

# Virtual intersection theory on the space of lines in the plane

Joint work with Rahul (ETH) and Dhruv (Cambridge)



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- With Dhruv, Rahul<sup>1</sup>
- Knudsen's  $\overline{\mathcal{M}}_{0,n}$  is the moduli space of  $(C, p_1, \dots, p_n)$ 
  - ▶ Curves of  $p_a(C) = 0$
  - ▶ At most nodes,  $p_i$  distinct,  $p_i \notin \text{sing}(C)$  — **locally stable**.
  - ▶  $\omega_C((1 - \epsilon) \sum p_i)$  is ample — **globally stable**.
- $\overline{\mathcal{M}}_{0,n}$  is a smooth scheme with simple and completely computable enumerative geometry.

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<sup>1</sup>thank you, guys!

# Generalizations of $\overline{\mathcal{M}}_{0,n}$

- Generalizes to  $\mathcal{M}_{g,n}$  (Deligne–Knudsen–Mumford),
  - ▶ a smooth **stack**
  - ▶ with challenging **tautological ring**,
  - ▶ the subject of **Witten's conjectures / Kontsevich's theorem**
  - ▶ and more recent **Faber's conjectures** etc.
- Generalizes to  $\overline{\mathcal{M}}_{g,n}(X)$  (Kontsevich)
  - ▶ a **virtually** smooth stack
  - ▶ The subject of **GW theory**
  - ▶ The careers of my collaborators.

# Goals of this work: part 1

- I am not a true GW theorist — I have never written down a GW generating function.
- Dhruv finds it ridiculous.
- **Dhruv's Goal 1:** “Get Dan to write down a GW generating function.”
- **Status report:** in progress, not there yet, wait for part 2 of the project.

## Goals of this work: part 2

- Dhruv is irked that I have not managed to cram matroid theory into my head.
- Dhruv is equally irked that Rahul is on record disparaging tropical geometry.
- **Dhruv's Goal 2:** instigate a **painless** life-changing experience for Rahul and Dan, to come out of the closet and embrace **tropical matroids**.
- **Status report:** resounding success.

# Dan and Rahul embracing a tropical matroid (the dressian)



# Dan embracing a tropical matroid, by Renzo



## Goals of this work: official goal

- Consider the space  $\mathcal{P}_n^\circ$  of configurations  $(S = \mathbb{P}^2, L = (\ell_1, \dots, \ell_n))$
- of  $n$  transversal numbered lines on  $\mathbb{P}^2$ , up to projective equivalence.
- For  $n \geq 4$  this is indeed a quasi-projective scheme, of dimension  $2n - 8$ .
- We find a **projective moduli space**  $\mathcal{P}_n \xleftarrow[\text{open}]{} \mathcal{P}_n^\circ$ ,
- which affords
  - ▶ tautological maps,
  - ▶ tautological cycles,
  - ▶ virtual fundamental class, and
  - ▶ enumerative geometry.  $\mathcal{C}$

# One result

## Theorem (PanDA)

for  $n \geq 4$ ,

- 1  $\mathcal{P}_n$  is a projective scheme.
- 2  $\mathcal{P}_n^\circ \subset \mathcal{P}_n$  is open.
- 3 It is a moduli space: its universal family  $(S_n, \mathcal{L}_n) \rightarrow \mathcal{P}_n$  is the family of  $(1 - \epsilon)$ -weighted log smooth very stable surface pairs<sup>a</sup>.
- 4  $\mathcal{P}_n$  carries a virtually log smooth log structure.
- 5 The constituents  $\ell_i \subset S_n$  of  $\mathcal{L}_n$  are reduced Cartier divisors on fibers.
- 6 There are **forgetful maps**  $\mathcal{P}_{n+1} \rightarrow \mathcal{P}_n$  forgetting  $\ell_{n+1}$  and stabilizing.
- 7 There are **rememberful maps**<sup>b</sup>  $\mathcal{P}_n \rightarrow \overline{\mathcal{M}}_{0,n-1}$  remembering  $\ell_n$  and stabilizing.
- 8 The symmetric group  $\Sigma_n$  acts by permuting the lines.

<sup>a</sup>embedded in the Dressian family of  $(1 - \epsilon)$ -weighted Alexeev-stable toric varieties.

<sup>b</sup>terminology?

# table

We are not the first here, and we use everyone's work.

- Kapranov: Chow quotients of Grassmannians.
- Kollár–Shepherd-Barron–Alexeev: stable pairs.
- Hacking–Keel–Tevelev: very stable pairs.
- Lafforgue: “an open subspace of the space of projective schemes . . .”
- Alexeev: weighted stable pairs and weighted stable toric varieties.
- Gross-Siebert, Chen: stable log maps.

