RAYNAUD'S GROUP-SCHEME AND REDUCTION OF COVERINGS

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WITH AN APPENDIX BY JONATHAN LUBIN

In grateful memory of Serge Lang

1. Introduction

1.1. **Reduction of coverings of degree divisible by p.** Let R be a discrete valuation ring of mixed characteristics, with spectrum $S = \operatorname{Spec} R$. Denote the generic point η with fraction field K, and the special point s with residue field k of characteristic $p > 0$. Consider a generically smooth, stable pointed curve $Y \to S$ with an action of a finite group G of order divisible by p. Denote $X = Y/G$. We assume that G acts freely on the complement of the marked points in Y_η - it then follows that G respects the branches of all nodes of Y_s.

In situations where the order of G is prime to the residue characteristic, the reduced covering $Y_s \to X_s$ is an admissible G-covering, and a nice complete moduli space of admissible G-coverings exists. An extensive literature exists describing that situation, see e.g. [H-M, Mo, Ek, W, ℵ-C-V]. However, in our case where the residue characteristic divides the order of G, interesting phenomena occur (see e.g. [ℵ-Oo]). The situation was studied by a number of people; we will concern ourselves with results of Raynaud [Ra] and, in a less direct way, Henrio [He] and more recently Maugeais [Ma]. Related work of Saidi [Sa1, Sa2, Sa3], Wewers and Bouw [W1, W2, W3, Bo, B-W1, B-W2], Romagny [Ro] and others provides additional inspiration. In [ℵ-O-V2, Section 5] the curve Y is replaced by something which could be much more singular, and therefore the results are somewhat orthogoal to the situation here.

Thus, in our case where $p \,\big|\, |G|$, the covering $Y \to X$ is no longer generically étale on each fiber. It is natural to consider some sort of group-scheme degeneration $\mathcal{G} \to X$ of G, in such a way that Y might

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be considered something like an admissible $\mathcal{G}$-covering. In our main theorem we show this is the case under appropriate assumptions:

Main Theorem (=Theorem 3.2.2)**.** *Assume $p^2 \nmid |G|$ and the p-Sylow subgroup of G is normal.*

There exist

(1) a twisted curve $\mathcal{X} \to X$,
(2) a finite flat group scheme $\mathcal{G} \to \mathcal{X}$,
(3) a homomorphism $G_{\mathcal{X}} \to \mathcal{G}$ which is an isomorphism on $\mathcal{X}_K$,
(4) a lifting $Y \to \mathcal{X}$ of $Y \to X$, and
(5) an action of $\mathcal{G}$ on Y through which the action of G factors,

such that $Y \to \mathcal{X}$ is a principal $\mathcal{G}$-bundle.

The formation of $\mathcal{G}$ commutes with any flat and quasi-finite base change $R \subset R'$.

It is important to note that, unlike the characteristic 0 case, X is not a stable pointed curve in general.

1.2. **Background.** Raynaud ([Ra], Proposition 1.2.1) considered such a degeneration locally at the generic points of the irreducible components of X_s, in the special case where $|G| = p$; our first goal, see Theorem 3.1.1 below, is to work out its extension to the smooth locus of X, and slightly more general groups, where as above $p^2 \nmid |G|$ and the p-Sylow subgroup of G is normal. The case where $p^2 \mid |G|$ remains a question which I find very interesting. See example 2.1.7 and the appendix by J. Lubin for a negative result in general, the discussion of question 2.1.5 for positive results in the literature, and remark 2.1.8 for a positive result for small ramification.

One still needs to understand the structure of $Y \to X$ at the nodes of X_s and Y_s. Henrio, working p-adic analytically, derived algebraic data along X_s, involving numerical combinatorial invariants and differential forms, which in some sense classify $Y_s \to X_s$. Our second goal in this note is to present a different approach to such degenerations at a node, modeled on *twisted curves*, i.e. curves with algebraic stack structures. Borrowing a metaphor from A. Ogus, these twisted curves have served well in the past as a sort of "magic powder" one sprinkles over the "bad locus" of certain structures, which brings about a hidden good property. The point here is that, just as in [ℵ-C-V], the introduction of twisted curves allows to replace $Y \to X$ by something that is actually a principal bundle. Unlike the case of residue characteristics prime to $|G|$, the twisted curves will in general be Artin stacks rather than Deligne–Mumford stacks. See [Ol], [ℵ-O-V2].

1.3. **Towards a proper moduli space.** The main theorem should be thought of as a first step in constructing a nice proper moduli space of degenerate coverings in mixed characteristics - it gives a special case of the valuative criterion for properness. In joint work with M. Romagny we plan to complete this task. Foundations have only recently been developed in [ℵ-O-V, ℵ-O-V2].

1.4. **Brief introduction to twisted curves.** A twisted pointed curve over a base scheme S is a diagram as follows:

$$\begin{tikzcd} \Sigma_i^{\mathcal{C}} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\ \Sigma_i^C \arrow[r, hook] & C \arrow[d] \\ & S. \end{tikzcd}$$

Here we follow [ℵ-O-V2, Section 2]:

- $C \to S$ is a usual n-pointed nodal curve, with sections $\Sigma_i^C, i = 1, \dots n$;
- $\mathcal{C}$ is an algebraic stack with finite diagonal having C as its coarse moduli space;
- $\Sigma_i^{\mathcal{C}} \subset \mathcal{C}$ are its markings, each of which a gerbe banded by some $\boldsymbol{\mu}_{r_i}$ over $\Sigma_i^C \simeq S$;
- $\mathcal{C} \to C$ is an isomorphism away from nodes and markings of C;
- at a marking of C, where the strict henselization $C^{\rm sh}$ is described by $(\operatorname{Spec}_S \mathcal{O}_S[x])^{\rm sh}$ and Σ_i^C is the vanishing locus of x, the twisted curve $\mathcal{C}$ is described as
$$\left[\, (\operatorname{Spec}_S \mathcal{O}_S[u])^{\rm sh} \,/\, \boldsymbol{\mu}_{r_i} \,\right],$$
where $\boldsymbol{\mu}_{r_i}$ acts on u via the standard character and $u^{r_i} = x$, and $\Sigma_i^{\mathcal{C}}$ is the quotient of the vanishing locus of u;
- at a node of C, where the strict henselization $C^{\rm sh}$ is described by $(\operatorname{Spec}_S \mathcal{O}_S[x,y]/\,(xy - f))^{\rm sh}$ with $f \in (\mathcal{O}_S)^{\rm sh}$, the twisted curve $\mathcal{C}$ is described as
$$\left[\, (\,\operatorname{Spec}_S \mathcal{O}_S\,[u,v]\,/\,(uv-g)\,)^{\rm sh} \,/\, \boldsymbol{\mu}_r \,\right]$$
for some r, where $\boldsymbol{\mu}_r$ acts via
$$(u,v) \mapsto (\zeta_r u, \zeta_r^{-1} v),$$
and $u^r = x, v^r = y$ and $g^r = f$.

