Moduli techniques in resolution of singularities

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Harvard-MIT algebraic geometry seminar

February 12, 2019
What is resolution of singularities?

**Definition**

A resolution of singularities $X' \to X$ is a modification\(^a\) with $X'$ nonsingular inducing an isomorphism over the smooth locus of $X$.

\(^a\)proper birational map. For instance, blowing up.
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\[\text{proper birational map. For instance, blowing up.}\]

**Theorem (Hironaka 1964)**

A variety $X$ over a field of characteristic 0 admits a resolution of singularities $X' \to X$, so that the critical locus $E \subset X'$ is a simple normal crossings divisor.

\[\text{Codim. 1, smooth components meeting transversally - as simple as possible}\]

Always characteristic 0 . . .
Compactifications

“Working with noncompact spaces is like trying to keep change with holes in your pockets”

Angelo Vistoli
Compactifications

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Corollary (Hironaka)

A smooth quasiprojective variety $X^0$ has a smooth projective compactification $X$ with $D = X \smallsetminus X^0$ a simple normal crossings divisor.
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Resolution of families: \( \dim B = 1 \)

**Key Question**

When are the singularities of a morphism \( X \to B \) simple?

**Theorem (Kempf–Knudsen–Mumford–Saint-Donat 1973)**

If \( \dim B = 1 \) by modifying \( X \) one can get

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\prod x_a^i t
\]

With base change \( t = s k \), can have \( s = \prod x_i \).
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- If $\dim B = 1$ by modifying $X$ one can get $t = \prod x_i^{a_i}$,
- With base change $t = s^k$, can have $s = \prod x_i$.

**Question**

What makes these special?
Log smooth schemes and log smooth morphisms

- A toric variety is a normal variety on which $T = (\mathbb{C}^*)^n$ acts algebraically with a dense free orbit.
- Zariski locally defined by equations between monomials.
Log smooth schemes and log smooth morphisms

- **A toric variety** is a normal variety on which $T = \left( \mathbb{C}^* \right)^n$ acts algebraically with a dense free orbit.
- Zariski locally defined by equations between monomials.
- A variety $X$ with divisor $D$ is toroidal or log smooth if étale locally it looks like a toric variety $X_\sigma$ with its toric divisor $X_\sigma \setminus T$.
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Log smooth schemes and log smooth morphisms

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- Étale locally it is defined by equations between monomials.
- A morphism $X \to Y$ is **toroidal** or **log smooth** if étale locally it looks like a torus equivariant morphism of toric varieties.
- The inverse image of a monomial $^1$ is a monomial.

$^1$defining equation of part of $D_Y$
Resolution of families: higher dimensional base

Question

When are the singularities of a morphism $X \to B$ simple?

The best one can hope for, after base change, is a semistable morphism:

Definition ($\mathcal{X}$-Karu 2000)

A log smooth morphism, with $B$ smooth, is semistable if locally it is a product of one-parameter semistable families.

$$t_1 = x_1 \cdots x_{l_1}$$

$$\vdots$$

$$t_m = x_{l_{m-1} + 1} \cdots x_{l_m}$$

In particular log smooth.

Similar definition by Berkovich, all inspired by de Jong.
Semistable reduction

Theorem (Many credits to be specified)

Let $\pi : X \to B$ be a dominant morphism of varieties in characteristic 0. Let $B^\circ \subset B$ be the locus where $\pi$ is smooth. There is an alteration $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{\text{main}}$, which is trivial over $B^\circ$, such that $X_1 \to B_1$ is semistable.
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*Proper, surjective, generically finite*

- One wants the **tight** result, with triviality over $B^\circ$ in order to compactify smooth families.
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- Major early results by [KKMS 1973], [de Jong 1997].
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- One wants the *tight* result, with triviality over $B^\circ$ in order to compactify smooth families.
- Some of this is work in preparation.
- Major early results by [KKMS 1973], [de Jong 1997].
- Wonderful results in positive and mixed characteristics by de Jong, Gabber, Illusie and Temkin.
Toroidalization and weak semistable reduction

Key results in characteristic 0:

**Theorem (Toroidalization, \(\aleph\)-Karu 2000, \(\aleph\)-K-Denef 2013)**

There is a modification \(B_1 \to B\) and a modification \(X_1 \to (X \times_B B_1)_{\text{main}}\) such that \(X_1 \to B_1\) is log smooth and flat.