Space	$\mathcal{P}_n^{\text{Kap}}$	$\mathcal{P}_n$	$\mathcal{P}_n^{\text{Laff}}$	$\mathcal{P}_n^{\text{HKT}}$	$\mathcal{P}_n^{\text{KSBA}}$
who	Kapranov	PanDA	Lafforgue	H–K–T	K–S–B–A
description	$Gr(3, n) // T$	<b>today</b>	Matroid/Log	Very stable	stable
modular?	not really	<b>YES!!</b>	sort of, yes	yes	yes
explicit?	yes	<b>YES!!</b>	??	yes	yes
$\int \int$ ?	??	<b>YES!!</b>	??	no	no
maps?	??	<b>YES!!</b>	no?	no	no
classes?	??	<b>YES!!</b>	??	no	no

# Knudsen's construction of $\overline{\mathcal{M}}_{0,n}$

- 1  $\overline{\mathcal{M}}_{0,3} = \{pt\}$  and the universal family is  $\mathcal{C}_{0,3} = (\mathbb{P}^1, (0, 1, \infty))$ .
- 2 Given  $(\mathcal{C}_{0,n}, p_1, \dots, p_n) \rightarrow \overline{\mathcal{M}}_{0,n}$  define  $\overline{\mathcal{M}}_{0,n+1} = \mathcal{C}_{0,n}$ , with the pullback family.

The identity map gives an  $(n+1)$ -th section with image the diagonal.  
Trouble is, it is not stable

- 3 Define  $\mathcal{C}_{0,n+1}$  by
  - (a) blowing up the diagonal
  - (b) Blowing up  $\tilde{\Delta} \cap p_j$ .

This is stable!

A subtle point: Knudsen shows (3) can be done on moduli.

# PanDA's construction of $\mathcal{P}_n$

- 1  $\mathcal{P}_4 = \{pt\}$  and  $(\mathcal{S}_4, \mathcal{L}_4) = (\mathbb{P}^2, V(XYZ(X+Y+Z)))$ .
- 2 Given  $(\mathcal{S}_n, \mathcal{L}_n) \rightarrow \mathcal{P}_n$  define  $\mathcal{P}_{n+1} = \overline{\mathcal{M}}_{0,n}((\mathcal{S}_n, \mathcal{L}_n)/\mathcal{P}_n, [\ell])$ .
  - ▶ The notation stands for the space of *logarithmic* stable maps,
  - ▶ and the “class of a line”  $[\ell]$  is determined by its contact orders using matroid theory.

$$\begin{array}{ccc} \ell & \longrightarrow & (\mathcal{S}_n, \mathcal{L}_n) \\ \downarrow & & \downarrow \pi_n \\ \mathcal{P}_{n+1} & \xrightarrow{\phi_{n+1}} & \mathcal{P}_n \end{array}$$

- 3 The pullback of  $\mathcal{S}_n$  now inherits a new  $(n+1)$ -st line  $\ell$ .
- 4 It is not locally stable, but the **Knudsen–Ranganathan expansion**  $(\mathcal{S}_{n+1}, \mathcal{L}_{n+1})$  of  $(\mathcal{S}_n, \mathcal{L}_n)$  determined by  $\ell$  becomes locally stable.
- 5 The sheaf  $\omega_{\mathcal{S}_{n+1}} \left( (1-\epsilon) \sum_{i=1}^{n+1} \ell_i \right)$  is ample.

- The space  $\mathcal{P}_5 = \overline{\mathcal{M}}_{0,4}(\mathcal{P}^2, \mathcal{L}_4), [\ell]$  is the blowup of  $(\mathbb{P}^2)^\vee$  at the four points where  $\ell$  coincides with  $\ell_i$ .
- The exceptionals indicate the marking  $\ell \cap \ell_i$  on  $\ell$ .
- The universal family  $\mathcal{S}_5$  is obtained by blowing up these loci, as well as the triple intersection points.
- In  $\mathcal{S}_5^{\text{Laff}} = \mathcal{P}_5^{\text{HKT}} = \mathcal{P}_5^{\text{KSBA}}$ , there is also a blowdown where  $\ell$  had two triple points. But then  $\ell_5$  is no longer Cartier, and one loses the forgetful map to  $\mathcal{S}_4$ .