Of course the description on the level of strict henselization descends to some étale neighborhoods. In case p divides r_i or r, the twisted curve $\mathcal{C}$ is not a Deligne–Mumford stack, and it is a little bit of a miracle, following from [ℵ-O-V2, Proposition 2.3], that one can use such a nice description locally in the étale topology (or on strict henselizations) rather than the f.p.p.f. topology. The reader is warned that transition isomorphisms between the étale local charts are in general not given in étale neighborhoods but rather in smooth or f.p.p.f. charts.

Near a marking Σ_i^C, the twisted curve is determined, uniquely up to a unique isomorphism, by the choice of r_i. In fact locally in the Zariski topology of C we can write $\mathcal{C} = \sqrt[r_i]{(C, \Sigma_i^C)}$, see [ℵ-G-V] for notation and proof. Near a node, $\mathcal{C}$ is still uniquely determined by r, but not up to a unique isomorphism - in fact $\mathrm{Aut}_C\mathcal{C}$ has a factor $\boldsymbol{\mu}_r$ for each such twisted node of index r, see [ℵ-C-V].

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2. Extensions of group schemes and their actions in dimension 1 and 2

2.1. **Raynaud's group scheme.** Raynaud (see [Ra], Proposition 1.2.1, see also Romagny, [Ro]) considers the following construction: let U be integral and let V/U be a finite flat G-invariant morphism of schemes, with G finite. Assume that the action of G on the generic fiber of V/U is faithful. We can view this as an action of the constant group scheme G_U on V, and we consider the schematic image $\mathcal{G}$ of the associated homomorphism of group schemes

$$G_U \to \mathrm{Aut}_U V.$$

Since, by definition, the image $G_U \to U$ is finite, we have that $\mathcal{G} \to U$ is finite as well. The scheme $\mathcal{G} \to U$ can also be recovered as the closure of the image of the generic fiber of G_U, which is, by the faithfulness assumption, a subscheme of $\mathrm{Aut}_U V$. By definition $\mathcal{G}$ acts faithfully on V.

Definition 2.1.1. We call the scheme $\mathcal{G}$ the effective model of G acting on V/U.

Note that a-priori we do not know that $\mathcal{G}$ is a group-scheme. It is however automatically a flat group scheme if U is the spectrum of a Dedekind domain. This follows because, in that case, the image of $\mathcal{G} \times_U \mathcal{G} \to \mathrm{Aut}_U V$ is also flat, and therefore must coincide with $\mathcal{G}$.

Also note that, if s is a closed point of U whose residue characteristic is prime to the order of G, then the fiber of $\mathcal{G}$ over s is simply G. So this effective model is only of interest when the residue characteristic divides $|G|$.

The following is a result of Raynaud, see [Ra], Proposition 1.2.1. The statement here is slightly extended as Raynaud assumes $|G| = p$:

Proposition 2.1.2. *Let U be the spectrum of a discrete valuation ring, with special point s of residue characteristic p and generic point η. Let $V \to U$ be a finite and flat morphism, and assume that the fiber V_s of V over s is reduced (but not assuming geometrically reduced). Assume given a finite group G, with normal p-Sylow subgroup,* such that $p^2 \nmid |G|$, *and an action of G on V such that $V \to U$ is G-invariant, and such that the generic fiber $V_\eta \to \{\eta\}$ is a principal homogeneous space. Let $\mathcal{G} \to U$ be the effective model of G acting on V/U.*

Then V/U is a principal bundle under the action of $\mathcal{G} \to U$.

Remark 2.1.3. An analogous construction in a wider array of cases is given in Romagny's [Ro1, Theorem A]. Romagny does not aim to construct a principal bundle; on the other hand he shows that an effective model for an action exists even if V/U and G/U are not finite, under very mild hypotheses.

Proof. As in Raynaud's argument, it suffices to show that the stabilizer of the diagonal of $V_s \times_U V_s$ inside the group scheme $V_s \times_U \mathcal{G}$ is trivial. Since G acts transitively on the closed points t_i of V_s sending the stabilizer on t_i to that over t_j, it is enough to show that one of these stabilizers, say over $t \in V_s$, is trivial. But this stabilizer P is a group scheme over the residue field $k(t)$ with degree $\deg P \mid p$, and if nontrivial it is of degree exactly p. In such a case it must coincide with the pullback of the unique p-Sylow group-subscheme of $\mathcal{G}$, therefore that p-sylow acts trivially, contradicting the fact that $\mathcal{G}$ acts effectively. ♣

Remark 2.1.4. In case the inertia group is not normal, Raynaud passes to an auxiliary cover, which encodes much of the behavior of $V \to U$.

Question 2.1.5. *What can one say on the action of $\mathcal{G}$ on V in case the order of G (and the degree of $V \to U$) is divisible by p^2, but the inertia group is still normal? Specifically, what happens if $|G| = p^2$?*

In the latter case, consider a subgroup $P \subset G$ of order p. It can be argued as in Raynaud's proof, that the effective model $\mathcal{P} \to U$ of P acts freely on V, and thus $V \to V/P$ is a principal $\mathcal{P}$-bundle. Similarly, if $\mathcal{Q}$ is the effective model of G/P acting on V/P, then $V/P \to U$ is a principal $\mathcal{Q}$-bundle. At the same time, we have an action of the effective model $\mathcal{G}$ of G on V/P, but it is not necessarily the case that $\mathcal{G}/\mathcal{P} \to \mathcal{Q}$ is an isomorphism.

