb}(\mathtt{RNC}(n,d)).$$

By definition the Hilbert quotient $\mathtt{RNC}(n,d)/\!/\!/\mathtt{SL}_{d+1}$ is the closure of $U/\mathtt{SL}_{d+1}$ inside this Hilbert scheme, so our goal is to extend this embedding to a morphism $\overline{\mathcal{M}}_{0,n} \to \mathtt{Hilb}(\mathtt{RNC}(n,d))$ whose image is contained in this closure.

In other words, we need to produce a flat family of n-pointed degree d generalized rational normal curve orbit closures over $\overline{\mathcal{M}}_{0,n}$ in such a way that the fiber over each nonsingular curve with distinct points is the closure of the $\mathtt{SL}_{d+1}$-orbit of the image of this curve in its d^{th} Veronese embedding.

To see how to do this, let us first consider the case $d = 1$. Here $\mathtt{RNC}(n,1) = (\mathbb{P}^1)^n$ and $U = \{(p_1, \ldots, p_n) \in (\mathbb{P}^1)^n \mid p_i \neq p_j\}$. Moreover, Kapranov's results from [Kap93] apply so that $\mathtt{RNC}(n,1)/\!/\!/\mathtt{SL}_2 \cong \overline{\mathcal{M}}_{0,n}$. Over a configuration $p = (p_1, \ldots, p_n) \in U/\mathtt{SL}_2 = \mathcal{M}_{0,n}$ we have the orbit closure $\overline{\mathtt{SL}_2 \cdot p}$. To see what closed subset of $(\mathbb{P}^1)^n$ should lie over a singular curve $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ consider the simplest case of a curve with one node and two components $C = C' \cup C''$. By choosing the linearization L appropriately one can force the GIT morphism $\overline{\mathcal{M}}_{0,n} \cong \mathtt{RNC}(n,1)/\!/\!/\mathtt{SL}_2 \to \mathtt{RNC}(n,1)/\!/_L\mathtt{SL}_2$ to contract either C' or C''. Thus the Hilbert quotient needs to somehow remember that both contractions are possible. A reasonable guess, therefore, is that the fiber over $C = C' \cup C''$ is the union of the orbit closure corresponding to the configuration of points on $\mathbb{P}^1$ obtained by contracting C' with the orbit closure corresponding to the configuration obtained by contracting C''.

Check that this actually produces a flat family, i.e. that all the fibers have the same Hilbert polynomial! Danny points out that for $n = 4$ this should be very easy to check because these orbit closures are all hypersurfaces.

Assuming the above check works, i.e. that for $d = 1$ we actually get a flat family and hence morphism $\overline{\mathcal{M}}_{0,n} \to \mathtt{RNC}(n,1)/\!\!/\!\!/\mathtt{SL}_2$, we can describe how to generalize this procedure to arbitrary d. For any curve $(C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{0,n}$ we let $\phi_I : C \to \mathbb{P}^d$ for $I \subset \{1, \dots, n\}$ with $|I| = d$ denote the morphism associated to the complete linear series $|\mathcal{O}_C(\sum_{i\in I} p_i)|$. The image $\phi_I(C)$ is an n-pointed generalized rational normal curve, so we define the fiber over $(C, p_1, \dots, p_n)$ to be

$$\bigcup_{I \in \binom{[n]}{d}} \overline{\mathtt{SL}_{d+1} \cdot \phi_I(C)}$$

Note that if $(C, p_1, \dots, p_n) \in U$ so that $C \cong \mathbb{P}^1$ and $p_i \neq p_j$ then all the maps ϕ_I are simply the d^{th} Veronese map on $\mathbb{P}^1$ so this union of orbit closures is just the single orbit closure that it is supposed to be for this flat family to extend the family on $\mathcal{M}_{0,n}$ corresponding to the embedding $\mathcal{M}_{0,n} = U/\mathtt{SL}_{d+1} \hookrightarrow \mathtt{Hilb}(\mathtt{RNC}(n,d))$.

To prove the conjecture it just remains to verify that all fibers in the family defined this way have the same Hilbert polynomial, or if they do not then to find a way to define a family for which they do.

Remark 0.5. For a fixed $I \subset \{1, \dots, n\}$ the collection of n-pointed curves obtained by applying ϕ_I to each curve of $\overline{\mathcal{M}}_{0,n}$ is precisely the Kontsevich-Boggi moduli space $\overline{\mathcal{M}}_{0,I}$ defined in [Bog99]. Intuitively, the contractions involved in mapping to each of these Boggi spaces are the contractions that arise in the maps to the various GIT quotients, so by taking the union over all $\binom{n}{d}$ Boggi spaces we in essence allow the Hilbert quotient to "remember" all the GIT quotients.

Remark 0.6. On the generic locus $U \subset \mathtt{RNC}(n,d)$ the $\mathtt{SL}_{d+1}$ action is free so we get an embedding $\mathtt{SL}_{d+1} \hookrightarrow \mathtt{RNC}(n,d)$ defined by $g \mapsto g \cdot u$ for any fixed $u \in U$. Danny asks whether the associated compactification (at least in the case $d = 1$ so $\mathtt{RNC}(n,1) = (\mathbb{P}^1)^n$) is the so-called *wonderful* compactification.

References

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