## Virtual structure of $\mathcal{P}_n$

- $(\mathcal{S}_n, \mathcal{L}_n) \rightarrow \mathcal{P}_n$  is proper and log smooth.
- The sheaf  $T_{(\mathcal{S}_n, \mathcal{L}_n)/\mathcal{P}_n}^{\log}$  is locally free of rank 2.
- Stability implies that  $\pi_{n*} T_{(\mathcal{S}_n, \mathcal{L}_n)/\mathcal{P}_n}^{\log} = 0$ .
- Therefore  $R\pi_{n*} T_{(\mathcal{S}_n, \mathcal{L}_n)/\mathcal{P}_n}^{\log}$  is a perfect obstruction theory  $\mathcal{F}_n^\bullet$  on  $\mathcal{P}_n$ , giving rise to a virtual fundamental class  $[\mathcal{P}_n]^{\text{virt}}$ , of degree  $2n - 8$ .
- By Gross–Siebert  $\mathcal{P}_{n+1} = \overline{\mathcal{M}}_{0,n}((\mathcal{S}_n, \mathcal{L}_n)/\mathcal{P}_n, [\ell])$  carries a logarithmic perfect obstruction theory  $\mathcal{E}_{n+1}^\bullet$  **relative to**  $\mathcal{P}_n$ .
- Manolache constructs a virtual pullback map  $(\phi_{n+1})_{\mathcal{E}_{n+1}^\bullet}^! : CH_i(\mathcal{P}_n) \rightarrow CH_{i+2}(\mathcal{P}_{n+1})$ .

### Theorem

$$(\phi_{n+1})_{\mathcal{E}_{n+1}^\bullet}^! ([\mathcal{P}_n]^{\text{vir}}) = [\mathcal{P}_{n+1}]^{\text{vir}}.$$

# Proof of compatibility: setup

Set  $\mathcal{L}_{n+1}^{(n)} = \ell_1 \cup \dots \cup \ell_n$ , and

$$\mathcal{S}_n^{(n+1)} = \mathcal{P}_{n+1} \times_{\mathcal{P}_n} \mathcal{S}_n, \quad \mathcal{L}_n^{(n+1)} = \mathcal{P}_{n+1} \times_{\mathcal{P}_n} \mathcal{L}_n.$$

$$\begin{array}{ccccccc}
 (\mathcal{S}_{n+1}, \mathcal{L}_{n+1}) & \xrightarrow{\alpha} & (\mathcal{S}_{n+1}, \mathcal{L}_{n+1}^{(n)}) & \xrightarrow{\beta} & (\mathcal{S}_n^{(n+1)}, \mathcal{L}_n^{(n+1)}) & \xrightarrow{\varphi_{n+1}} & (\mathcal{S}_n, \mathcal{L}_n) \\
 \pi_{n+1} \downarrow & & \pi_{n+1}^{(n)} \downarrow & & \pi_n^{(n+1)} \downarrow & & \downarrow \pi_n \\
 \mathcal{P}_{n+1} & \xlongequal{\quad} & \mathcal{P}_{n+1} & \xlongequal{\quad} & \mathcal{P}_{n+1} & \xrightarrow{\phi_{n+1}} & \mathcal{P}_n
 \end{array}$$

# Proof of compatibility: identifying the acting characters

$$\begin{array}{ccccccc}
 (\mathcal{S}_{n+1}, \mathcal{L}_{n+1}) & \xrightarrow{\alpha} & (\mathcal{S}_{n+1}, \mathcal{L}_{n+1}^{(n)}) & \xrightarrow{\beta} & (\mathcal{S}_n^{(n+1)}, \mathcal{L}_n^{(n+1)}) & \xrightarrow{\varphi_{n+1}} & (\mathcal{S}_n, \mathcal{L}_n) \\
 \pi_{n+1} \downarrow & & \pi_{n+1}^{(n)} \downarrow & & \pi_n^{(n+1)} \downarrow & & \downarrow \pi_n \\
 \mathcal{P}_{n+1} & \xlongequal{\quad} & \mathcal{P}_{n+1} & \xlongequal{\quad} & \mathcal{P}_{n+1} & \xrightarrow{\phi_{n+1}} & \mathcal{P}_n
 \end{array}$$

- 1  $\varphi_{n+1}^* T_{\pi_n}^{\log} = T_{\pi_n^{(n+1)}}^{\log}$
- 2 Since  $\beta$  is log étale  $\beta^* T_{\pi_n^{(n+1)}}^{\log} = T_{\pi_{n+1}^{(n)}}^{\log}$ .
- 3  $\beta$  is a toric modification so  $R\beta_* = \beta_*$ . So  $T_{\pi_n^{(n+1)}}^{\log} = \beta_* T_{\pi_{n+1}^{(n)}}^{\log}$ .
- 4  $N_{\ell_{n+1}} = N_{\ell_{n+1}}^{\log}$ .

It follows that  $\phi_{n+1}^* \mathcal{F}_n^\bullet = \phi_{n+1}^* R\pi_{n*} = R\pi_{n+1*} T_{\pi_{n+1}^{(n)}}^{\log}$ .