While the statement of Question 2.1.5 is somewhat vague, two definite answers can already be given. First, if one concentrates on effective models of the action in the sense of Romagny, a great deal can be said. The recent work of Tossici [To1, To2] concentrates on the case where $G_K = \mathbb{Z}/p^2\mathbb{Z}$ and $\mathcal{O}_U$ contains a primitive root of unity of order p^2. The paper [To1] describes explicitly the possible models $\mathcal{G}$ of G_K; in [To2] an explicit description of the effective model of G_K acting on V is provided. I think it would be of interest to see if results like Theorems 3.1.1 and 3.2.2 can be obtained for more general effective models such as these.

Second, in general no model of $\mathcal{G}$ will act freely on V. This is the case even for some of the prettiest actions one can consider. This makes giving a complete answer to the previous Question 2.1.5 tricky, and underscores the importance of work such as Tossici's.

As probably the simplest example, consider an action of $\mathcal{G}_0 = (\alpha_p)^2 = \operatorname{Spec} k[a,b]/(a^p, b^p)$ on $k(t)$. Examples of liftings of a non-free action of the type

$$t \ \mapsto \ t + a + f(t)\, b$$

for any residue characteristic have been written down by Romagny (personal communication) and Saidi (see [Sa4]). The case of

$$t \ \mapsto \ t + a + t^p\, b \tag{1}$$

is particularly appealing, as it involves torsion and endomorphisms of a formal group. I therefore ask

Question 2.1.6. *Can one lift the action (1) to characteristic 0?*

A formal positive answer in arbitrary residue characteristics is given by Jonathan Lubin in the appendix. Here I discuss explicitly the case of residue characteristic 2, where this action can be obtained as a reduction of an action of $(\mathbb{Z}/2\mathbb{Z})^2$ on a smooth curve. I concentrate on the local picture (making it global is not difficult):

Example 2.1.7. Let $R = \mathbb{Z}_2[\sqrt{2}]$. Consider the group-scheme Y/R defined by

$$t * t' \;=\; t + t' + \sqrt{2}\, t\, t'$$

This is an additive reduction of the multiplicative group. The reduction of the subgroup μ_2 is given as

$$\text{Spec } R[a] \Big/ \big(a\,(a+\sqrt{2})\,\big),$$

reducing to α_2. It acts on Y by translation via the addition law as above:

$$t \;\mapsto\; t + a + \sqrt{2}\, a\, t.$$

The reduction of the action of $\mathbb{Z}/2\mathbb{Z}$ by inversion is the same group scheme, again reducing to α_2, which we write as

$$\text{Spec } R[b] \Big/ \big(b\,(b+\sqrt{2})\,\big).$$

This time the action is given by

$$t \;\mapsto\; \Big(1+\sqrt{2}\, b\Big)\, t \;-\; \frac{b\, t^2}{1+\sqrt{2}\, t}.$$

Since 2-torsion is fixed by inversion, these actions commute. Explicitly, the action of the product is given by

$$\begin{aligned} t \;\mapsto\; a \;+\; &\Big(1 + \sqrt{2}\, b\Big)\, t \;-\; \frac{b\, t^2}{1 + \sqrt{2}\, t} \\ &+ \sqrt{2}\, a\, \left(\Big(1 + \sqrt{2}\, b\Big)\, t \;-\; \frac{b\, t^2}{1 + \sqrt{2}\, t}\right). \end{aligned}$$

The reduction modulo $\sqrt{2}$ is given by

$$t \;\mapsto\; t + a + t^2\, b,$$

as required.

Remark 2.1.8. Raynaud's arguments do work when $p^2 \big| \, |G|$ if the p-Sylow group-scheme of $\mathcal{G}$ has only étale and cyclotomic Jordan–Hölder factors. This is because, in that case, there are no nonconstant group subschemes in the reduction. In particular this works whenever the absolute ramification index over $\mathbb{Z}_p$ is $< p$.

2.2. **Extension from codimension 1 to codimension 2.** Consider now the case where U a Gorenstein noetherian scheme, $\dim U = 2$, and V/U finite flat and G-invariant as above. Consider the S2-saturation $\mathcal{G}' \to \mathcal{G}$ of the effective model $\mathcal{G}$ of the G action on V/U. In Section 6.1.2 of [Va] Vasconcelos considers such saturation (S_2-ification in his terminology). His Proposition 6.21 on page 318 applies in our situation, and gives the existence and a characterization of the S2-saturation. We have

Lemma 2.2.1. *If $\mathcal{G}' \to U$ is flat then $\mathcal{G}'$ is a group-scheme acting on V.*

Proof. We claim that the rational map $\mathcal{G}' \times_U \mathcal{G}' \dashrightarrow \mathcal{G}'$ induced by multiplication in $Aut_U V$ is everywhere defined. Indeed the graph of this map is finite over $\mathcal{G}' \times_U \mathcal{G}'$ and isomorphic to it over the locus where $\mathcal{G}' \to \mathcal{G}$ is an isomorphism, whose complement has codimension ≥ 2. Now $\mathcal{G}'$ is S2 and of dimension 2, hence Cohen–Macaulay. Pulling back to $\mathcal{G}'$ the flat Cohen–Macaulay $\mathcal{G}' \to U$ we get that $\mathcal{G}' \times_U \mathcal{G}'$ is Cohen–Macaulay, in particular S2. This implies that the graph of $\mathcal{G}' \times_U \mathcal{G}' \dashrightarrow \mathcal{G}'$ is isomorphic to $\mathcal{G}' \times_U \mathcal{G}'$, and therefore the map is regular. The same works for the map defined by the inverse in $Aut_U V$. This makes $\mathcal{G}'$ a group-scheme, and the map $\mathcal{G}' \to Aut_U V$ into a group-homomorphism. ♣

This applies, in particular, when U is regular:

Lemma 2.2.2. *If U is regular, the S2-saturation $\mathcal{G}'$ of the effective model $\mathcal{G}$ is a finite flat group scheme acting on V.*

Proof. Again $\mathcal{G}'$, being 2-dimensional and S2, is Cohen-Macaulay, and being finite over the nonsingular scheme U, it is finite *and flat* over U (indeed its structure sheaf, being saturated, is locally free over the nonsingular 2-dimensional scheme U). The result follows from Lemma 2.2.1. ♣

When the action on the generic fiber is free, we have more:

Proposition 2.2.3. *Let U be a Cohen-Macaulay integral scheme with $\dim U = 2$. Let $V \to U$ be a G invariant, finite, flat and Cohen–Macaulay morphism, and assume the action of G on the generic fiber is free. Let $\mathcal{G} \to U$ be the effective model of the action. Assume that for every codimension-1 point ξ, the action of the fiber $\mathcal{G}_\xi$ on V_ξ is free.*

Then

(1) $\mathcal{G} \to U$ is a flat group-scheme, and
(2) The action of $\mathcal{G}$ on V is free.