**Theorem (Weak semistable reduction, \(\aleph\)-Karu 2000)**

There is an alteration \(B_2 \to B_1\) and a modification \(X_2 \to (X_1 \times_{B_1} B_2)\), trivial over \(B_1^0\), such that \(X_2 \to B_2\) is log smooth, flat, with reduced fibers.
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Theorem (Semistable reduction, Adiprasito-Liu-Temkin 2018)

There is an alteration $B_3 \to B_2$ and a modification $X_3 \to (X_2 \times_{B_2} B_3)$, trivial over $B_2^\circ$, such that $X_3 \to B_3$ is semistable.
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Passing from toroidalization to weak semistable reduction to semistable reduction was a purely combinatorial question [\(\mathbb{A}\)-Karu 2000].
Applications of loose semistable reduction

This is already useful for studying families:

Theorem (Karu 2000; K-SB 97, Alexeev 94, BCHM 11)

The moduli space of stable smoothable varieties is projective\(^a\).

\(^a\)in particular bounded and proper
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**Theorem (Viehweg-Zuo 2004)**

*The moduli space of canonically polarized manifolds is Brody hyperbolic.*
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Theorem (Fujino 2017)

*Nakayama’s numerical logarithmic Kodaira dimension is subadditive in families* \(X \to B\) *with generic fiber* \(F\):

\[
\kappa_\sigma(X, D_X) \geq \kappa_\sigma(F, D_F) + \kappa_\sigma(B, D_B).
\]
Main result
The following result is work-in-progress.

Main result (Functorial toroidalization, \(\aleph\)-Temkin-Włodarczyk)
Let \(X \to B\) be a dominant morphism.

- There are modifications \(B_1 \to B\) and \(X_1 \to (X \times_B B_1)_{\text{main}}\) such that \(X_1 \to B_1\) is log smooth and flat;
- this is compatible with base change \(B' \to B\);
- this is functorial, up to base change, with log smooth \(X'' \to X\).

Corollary
Tight semistable reduction holds in characteristic 0.
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Application:

Theorem (Deng 2018)

The moduli space of minimal complex projective manifolds of general type is Kobayashi hyperbolic.
dim $B = 0$: log resolution via principalization

- To resolve log singularities, one embeds $X$ in a log smooth $Y$...
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- One reduces to principalization of $\mathcal{I}_X$ (Hironaka, Villamayor, Bierstone–Milman).

**Theorem (Principalization ... Â-T-W)**

Let $\mathcal{I}$ be an ideal on a log smooth $Y$. There is a functorial logarithmic morphism $Y' \to Y$, with $Y'$ logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.
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**Figure:** The ideal $(u^2, x^2)$

Here $u$ is a monomial but $x$ is not.
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**Figure:** The ideal $(u^2, x^2)$ and the result of blowing up the origin, $\mathcal{I}_E^2$. Here $u$ is a monomial but $x$ is not.
Logarithmic order

Principalization is done by order reduction, using logarithmic derivatives.

- for a monomial $u$ we use $u \frac{\partial}{\partial u}$.
- for other variables $x$ use $\frac{\partial}{\partial x}$.  

Definition

Write $D \leq a$ for the sheaf of logarithmic differential operators of order $\leq a$.

The logarithmic order of an ideal $I$ is the minimum $a$ such that $D \leq a I = (1)$.

Take $u, v$ monomials, $x$ free variable, $p$ the origin.

$\logord_p(u^2, x) = 1$ (since $\frac{\partial}{\partial x} x = 1$)

$\logord_p(v^2, x^2) = 2$

$\logord_p(v + u) = \infty$ since $D \leq 1 I = D \leq 2 I = \cdots = (u, v)$. 