## Proof of compatibility: conclusion

We have an exact sequence

$$0 \rightarrow T_{\pi_{n+1}}^{\log} \rightarrow T_{\pi_{n+1}}^{\log(n)} \rightarrow N_{\ell_{n+1}} \rightarrow 0,$$

giving rise to a triangle

$$R\pi_{n+1}^* T_{\pi_{n+1}}^{\log} \longrightarrow R\pi_{n+1}^* T_{\pi_{n+1}}^{\log(n)} \longrightarrow R\pi_{n+1}^* N_{\ell_{n+1}} \xrightarrow{[+1]},$$

in other words

$$\mathcal{F}_{n+1}^{\bullet} \longrightarrow \phi_{n+1}^* \mathcal{F}_n^{\bullet} \longrightarrow \mathcal{E}_{n+1}^{\bullet} \xrightarrow{[+1]} .$$

By Bryan-Leung this gives compatibility.

# A tautological class of its own



# Tautological classes

There are two sources of tautological classes on  $\mathcal{P}_n$ :

- 1 Mumford–Donaldson  $\kappa$  classes: on  $\mathcal{S}_n$  we have
  - ▶ line bundles  $\Lambda_i := \mathcal{O}_{\mathcal{S}_n}(\ell_i)$  with  $c_1(\Lambda_i) = \lambda_i$
  - ▶ rank-2 vector bundle  $\Theta := T_{\mathcal{S}_n/\mathcal{P}_n}^{\log}$  with  $\theta_1 = c_1(\Theta)$ ,  $\theta_2 = c_2(\Theta)$ .

Set

$$\kappa(a_1, \dots, a_n; b, c) = \pi_{n*} \left( \lambda_1^{a_1} \cdots \lambda_n^{a_n} \cdot \theta_1^b \cdot \theta_2^c \right).$$

These are in  $CH^{a+b+2c-2}(\mathcal{P}_n)$ . We write  $R_\kappa(\mathcal{P}_n)$  for the subring generated by these.

- 2 Boundary pushforward maps. These land in  $CH_*$  and can be promoted to a subring  $R(\mathcal{P}_n) \subset \text{Log}CH^*(\mathcal{P}_n)$ .

## Theorem

*$R(\mathcal{P}_n)$  is generated by boundary classes, and is recursively computable.*

## Reminder: $\overline{\mathcal{M}}_{0,n}$

- We have

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} = \binom{n-3}{k_1, \dots, k_n},$$

when the sum  $\sum_i k_i$  is equal to  $n-3$ .

All you need to know is that  $\psi_i = \phi_n^* \psi_i^{(n-1)} + \Delta_{i,n}$ , and  $\Delta_{i,n} \Delta_{j,n} = -\delta_{ij} \Delta_{i,n} \psi_i$ .

## $\mathcal{P}_4$ and other codimension 0 cases

- 1 The sheaf  $\mathcal{O}_{S_4}(\ell_i)$  is  $\mathcal{O}_{\mathbb{P}^2}(1)$  with first Chen class  $[\ell]$ .
- 2 For the log Chern classes use the exact sequence  $0 \rightarrow \Theta \rightarrow T_{(\mathbb{P}^2, \ell_1 + \ell_2 + \ell_3)}^{\log} \rightarrow N_{\ell_4} \rightarrow 0$ , with  $T_{(\mathbb{P}^2, \ell_1 + \ell_2 + \ell_3)}^{\log} = \mathcal{O}_{\mathbb{P}^2}^2$ .
- 3 So  $c_*(\Theta) = c_*(N_{\ell_4})^{-1} = 1 - [\ell] + [\ell]^2$ .
- 4 Also the sequence  $0 \rightarrow T_{\ell_i}^{\log} \rightarrow \Theta|_{\ell_i} \rightarrow \mathcal{O}_{\ell_i} \rightarrow 0$  helps.
  - $\kappa(1, 1, 0, 0, 0; 0, 0) = 1$ . In general it is the class  $\mathbf{1}$  of a section.
  - $\kappa(1, 0, 0, 0, 0; 1, 0) = -1$ . In general  $(3 - n) \cdot \mathbf{1}$ .
  - $\kappa(0, 0, 0, 0, 0; 2, 0) = 1$ . In general  $(3 - n)^2 \cdot \mathbf{1}$ .
  - $\kappa(2, 0, 0, 0, 0; 0, 0) = 1$ . In general  $\mathbf{1}$ .
  - $\kappa(0, 0, 0, 0, 0; 0, 1) = 1$ . In general  $\chi_{\text{top}}(\mathbb{P}^2 \setminus L) \cdot \mathbf{1} = (3 - 2n + \binom{n}{2}) \cdot \mathbf{1}$ .

The end

Thank you for your attention