Note that, by Raynaud's result 2.1.2, the assumptions hold when $U = V/G$ is local of mixed characteristics $(0, p)$, the fibers V_ξ are reduced, the p-Sylow of G is normal and $p^2 \nmid |G|$.

Proof. Consider the S2-saturation $\mathcal{G}'$ of $\mathcal{G}$. Since $V \to U$ is flat and Cohen-Macaulay, the same is true for $V \times_U V \to V$ and for $\mathcal{G}' \times_U V \to \mathcal{G}'$. Since $\mathcal{G}'$ and V are Cohen-Macaulay, we have that $V \times_U V$ and $\mathcal{G}' \times_U V$ are Cohen-Macaulay, hence S2, as well. The morphism $\mathcal{G}' \times_U V \to V \times_U V$ induced by the action $\mathcal{G}' \to \mathrm{Aut}_U V$ is finite birational and restricts to an isomorphism in condimension 1. By the S2 property it is an isomorphism. In particular we have that $\mathcal{G}' \times_U V \to V$ is flat, and since $V \to U$ is faithfully flat we have that $\mathcal{G}' \to U$ is flat. By Lemma 2.2.1 we have that $\mathcal{G}' \to U$ is a finite flat group scheme acting on V, and the isomorphism $\mathcal{G}' \times_U V \to V \times_U V$ shows that the action is free, in particular $\mathcal{G}' \to \mathcal{G}$ is an isomorphism. ♣

3. Curves

3.1. **The smooth locus.** The main case of interest for us is the following:

Let R be a complete discrete valuation ring of mixed characteristic, with fraction field K of characteristic 0, residue field k of characteristic $p > 0$, and spectrum S. Assume $Y \to S$ is a stable pointed curve with smooth generic fiber, G a finite group acting on Y over S, and denote

$$X = Y/G.$$

We assume that the closure of the locus of fixed points of G in Y_K forms a disjoint union of marked sections of the smooth locus Y_{sm}. Hence for every node $y \in Y$, the stabilizer in G of y keeps the branches of Y at y invariant. We denote the complement of the closure in Y_{sm} of the fixed locus of the generic fiber by Y_{gen}, and the image in X by X_{gen} - the so called *general locus*.

Note that the morphism $Y_{\mathrm{sm}} \to X_{\mathrm{sm}}$ is flat.

The propositions above give:

Theorem 3.1.1. *Assume $p^2 \nmid |G|$ and the p-Sylow subgroup of G is normal.*

There exist

1. *a finite flat group scheme $\mathcal{G}_{\mathrm{sm}} \to X_{\mathrm{sm}}$,*
2. *a homomorphism $G_{X_{\mathrm{sm}}} \to \mathcal{G}_{\mathrm{sm}}$ which is an isomorphism over X_K, and*
3. *an action of $\mathcal{G}_{\mathrm{sm}}$ on Y_{sm} through which the action of G factors,*

such that $Y_{\mathrm{gen}} \to X_{\mathrm{gen}}$ is a principal $\mathcal{G}_{\mathrm{sm}}$-bundle.

The formation of $\mathcal{G}_{\mathrm{sm}}$ commutes with any flat and quasi-finite base change $R \subset R'$.

Proof. Let $\mathcal{G}_{\mathrm{sm}} \to X_{\mathrm{sm}}$ be the S2-saturation of the effective model of the action of G on Y_{sm}. As X_{sm} is smooth we can apply Lemma 2.2.2, therefore $\mathcal{G}_{\mathrm{sm}} \to X_{\mathrm{sm}}$ is a finite flat group scheme acting on Y_{sm}, giving (1) and (3). Part(2) applies since over K the group G does not degenerate.

The assumptions on G mean we can apply Proposition 2.1.2, so the action of $\mathcal{G}_{\mathrm{sm}}|X_{\mathrm{gen}}$ on Y_{gen} is free in codimension 1. We can therefore apply Proposition 2.2.3, and obtain that $Y_{\mathrm{gen}} \to X_{\mathrm{gen}}$ is a principal bundle.

The formation of $\mathcal{G}_{\mathrm{sm}}$ clearly commutes with base change when restricted to the locus where it acts freely, and also over X_K. As it is flat and $S2$, its formation also commutes with base change across the remaining codimension 2 locus. ♣

It would be really interesting to see what happens for other groups G.

3.2. **The structure of Y and G over nodes and markings of X.** What can be done about the singular points and markings of X and Y? It is easy to see that even in the case of characteristic 0, the cover $Y \to X$ is not a principal bundle in general; it is already not a principal bundle at the fixed points of Y_K, and rarely a principal bundle at the nodes. However, the behavior of $Y \to X$ at the nodes is very interesting. My suggested approach here is to follow the method of [ℵ-V1, ℵ-V2, ℵ-C-V, Ol, ℵ-O-V2] using twisted curves. Let us first consider the cover $Y \to X$ itself and investigate its structure from this point of view.

Consider first a node $P \in X$ where étale locally X^{sh} is described by the equation $xy = \pi^m$, with π a uniformizer in S. Similarly take a node $Q \in Y$ over P with local equation $st = \pi^n$. Say the local degree of $Y \to X$ at Q is d, so without loss of generality we can write $x = s^d\mu$ and $y = t^d\nu$, where μ and ν are units on Y^{sh}. Comparing the Cartier divisors of x, y, s, t and π on Y^{sh} we get that $m = dn$, and $\mu\nu = 1$. Note that, since G acts transitively on the points of Y lying over $P \in X$, the degree d is independent of the choice of Q, and we may denote it d_P, to indicate its dependence on P.