Abramovich

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- $\logord_p(u^2, x) = 1$ (since $\frac{\partial}{\partial x} x = 1$)
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- $\logord_p(v, x^2) = 2$
- $\logord_p(v + u) = \infty$ since $\mathcal{D}^{\leq 1} \mathcal{I} = \mathcal{D}^{\leq 2} \mathcal{I} = \cdots = (u, v)$. 
dim $B = 0$: sketch of argument, logord $< \infty$

- In characteristic 0, if $\logord_p(I) = a < \infty$, then $D^{\leq a^{-1}}I$ contains an element $x$ with derivative 1, a maximal contact element.
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- Carefully applying induction on dimension to an ideal on $\{x = 0\}$ gives order reduction (Encinas–Villamayor, Bierstone–Milman, Włodarczyk):
**dim \( B = 0 \): sketch of argument, logord < \( \infty \)**

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**Proposition (\ldots \&-T-W)**

*Let \( I \) be an ideal on a logarithmically smooth \( Y \) with* 

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- In characteristic 0, if $\text{logord}_p(I) = a < \infty$, then $D^{\leq a-1}I$ contains an element $x$ with derivative 1, a maximal contact element.
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**Proposition (…ℕ-T-W)**

Let $I$ be an ideal on a logarithmically smooth $Y$ with

$$\max_p \text{logord}_p(I) = a.$$  

There is a functorial logarithmic morphism $Y_1 \to Y$, with $Y_1$ logarithmically smooth, such that $\mathcal{O}_Y = M \cdot I_1$ with $M$ an invertible monomial ideal and

$$\max_p \text{logord}_p(I_1) < a.$$
Order reduction: Example 1

- Consider $Y_1 = \text{Spec} \mathbb{C}[u, x]$ and $D = \{u = 0\}$.
- Let $\mathcal{I} = (u^2, x^2)$.
- If one blows up $(u, x)$ the ideal is principalized:
Order reduction: Example 1

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$$IO Y'_1 = (u^2),$$

which is exceptional hence monomial.

This is in fact the only functorial admissible blowing up.
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  - on the $u$-chart $\text{Spec } \mathbb{C}[u, x']$ with $x = x'u$ we have $\mathcal{I} \mathcal{O}_{Y_1} = (u^2)$,
  - on the $x$-chart $\text{Spec } \mathbb{C}[u', x]$ with $u' = xu'$ we have $\mathcal{I} \mathcal{O}_{Y_1} = (x^2)$,
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  which is exceptional hence monomial.

- This is in fact the only functorial admissible blowing up.
Consider $Y_2 = \text{Spec } \mathbb{C}[\nu, x]$ and $D = \{\nu = 0\}$.

Let $\mathcal{I} = (\nu, x^2)$. 

Order reduction: Example 2

- Consider $Y_2 = \text{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$.
- Example 1 is the pullback of this via the log smooth $v = u^2$.
- Functoriality says: we need to blow up an ideal whose pullback is $(u, x)$.
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This means we need to blow up $(v^{1/2}, x)$. 
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Functoriality says: we need to blow up an ideal whose pullback is $(u, x)$.

This means we need to blow up $(v^{1/2}, x)$.

What is this? What is its blowup?
A Kummer monomial is a monomial in the Kummer-étale topology of $Y$ (like $v^{1/2}$).
Kummer ideals

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- A **Kummer monomial ideal** is a monomial ideal in the Kummer-étale topology of $Y$. 

