Resolution in characteristic 0 — why does it work?

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with Michael Tëmkin and Jarosław Włodarczyk
parallel work by M. McQuillan

Harmonies in Moduli Spaces

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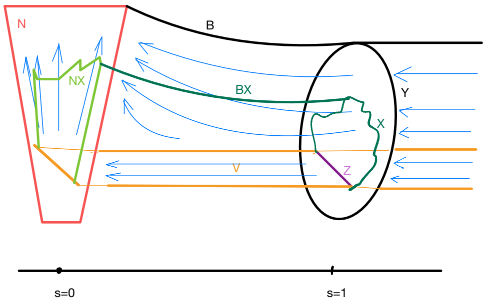
- in general stabilized by $\boldsymbol{\mu}_{w_i}$:
- $x_i = s^{w_i} \quad \rightsquigarrow \quad x_j = s^{w_j} x_j'$, not the usual charts.

Picture



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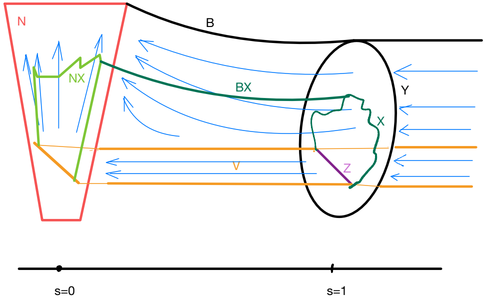

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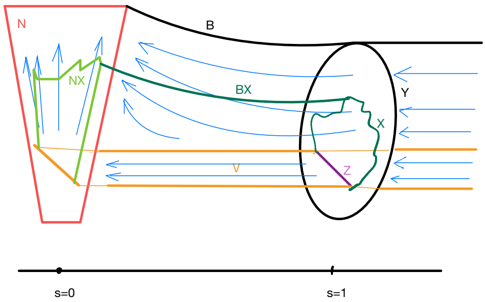

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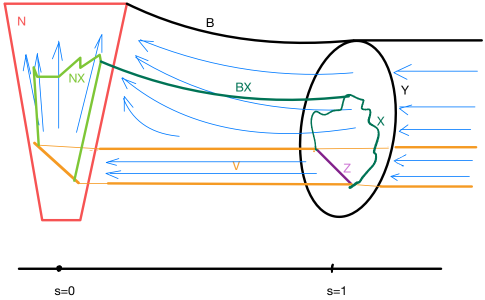

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Weighted blowups — global meaning

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- $V \ = \ V(\widetilde{R}_+), \qquad \widetilde{R}_+ := \bigoplus_{a>0} R_a.$

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- Smooth invariant if the maximal locus

$$\{p \in X \ : \ \mathrm{inv}_X(p) \text{ is maximal } \}$$

in X is smooth.

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2. *Assume a center $\bar{J}$ satisfying these conditions exists for every resolution situation $X \subset Y$. Then resolution of singularities holds.*

Proof of criterion

Proof of 2: denoting $X' = Bl_{\bar{J}}(X)$, and then $X^{(n)} = Bl_{\bar{J}_{X^{(n-1)}}}(X^{(n-1)})$, the invariants drop, and must stop when $X^{(n)}$ is smooth.

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Proof of 1:
Claim: The maximal locus W equals V.
With this, the invariant on B_+X drops. But $B_+X \to X'$ is smooth, so the invariant on X' drops.

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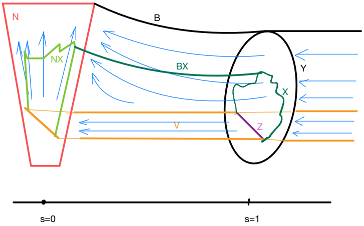

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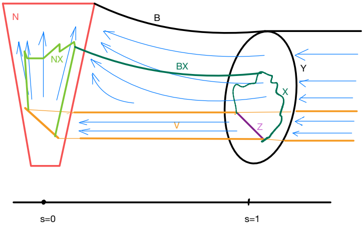

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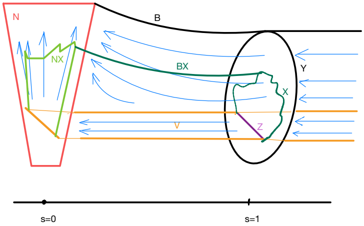