Consider now the twisted curve $\mathcal{X}$ having index d_P at each node P. Recall from above that it has local description

$$\left[\,(\,\mathrm{Spec}_S\,\mathcal{O}_S\,[u,v]\,/\,(uv-\pi^n)\,)^{\mathrm{sh}}\,/\,\boldsymbol{\mu}_d\,\right].$$

Write $Z = (\mathrm{Spec}_S\,\mathcal{O}_S\,[u,v]\,/\,(uv-\pi^n))^{\mathrm{sh}}$. We stress again that up to a non-unique isomorphism $\mathcal{X}$ does not depend on the choice of local coordinates.

Lemma 3.2.1. *There is a lifting, unique up to a unique isomorphism, of $Y \to X$ to a finite flat Cohen-Macaulay morphism $Y \to \mathcal{X}$.*

Proof. Recall that the coordinate s on Y^{sh} is related to x via $x = s^d\mu$ with μ a unit. Consider the $\boldsymbol{\mu}_d$-cover $P \to Y^{\mathrm{sh}}$ given by

$$P = \mathrm{Spec}\,\mathcal{O}_{Y^{\mathrm{sh}}}[w]/(w^d-\mu),$$

using the same unit μ, where $\boldsymbol{\mu}_d$ acts via $w \mapsto \zeta_d w$. Define a morphism $P \to Z$ via $u = sw$ and $v = t/w$. This morphism is clearly equivariant, giving a morphism $Y^{\mathrm{sh}} \to [Z/\mu_p] = \mathcal{X}^{\mathrm{sh}}$. Since $(sw)^p = x = u^p$ and $(t/w)^p = y = v^p$ this lifts the given map $Y \to X$. It is a tedious but straightforward exercise to show that the morphism on strict henselization descends to give the required morphism $Y \to \mathcal{X}$. The uniqueness statement follows from the fact that $\mathcal{X}$ is a separated stack. To check that $Y \to \mathcal{X}$ is flat it suffices to show $P \to Z$ flat. This follows from the local criterion for flatness: the fiber over $u = v = 0$ is given by $s = t = 0, w^d = c$ where c is the constant coefficient of μ at $s = t = 0$. This is a scheme of degree precisely d as required. Since Y and $\mathcal{X}$ (or, for that matter, P and Z) are Cohen-Macaulay, the morphism is Cohen-Macaulay. ♣

We now have our main theorem:

Theorem 3.2.2. *Assume $p^2 \nmid |G|$ and the p-Sylow subgroup of G is normal.*

There exist

1. *a twisted curve $\mathcal{X} \to X$,*
2. *a finite flat group scheme $\mathcal{G} \to \mathcal{X}$,*
3. *a homomorphism $G_{\mathcal{X}} \to \mathcal{G}$ which is an isomorphism on $\mathcal{X}_K$,*
4. *a lifting $Y \to \mathcal{X}$ of $Y \to X$, and*
5. *an action of $\mathcal{G}$ on Y through which the action of G factors,*

such that $Y \to \mathcal{X}$ is a principal $\mathcal{G}$-bundle.

The formation of $\mathcal{G}$ commutes with any flat and quasi-finite base change $R \subset R'$.

Proof. There are two issues we need to resolve here: the construction of $\mathcal{G}$ at the nodes, and the construction of $\mathcal{X}$ and $\mathcal{G}$ at the markings.

First we need to extend $\mathcal{G}$ over nodes. We have that $Y \to \mathcal{X}$ is flat and Cohen-Macaulay at the nodes; by Theorem 3.1.1 we have that $Y_{\mathrm{gen}} \to X_{\mathrm{gen}}$ is a principal bundle under $\mathcal{G}_{\mathrm{gen}}$. By Proposition 2.2.3 the effective model $\mathcal{G} \to \mathcal{X}$ of the action of $G_{\mathcal{X}}$ on the $\mathcal{X}$-scheme Y is a finite flat group scheme over $\mathcal{X}$, and away from the markings Y is a principal bundle.

Next, we deal with the markings: the local picture of $Y_K \to X_K$ at a marking is $Y^{\mathrm{sh}} = (\operatorname{Spec} R[s])^{\mathrm{sh}}$ and $X^{\mathrm{sh}} = (\operatorname{Spec} R[x])^{\mathrm{sh}}$ where $x = s^d$, and the stabilizer in G of $s = 0$ on Y is identified with $\boldsymbol{\mu}_d$, acting via $s \mapsto \zeta s$. We give $\mathcal{X}$ the unique structure of a twisted curve with index d along this marking; locally around the marking $s = 0$ we have $\mathcal{X}^{\mathrm{sh}} = [(\operatorname{Spec} \mathcal{O}_S[u])^{\mathrm{sh}}/\boldsymbol{\mu}_d]$. The discussion above shows that $Y_K \to \mathcal{X}_K$ is a principal G-bundle. Applying Proposition 2.2.3 again we obtain that $\mathcal{G}$ is a group scheme and Y is a principal bundle. ♣

So, in view of the characteristic 0 discussion in [ℵ-C-V], we might call $\mathcal{Y} \to \tilde{\mathcal{X}}$ a twisted $\mathcal{G}$-bundle.

This suggests an approach to lifting covers from characteristic p to characteristic 0, by breaking it in two stages: (1) lifting group-schemes over $\mathcal{X}$, and (2) lifting the covers. Recent work of Wewers [W3] seems to support such an approach.

Appendix A. Lifting a non-free action on a formal group

by Jonathan Lubin[1]

The Question. In characteristic $p > 0$, consider the substitution $t \mapsto a + t + bt^p$, where $a^p = b^p = 0$. This clearly defines a groupscheme of rank p^2, isomorphic to $\alpha_p \times \alpha_p$, and an action of the groupscheme on a curve, in this case the affine line. Question 2.1.6 asked whether this group-scheme and this action can be lifted to characteristic zero, over a suitably ramified extension of $\mathbb{Z}_p$.