A Kummer center is the sum of a Kummer monomial ideal and the ideal of a log smooth subscheme.
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- A **Kummer monomial ideal** is a monomial ideal in the Kummer-étale topology of $Y$.
- A **Kummer center** is the sum of a Kummer monomial ideal and the ideal of a log smooth subscheme.
- Locally $(x_1, \ldots, x_k, u_1^{1/d}, \ldots, u_{\ell}^{1/d})$. 
Blowing up Kummer centers

Proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth $Y$. There is a universal proper birational $Y' \to Y$ such that $Y'$ is logarithmically smooth and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.
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Example 0

$Y = \text{Spec } \mathbb{C}[v]$, with toroidal structure associated to $D = \{v = 0\}$, and $\mathcal{J} = (v^{1/2})$. 
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Example 0
$Y = \text{Spec } \mathbb{C}[v]$, with toroidal structure associated to $D = \{v = 0\}$, and $\mathcal{J} = (v^{1/2})$.

- There is no log scheme $Y'$ satisfying the proposition.
- There is a stack $Y' = Y(\sqrt{D})$, the Cadman–Vistoli root stack, satisfying the proposition!
Consider $Y_2 = \text{Spec } \mathbb{C}[v, x]$ and $D = \{v = 0\}$.

Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$. 

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- Consider $Y_2 = \text{Spec } \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.
- associated blowing up $Y' \to Y_2$ with charts:
  - $Y'_{x} := \text{Spec } \mathbb{C}[v, x, v']/(v'x^2 = v)$, where $v' = v/x^2$ (nonsingular scheme).
    - Exceptional $x = 0$, now monomial.
    - $\mathcal{I} = (v, x^2)$ transformed into $(x^2)$, invertible monomial ideal.
    - Kummer ideal $(v^{1/2}, x)$ transformed into monomial ideal $(x)$. 


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  - The $v^{1/2}$-chart:
    - stack quotient $X'_{v^{1/2}} := \left[\text{Spec } \mathbb{C}[w, y]/\mu_2\right]$, where $y = x/w$ and $\mu_2 = \{\pm 1\}$ acts via $(w, y) \mapsto (-w, -y)$.
    - Exceptional $w = 0$ (monomial).
    - $(v, x^2)$ transformed into invertible monomial ideal $(v) = (w^2)$.
    - $(v^{1/2}, x)$ transformed into invertible monomial ideal $(w)$. 

Abramovich
Proof of proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth $Y$. There is a universal proper birational $Y' \to Y$ such that $Y'$ is a logarithmically smooth stack and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.
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- Choose a stack $\tilde{Y}$ with coarse moduli space $Y$ such that $\tilde{\mathcal{J}} := \mathcal{J} \mathcal{O}_{\tilde{Y}}$ is an ideal.
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- Let $\tilde{Y}' \to \tilde{Y}$ be the blowup of $\tilde{\mathcal{J}}$, with exceptional $E$.
- Let $\tilde{Y}' \to B\mathbb{G}_m$ be the classifying morphism of $\mathcal{I}_E$. 
Let \( \mathcal{J} \) be a Kummer center on a logarithmically smooth \( Y \). There is a universal proper birational \( Y' \to Y \) such that \( Y' \) is a logarithmically smooth stack and \( \mathcal{J} O_{Y'} \) is an invertible ideal.

- Choose a stack \( \tilde{Y} \) with coarse moduli space \( Y \) such that \( \tilde{\mathcal{J}} := \mathcal{J} O_{\tilde{Y}} \) is an ideal.
- Let \( \tilde{Y}' \to \tilde{Y} \) be the blowup of \( \tilde{\mathcal{J}} \), with exceptional \( E \).
- Let \( \tilde{Y}' \to B\mathbb{G}_m \) be the classifying morphism of \( \mathcal{I}_E \).
- \( Y' \) is the relative coarse moduli space of \( \tilde{Y}' \to Y \times B\mathbb{G}_m \).
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Let \( \mathcal{J} \) be a Kummer center on a logarithmically smooth \( Y \). There is a universal proper birational \( Y' \to Y \) such that \( Y' \) is a logarithmically smooth stack and \( \mathcal{J}O_{Y'} \) is an invertible ideal.