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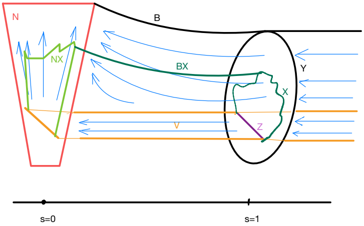

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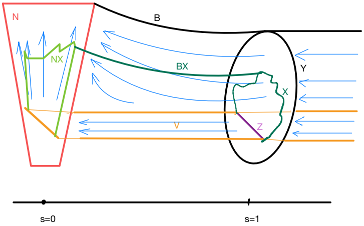

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- By functoriality any point $b \in N_Z \cap W$ comes along with its orbit.
- The limit point of the orbit lies in V, contradicting that V is a connected component of W.

Resolution exists

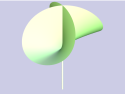
Theorem ($\aleph$-Tëmkin-Włodarczyk)

In characteristic 0, a smooth functorial singularity invariant and, for every $X \subset Y$, a center $\overline{J}$ satisfying (a) *and* (b) *exist.*

- In particular, by 2 resolution exists.
- I am not aware of a proof of Hironaka's theorem going through such criterion without weighted blowups.
- Let us see some examples.

Weighted blowup of Whitney's umbrella

- Let $X = V(x^2 - y^2 z)$.



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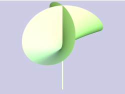

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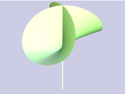

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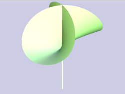

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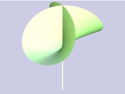

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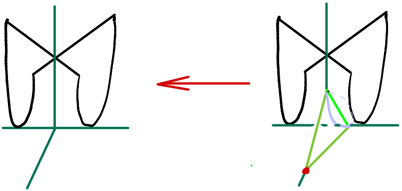
- This locus is removed in B_+, hence the invariant drops.

Weighted blowup of Whitney's umbrella — picture



The red dot is the μ_3 point $s = z' = y' = 0$.
The green triangle is the exceptional weighted projective plane $s = 0$.
The light green line is the μ_2-locus $s = x' = 0$.
The purple curve is the intersection of the umbrella with the exceptional.

Standard blowup of Whitney's umbrella

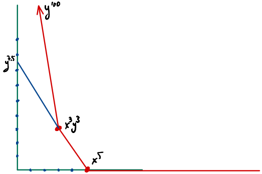
- Consider the same X but with the blowup but with $\overline{J} = (x, y, z)$.
- In this case the proper transform BX is defined by the equation

$$x'^2 + y'^2 z' s.$$

- We notice that at the point where $x' = y' = z' = s = 0$ has these variables appearing in degrees $2, 4, 4, 4$ which is larger than the original $2, 3, 3$.
- In other words, it does not satisfy the criterion.
- As is well-known, the standard blowup of X has an isomorphic singularity: for any $z' \neq 0$ the equation above is isomorphic to X, so the invariant stays the same after blowup.
- In the example of $x^2 - yzwt$, the standard blowup would actually get things worse.

Resolving $X = V(x^5 + x^3y^3 + y^{100})$

Consider the newton polyhedron



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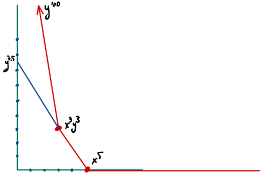

- $\mathrm{inv}(0) = (5, 7.5), J = (x^5, y^{7.5}), \overline{J} = (x^{1/3}, y^{1/2})$.
- The proper transform is $x'^5 + x'^3y'^3 + y'^{100}s^{185}$ and the criterion holds.

Resolving $X = V(x^5 + x^3y^3 + y^{100})$

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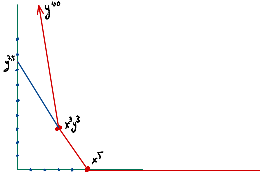

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- The proper transform is $x'^5 + x'^3y'^3 + y'^{100}s^{185}$ and the criterion holds.
- The invariant of the blowup is $(3, 185) < (5, 7.5)$ lexicographically.
- The next weighted blowup is nonsingular.

Thanks for years of friendship and inspiration

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. . . wishing you many more happy and productive years!