The Answer. It's a partial yes, in that the example presented here shows an action not on the affine line but on the formal version of this, the formal spectrum of $\mathfrak{O}[[t]]$, where $\mathfrak{O}$ is the ring of integers in a well-chosen ramified extension of $\mathbb{Q}_p$. But if the question is whether there is any example of an action of $\alpha_p \times \alpha_p$ on a genuine algebraic curve in characteristic zero, then I must plead ignorance.

In general, if R is a ring and f and g are power series in one variable over R, then it makes no sense to compose the series, $f \circ g$, unless g has zero constant term. Yet, there are situations where R has a suitable complete topology, when $f \circ g$ can make sense even when $g(0) \neq 0$. Let us detail one fairly general such situation:

If $(\mathfrak{o}, \mathfrak{m})$ is a complete local ring, then on the category of complete local $\mathfrak{o}$-algebras (R, M) we define a group functor denoted $\mathfrak{B}$ or $\mathfrak{B}_{\mathfrak{o}}$, such that $\mathfrak{B}(R)$ is the set of power series

$$f(t) = \sum_{j \geq 0} c_j t^j \quad \in \quad R[[t]]$$

for which $c_0 \in M$ and $c_1 \notin M$. Our desire is that $\mathfrak{B}_{\mathfrak{o}}(R)$ should be a group under composition of power series, and indeed the condition on c_0 guarantees that composition will be well-defined, while the condition on c_1 guarantees that the series will have an inverse in $\mathfrak{B}(R)$. One sees now that if κ is the characteristic-p field of definition in Question 2.1.6, and if R is the local κ-algebra $\kappa[a, b]/(a^p, b^p)$, then the series $a + t + bt^p$ is an element of $B_\kappa(R)$. The relation

$$(a + t + bt^p) \circ (a' + t + b't^p) = (a + a') + t + (b + b')t^p$$

shows that the groupscheme that's being described is finite and isomorphic to $\alpha_p \times \alpha_p$.

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The Method. We take a formal group F of finite height that has a subgroup of order p as well as a group of automorphisms of order p. Now, finite groups of automorphisms of a formal group of finite height are always étale, but by taking a slight blowup of F, we convert the automorphism subgroup to a local groupscheme, without going so far as to make the above group of torsion points of F étale as well. Then, allowing ourselves a slight abuse of language, our desired lifting consists of all substitutions

$$t \mapsto a \tilde{+} t \tilde{+} [b]_{F'}(t),$$

where a is a torsion point of the blow-up of F, and $1 + b$ is a p-th root of 1. In the displayed formula, F' is the blown-up version of F, the tilde over the plus-sign indicates addition with respect to F', and as usual, $[b]_{F'}(t)$ is the endomorphism whose first-degree term is bt. I suppose that very confident people may be able to look at the preceding explanation and say, Of Course, No Problem, End of Story. But I'm not so confident, and the rest of this note is devoted to filling in the gaps and making sure, to my own satisfaction at least, that everything is on the up and up. To those confident readers, everything from here on may thus be unnecessary, though the summary 1-7 at the end of this note may be an aid to flagging assurance.

A.1. **Some Algebra.** Let $\zeta = \zeta_p$ be a primitive p-th root of 1 in an algebraic extension of $\mathbb{Q}_p$, and let $\mathfrak{o} = \mathbb{Z}_p[\zeta]$. Let also $\pi = \zeta - 1$, a prime element of $\mathfrak{o}$, and let k be the fraction field of $\mathfrak{o}$. In the ring $\mathfrak{o}[T]/(T^p - 1)$, let us call Γ the image of T, and let us consider $\Delta = \frac{\Gamma - 1}{\pi}$. Then the minimal polynomial for Δ is

$$(*) \qquad T^p + \frac{p}{\pi}T^{p-1} + \frac{p(p-1)}{2\pi^2}T^{p-2} + \cdots + \frac{p(p-1)}{2\pi^{p-2}}T^2 + \frac{p}{\pi^{p-1}}T,$$

in which the coefficient of T is a unit in $\mathfrak{o}$ congruent to -1 modulo π. Let us call B the ring $\mathfrak{o}[\Delta]$; we need to establish a few facts about it. I will use capital Greek letters for elements of B, lower case Greek letters for elements of $\mathfrak{o}$.

Lemma A.1.1. *The ring B is isomorphic to $\mathfrak{o} \oplus \mathfrak{o} \oplus \cdots \oplus \mathfrak{o}$, with p factors. In B, every element Θ satisfies the condition that $\Theta^p - \Theta \in \pi B$.*

Proof. The minimal polynomial for Δ, described above, is $T^p - T$ modulo π. By Hensel's Lemma it splits into dinstinct linear factors over the complete local ring $\mathfrak{o}$, so that the first part of the statement is verified. Since each element $\beta \in \mathfrak{o}$ has the property that $\beta^p - \beta \in \pi\mathfrak{o}$, the corresponding property holds for elements of B as well. It may be of interest to note that this is not true of the subring $\mathfrak{o}[\Gamma]$ of B.

A.2. **Endomorphisms of the fundamental formal group.** We start with the polynomial $f(t) = \pi t + t^p \in \mathfrak{o}[[t]]$, which has associated to it a unique formal group $F(x, y) \in \mathfrak{o}[[x, y]]$ for which $f \in \mathrm{End}_{\mathfrak{o}}(F)$, as proved in [LT]. The following is hardly surprising:

Lemma A.2.1. *For each $\Theta \in B$, there is a unique series $[\Theta]_F(t) \in B[[t]]$ such that $[\Theta]'_F(0) = \Theta$ and $f \circ [\Theta]_F = [\Theta]_F \circ f$; this series is an element of $\mathrm{End}_B(F)$. In particular for $\Theta = \pi$ we have $[\pi]_F = f$.*

This may be proved by using either of the halves of Lemma A.1.1; if one wishes to use the fact that $\Theta^p - \Theta$ is always in πB, then the proof of the first Lemma in [LT] goes through word for word. The statement $[\pi]_F = f$ follows by uniqueness.