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- \( Y' \) is the relative coarse moduli space of \( \tilde{Y}' \to Y \times B\mathbb{G}_m \).
- One shows this is independent of choices. ♠
**Key new ingredient:** The monomial part of an ideal

**Definition**

\( \mathcal{M}(\mathcal{I}) \) is the minimal monomial ideal containing \( \mathcal{I} \).
Key new ingredient: The monomial part of an ideal

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\( M(I) \) is the minimal monomial ideal containing \( I \).

Proposition (Kollár, \( \& \)-T-W)

1. In characteristic 0, \( M(I) = D^\infty(I) \). In particular \( \max_p \log_\rho(I) = \infty \) if and only if \( M(I) \neq 1 \).
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2. *Let \( Y_0 \to Y \) be the normalized blowup of \( M(\mathcal{I}) \).*
Key new ingredient: The monomial part of an ideal

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\( \mathcal{M}(\mathcal{I}) \) is the minimal monomial ideal containing \( \mathcal{I} \).

Proposition (Kollár, K-T-W)

(1) \textit{In characteristic 0,} \( \mathcal{M}(\mathcal{I}) = D^\infty(\mathcal{I}) \). \textit{In particular}
\[ \max_p \log \text{ord}_p(\mathcal{I}) = \infty \text{ if and only if } \mathcal{M}(\mathcal{I}) \neq 1. \]

(2) \textit{Let } \mathcal{Y}_0 \to \mathcal{Y} \textit{ be the normalized blowup of } \mathcal{M}(\mathcal{I}). \textit{Then}
\[ \mathcal{M} := \mathcal{M}(\mathcal{I})\mathcal{O}_{\mathcal{Y}_0} = \mathcal{M}(\mathcal{I}\mathcal{O}_{\mathcal{Y}_0}), \text{ and it is an invertible monomial ideal,} \]
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2. Let \( Y_0 \to Y \) be the normalized blowup of \( \mathcal{M}(\mathcal{I}) \). Then \( \mathcal{M} := \mathcal{M}(\mathcal{I})\mathcal{O}_{Y_0} = \mathcal{M}(\mathcal{I}\mathcal{O}_{Y_0}) \), and it is an invertible monomial ideal, and so \( \mathcal{I}\mathcal{O}_{Y_0} = \mathcal{I}_0 \cdot \mathcal{M} \) with \( \max_p \logord_p(\mathcal{I}_0) < \infty \).
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Definition

\( \mathcal{M}(\mathcal{I}) \) is the minimal monomial ideal containing \( \mathcal{I} \).

Proposition (Kollár, \( \aleph \)-T-W)

(1) In characteristic 0, \( \mathcal{M}(\mathcal{I}) = \mathcal{D}^\infty(\mathcal{I}) \). In particular, \( \max_p \log \text{ord}_p(\mathcal{I}) = \infty \) if and only if \( \mathcal{M}(\mathcal{I}) \neq 1 \).

(2) Let \( Y_0 \to Y \) be the normalized blowup of \( \mathcal{M}(\mathcal{I}) \). Then \( \mathcal{M} := \mathcal{M}(\mathcal{I})\mathcal{O}_{Y_0} = \mathcal{M}(\mathcal{I}\mathcal{O}_{Y_0}) \), and it is an invertible monomial ideal, and so \( \mathcal{I}\mathcal{O}_{Y_0} = \mathcal{I}_0 \cdot \mathcal{M} \) with \( \max_p \log \text{ord}_p(\mathcal{I}_0) < \infty \).

(1) \( \Rightarrow \) (2)

\( \mathcal{D}_{Y_0} \) is the pullback of \( \mathcal{D}_Y \), so (2) follows from (1) since the ideals have the same generators.
The monomial part of an ideal - proof

Proof of (1), basic affine case.

