Singularities and their resolutions

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On Singularities - Part 1

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My subject: algebraic geometry

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- These sets are called *algebraic varieties*.\(^1\)

\(^1\)affine
Examples of algebraic varieties

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Singular and smooth points

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**Definition**

\{ V = f(x_1, \ldots, x_n) = 0 \} is singular at \( p \) if \( \frac{\partial f}{\partial x_i}(p) = 0 \) for all \( i \), namely \( \nabla f(p) = 0 \).

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(In codimension \( c \), the singular locus of \( \{ f_1 = \cdots = f_k = 0 \} \) is the set of points where \( d(f_1, \ldots, f_k) \) has rank \(< c \).)
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Looks like in general it might be hard to find the singularities. There is a theorem saying that it is.
Resolution of singularities

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A resolution of singularities $X' \to X$ is a modification\(^a\) with $X'$ nonsingular inducing an isomorphism over the smooth locus of $X$.

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**Theorem (Hironaka 1964)**

A complex algebraic variety $X$ admits a resolution of singularities $X' \to X$, so that the critical locus $E \subset X'$ is a simple normal crossings divisor.\(^a\)

\(^a\) Codimension 1, smooth components meeting transversally
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figures by Herwig Hauser, https://imaginary.org/gallery/herwig-hauser-classic
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Why should we ‘get rid of them’?
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Example: Stepanov’s theorem

If $X' \to X$ a resolution with $E \subset X'$ a simple normal crossings divisor, define $\Delta(E)$ to be the dual complex of $E$. 

Theorem (Stepanov 2006)

The simple homotopy type of $\Delta(E)$ is independent of the resolution $X' \to X$.

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- Stephen Obinna and Ming-Hao Quek are PhD students at Brown who will prove a generalization of that paper.
The end

Thank you for your attention