The endomorphism ring $\mathrm{End}_B(F)$ contains in particular the series $[\Delta]_F$ and $[\Gamma]_F$; the p-fold iterate of the latter series is the identity series t. And since $\Gamma = 1 + \pi\Delta$, our periodic series $[\Gamma]_F(t)$ may also be written as

$$F\,(\,t,\ ([\pi]_F \circ [\Delta]_F)\,(t)\,)\,.$$

If β is an element of $\ker([\pi]_F)$, then the series $\tau_\beta(t) = F(t, \beta)$ commutes with both

$$[\zeta]_F(t) \quad = \quad F\,(\,t,\ [\pi]_F(t)\,)$$

and

$$[\Gamma]_F(t) \quad = \quad F\,(\,t,\ ([\Delta]_F \circ [\pi]_F)\,(t)\,)\,.$$

If only $B = \mathfrak{o}[\Delta]$ had not been an étale $\mathfrak{o}$-algebra, we could have taken $\ker([\pi]_F)\times\mathrm{Spec}(B)$ as our desired lifting of $\alpha_p\times\alpha_p$. After all, the points of $\ker([\pi]_F)$ are the β's mentioned above, and the points of $\mathrm{Spec}(B)$ are essentially the p-th roots of unity ξ, and the substitution

$$t \mapsto F\,(\beta, [\xi]_F(t))$$

would be our lifting of the substitution mentioned in the introduction. There is the additional problem that in case $p = 2$, F is of height one and so $\ker[\pi]$ is not a lifting of α_p, but the étaleness of the other factor is a much bigger obstacle.

Because of the form of $f(t) = [\pi]_F(t) = \pi t + t^p$, not only F but also all the B-endomorphisms $[\Theta]_F$ have the property that the only nonzero terms are in degrees congruent to 1 modulo $p-1$. Any such series can be written, that is, in the form $\sum_{j\geq 0} H_j$ where each H_j is a form or monomial of degree $1 + j(p-1)$. For want of a better term, I'll call any series with this last property $(p-1)$-lacunary.

Now I want to let $\mathfrak{O}$ be any complete local $\mathfrak{o}$-algebra in which π is no longer indecomposable, $\pi = \lambda\mu$ where both λ and μ are nonunits. Minimally, one may take $\lambda = \mu = \sqrt{\pi}$ and $\mathfrak{O} = \mathfrak{o}[\sqrt{\pi}]$. Or we may let

$\mathfrak{O}$ be the ring of integers in any properly ramified algebraic extension K of k, and λ any element of K with valuation $0 < v(\lambda) < v(\pi) = 1$. Or, generically, we can take $\mathfrak{O} = \mathfrak{o}[[\lambda, \pi/\lambda]]$, a ring that can be described alternatively as $\mathfrak{o}[[\lambda, \mu]]/(\lambda\mu - \pi)$ or as the set of all doubly infinite Laurent series $\sum_{j\in\mathbb{Z}} \alpha_j \lambda^j$ in the indeterminate λ and with coefficients $\alpha_j \in \mathfrak{o}$ satisfying the additional condition that $j + v(\alpha_j) \geq 0$, where v is the (additive) valuation on $\mathfrak{o}$ and k normalized so that $v(\pi) = 1$.

If G is a $(p-1)$-lacunary series in one or more variables, I will call the λ-blowup of G, denoted $G^{(\lambda)}$, the series formed from G in the following way: if $G = \sum_{j\geq 0} H_j$, each H_j being homogeneous of degree $1 + j(p-1)$, then $G^{(\lambda)} = \sum_{j\geq 0} \lambda^j H_j$.

When we apply the above operation to F and its endomorphisms, here's what happens: $F^{(\lambda)}$ becomes a formal group whose reduction modulo the maximal ideal of $\mathfrak{O}$ is just the additive formal group $x+y$. The maps $\mathrm{End}_{\mathfrak{o}}(F) \to \mathrm{End}_{\mathfrak{O}}(F^{(\lambda)})$ and $End_B(F) \to \mathrm{End}_{B\otimes_{\mathfrak{o}}\mathfrak{O}}(F^{(\lambda)})$ that take $g(t)$ to $g^{(\lambda)}(t)$ are injections. For any $\Theta \in B$, I will write $[\Theta]^{(\lambda)}$ for $[\Theta]_{F^{(\lambda)}} = ([\Theta]_F)^{(\lambda)}$; then since $[\pi]^{(\lambda)}(t) = \pi t + \lambda t^p = \lambda(\mu t + t^p)$, the new formal group $F^{(\lambda)}$ has at least one nontrivial finite subgroup, namely the set of roots of $\mu t + t^p$, under the group law furnished by $F^{(\lambda)}$, and they certainly are the geometric points of $\mathrm{Spec}\,(\mathfrak{O}[t]/(\mu t + t^p))$, but this is not the kernel of $[\pi]^{(\lambda)}$, since the standard construction of kernel in that case leads to something that's not flat. Rather, if we call $g(t) = \mu t + t^p$, then the finite groupscheme we're talking about is the kernel of $g : F^{(\lambda)} \to F^{(\lambda^2)}$.

Seeing just how a group scheme lifting α_p acts on $F^{(\lambda)}$ is a little trickier and more unusual. Our aim is to show that the automorphism $[\Gamma]^{(\lambda)}(t)$ of $F^{(\lambda)}$ lies in $\mathfrak{O}[\Delta'][[t]]$, where $\Delta' = \lambda\Delta$ has the $\mathfrak{O}$-minimal polynomial

$$(**)\quad \begin{aligned} &T^p + \frac{p\lambda}{\pi}T^{p-1} + \frac{p(p-1)\lambda^2}{2\pi^2}T^{p-2} + \cdots + \frac{p(p-1)\lambda^{p-2}}{2\pi^{p-2}}T^2 + \frac{p\lambda^{p-1}}{\pi^{p-1}}T \\ &= T^p + \frac{p}{\mu}T^{p-1} + \frac{p(p-1)}{2\mu^2}T^{p-2} + \cdots + \frac{p(p-1)}{2\mu^{p-2}}T^2 + \frac{p}{\mu^{p-1}}T; \end{aligned}$$

Note that this polynomial is congruent to T^p modulo the maximal ideal $\mathfrak{M}$ of $\mathfrak{O}$.