Let $\mathcal{O}_Y = \mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ and assume $\mathcal{M} = \mathcal{D}(\mathcal{M})$. The operators $1, u_1 \frac{\partial}{\partial u_1}, \ldots, u_l \frac{\partial}{\partial u_l}$ commute and have distinct systems of eigenvalues on the eigenspaces $\mathbb{C}[x_1, \ldots, x_n]$, for distinct monomials $u$. Therefore $\mathcal{M} = \bigoplus u \mathcal{M}_u$ with ideals $\mathcal{M}_u \subset \mathbb{C}[x_1, \ldots, x_n]$ stable under derivatives, so each $\mathcal{M}_u$ is either $(0)$ or $(1)$. In other words, $\mathcal{M}$ is monomial.

The general case requires more commutative algebra.
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The monomial part of an ideal - proof

Proof of (1), basic affine case.

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- Therefore \( \mathcal{M} = \bigoplus u \mathcal{M}_u \) with ideals \( \mathcal{M}_u \subset \mathbb{C}[x_1, \ldots, x_n] \) stable under derivatives,
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The general case requires more commutative algebra.
Arbitrary $B$

(Work in progress)

**Main result (ℵ-T-W)**

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{I} \subset \mathcal{O}_Y$ an ideal. There is a log morphism $B' \to B$ and functorial log morphism $Y' \to Y$, with $Y' \to B'$ logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

- This is done by relative order reduction, using relative logarithmic derivatives.
Arbitrary $B$

(Work in progress)

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Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{I} \subset \mathcal{O}_Y$ an ideal. There is a log morphism $B' \to B$ and functorial log morphism $Y' \to Y$, with $Y' \to B'$ logarithmically smooth, and $I\mathcal{O}_{Y'}$ an invertible monomial ideal.

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Definition

Write $\mathcal{D}^{\leq a}_{Y/B}$ for the sheaf of relative logarithmic differential operators of order $\leq a$. The relative logarithmic order of an ideal $\mathcal{I}$ is the minimum $a$ such that $\mathcal{D}^{\leq a}_{Y/B} \mathcal{I} = (1)$. 
The new step

- $\mathcal{M} := \mathcal{D}_Y^\infty_B \mathcal{I}$ is an ideal which is monomial along the fibers.
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- $\mathcal{M} := D_{Y/B}^\infty \mathcal{I}$ is an ideal which is monomial along the fibers.
- $\text{relord}_p(\mathcal{I}) = \infty$ if and only if $\mathcal{M} := D_{Y/B}^\infty \mathcal{I}$ is a nonunit ideal.
The new step

- $\mathcal{M} := \mathcal{D}_{Y/B}^\infty \mathcal{I}$ is an ideal which is monomial along the fibers.
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**Monomialization Theorem [\$-T-W\]**

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{M} \subset \mathcal{O}_Y$ an ideal with $\mathcal{D}_{Y/B} \mathcal{M} = \mathcal{M}$. There is a log morphism $B' \to B$ with saturated pullback $Y' \to B'$, such that $\mathcal{M} \mathcal{O}_{Y'}$ a monomial ideal.

After this one can proceed as in the case “$\text{dim } B = 0$”.
Proof of Monomialization Theorem, special case

Let $Y = \text{Spec } \mathbb{C}[u, v] \to B = \text{Spec } \mathbb{C}[w]$ with $w = uv$, and $\mathcal{M} = (f)$. 

Every monomial is either $u^\alpha w^k$ or $v^\alpha w^k$. Once again the operators $1, u \partial_u - v \partial_v$ commute and have different eigenvalues on $u^\alpha, v^\alpha$. Expanding $f = \sum u^\alpha f^\alpha + \sum v^\beta f^\beta$, the condition $\mathcal{M} = \partial Y / B \mathcal{M}$ gives that only one term survives, say $f = u^\alpha f^\alpha$, with $f^\alpha \in \mathbb{C}[w]$. Blowing up $(f^\alpha)$ on $B$ has the effect of making it monomial, so $f$ becomes monomial. ♠

The general case is surprisingly subtle.

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Moduli techniques in resolution of singularities
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- Expanding \( f = \sum u^\alpha f_\alpha + \sum v^\beta f_\beta \), the condition \( \mathcal{M} = D_{Y/B} \mathcal{M} \) gives that only one term survives,
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Thank you for your attention!