Now recall that $\Gamma = 1+\Delta\pi$, so that the series $[\Gamma](t)$, which is periodic of period p with respect to substitution of series, whether we're talking about automorphisms of the original F or of the blown-up $F^{(\lambda)}$, can be written

$$[\Gamma](t) \;=\; F\,(\,t,\;([\Delta]\circ[\pi]\,)(t))\,.$$

Since every element of B is an $\mathfrak{o}$-linear combination of $\{1, \Delta, \ldots, \Delta^{p-1}\}$, we may write

$$[\Delta]_F(t) \;=\; \Delta t + \sum_{j\geq 1} C_j t^{j(p-1)+1} \quad\in\quad B[[t]],$$

where, as remarked, each coefficient C_j is an $\mathfrak{o}$-linear combination of the powers of Δ, up to Δ^{p-1}. It follows that $[\Delta]^{(\lambda)}$, the corresponding endomorphism of F, has the form

$$[\Delta]_{F^{(\lambda)}}(t) \;=\; \Delta t + \sum_{j\geq 1} C_j \lambda^j t^{j(p-1)+1} \quad\in\quad \mathfrak{O}[[t]],$$

where the C_j's are the same in both displayed formulas. Now, what of $[\Delta]^{(\lambda)} \circ [\pi]^{(\lambda)} = [\Delta]^{(\lambda)}(\pi t + \lambda t^p)$? Making the indicated substitution gives

$$\begin{aligned} &\Delta\;(\pi t + \lambda t^p) \;+\; \sum_{j\geq 1} C_j \lambda^j (\pi t + \lambda t^p)^{j(p-1)+1} \\ =\;&\Delta'\;(\mu t + t^p) \;+\; \sum C_j \lambda^{jp+1} (\mu t + t^p)^{j(p-1)+1}. \end{aligned}$$

But now because $C_j \in B = \mathfrak{o}[\Delta]$, we also have $C_j\lambda^{jp+1} \in \lambda\mathfrak{O}[\Delta'] = \lambda\mathfrak{O}[\lambda\Delta]$, since the j's all are at least 1. This shows that $[\pi\Delta]^{(\lambda)}(t)$ is a power series with coefficients in $\mathfrak{O}[\Delta']$, and indeed, modulo $\mathfrak{M}$, this series is just $\Delta' t^p$. Finally, when we add this series and the series t by means of the formal group $F^{(\lambda)}(x, y) \equiv x + y \mod \mathfrak{M}$, the result, namely $[\Gamma]^{(\lambda)}(t)$, has coefficients in $B' = \mathfrak{O}[\Delta']$, and is congruent modulo $\mathfrak{M}$ to $t + \Delta' t^p$. One more remark is necessary, the obvious one that if $\mu\alpha + \alpha^p = 0$, then $[\Delta\pi]^{(\lambda)}(\alpha) = 0$ and $[\Gamma]^{(\lambda)}(\alpha) = \alpha$.

In summary, this is what we now have:

(1) The ring $\mathfrak{o}$ is $\mathbb{Z}_p[\zeta]$, where $\zeta = \zeta_p$ is a primitive p-th root of unity, and we use the prime element $\pi = \zeta - 1$.

(2) The ring $\mathfrak{O}$ is any suitably ramified extension of $\mathfrak{o}$, the minimal example being $\mathfrak{O} = \mathfrak{o}[\sqrt{\pi}]$. This $\mathfrak{O}$ is the ring over which our liftings and action are defined, and we identify in it elements $\lambda, \mu \in \mathfrak{O}$ with $\lambda\mu = \pi$.

(3) The formal group F over $\mathfrak{o}$ has $\pi t + t^p$ as an endomorphism and thus has $\mathfrak{o}$ as its ring of "absolute" endomorphisms (over the ring of integers of any algebraic extension field of the fraction field of $\mathfrak{o}$). Allowing for abuse of language, there is a unique $\mathfrak{o}$-subgroupscheme of F of rank p, namely $\ker[\pi]_F = \mathrm{Spec}(A)$, where $A = \mathfrak{o}[[t]]/\,([\pi]_F(t))$.

(4) The finite $\mathfrak{o}$-algebra B is $\mathfrak{o}[\Delta]$, where the minimal polynomial for Δ over $\mathfrak{o}$ is given in formula $(*)$. Algebraically, B is $\mathfrak{o}^{\oplus p}$, and when we call $\Gamma = 1 + \pi\Delta \in B$, we have $\Gamma^p = 1$. The scheme $\mathrm{Spec}(B)$ is a finite étale groupscheme of order p; the element $\Gamma \in B$ is a generic p-th root of unity, and the operation of the étale groupscheme on the formal-affine line is $t \to [\Gamma]_F(t) = F\left(t, ([\pi] \circ [\Delta])(t)\right)$.

(5) We use $\lambda \in \mathfrak{O}$ to form a sort of blowup of F, which we call $F^{(\lambda)}$ and which is described on the preceding page. This formal group has the subgroup scheme $\mathrm{Spec}(A')$, where $A' = \mathfrak{O}[[t]]/(\mu t + t^p)$, and this groupscheme acts on the formal-affine line by the substitution $t \to F(\lambda)(a, t)$ when a is any root of $\mu t + t^p$.

(6) We define $\Delta' = \lambda\Delta \in B \otimes_{\mathfrak{o}} \mathfrak{O}$, and note that Δ' has the minimal polynomial over $\mathfrak{O}$ given by $(**)$ on the preceding page. Call $B' = \mathfrak{O}[\Delta']$. The periodic power series $[\Gamma]^{(\lambda)}(t)$, originally defined to be in $B \otimes_{\mathfrak{o}} \mathfrak{O}[[t]]$, actually is in $B'[[t]]$ and as an element of this ring, it becomes $t + \Delta' t^p$ in $B' \otimes_{\mathfrak{O}} \mathfrak{O}/\mathfrak{M}[[t]]$.

(7) Since the series $[\Gamma]^{(\lambda)}(t)$ and the $F^{(\lambda)}(a, t)$ mentioned in (5) commute, we do indeed have a finite groupscheme, namely $\mathrm{Spec}(A' \otimes_{\mathfrak{O}} B')$, acting on the formal-affine line in such a way that over $\mathfrak{O}/\mathfrak{M}$, the action is $t \mapsto a + t + \Delta' t^p$.

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