ON TAME STACKS IN POSITIVE CHARACTERISTIC

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Contents

1. Linearly reductive finite group schemes

1.1. **Equivariant sheaves.** All group schemes will be flat, finite and finitely presented over an arbitrary scheme.

Such a group scheme $G \to S$ will be called *constant* if G is the product of S by a finite group.[1] ←1

Let $\pi\colon G \to S$ be such a scheme. We will denote by $\mathrm{QCoh}(S)$ the category of quasi-coherent sheaves on S, and by $\mathrm{QCoh}^G(S)$ the category of G-equivariant quasi-coherent schemes over over S. We can think of $\mathrm{QCoh}^G(S)$ as the category of quasi-coherent schemes over the classifying stack $\mathcal{B}_S G$. When S is locally noetherian, we also denote by Coh and Coh^G the categories of coherent sheaves, respectively without and with a G-action.

There are three ways of defining $\mathrm{QCoh}^G(S)$.

(a) A quasi-coherent sheaf F on S extends naturally to a functor $F\colon (\mathrm{Sch}/S)^{\mathrm{op}} \to (\mathrm{Set})$. If $f\colon T \to S$ is a morphism of schemes, then we define $F(T)$ as $(f^*F)(T)$.

Then an action of G on F is an action of the functor $G\colon (\mathrm{Sch}/S)^{\mathrm{op}} \to (\mathrm{Grp})$ on F. In other words, for each $T \to S$ we have an action of the group $G(T)$ on the set $F(T)$, and that is functorial in $T \to S$.

The morphisms are defined in the obvious way.

Date: July 20, 2006.

[1](Angelo) added this

(b) We have a sheaf of commutative Hopf algebras $\pi_*\mathcal{O}_G$ on S. Then an action of G on a quasi-coherent sheaf F is defined as a coaction $F \to F \otimes_{\mathcal{O}_S} \pi_*\mathcal{O}_G$ of this sheaf on F. Equivalently, in terms of the dual Hopf algebra $\mathbf{H}_G = (\pi_*\mathcal{O}_G)^\vee$, the "convolution hyperalgebra of G", it is an action $F \otimes \mathbf{H}_G \to F$.

(c) Finally, $\mathrm{QCoh}^G(S)$ is the category of quasi-coherent sheaves on the classifying stack $\mathcal{B}_S G$. This means the following. An object F of $\mathrm{QCoh}^G(S)$ associates with each G-torsor $P \to T$ an $\mathcal{O}(T)$-module $F(P \to T)$; also, for each commutative diagram

$$\begin{array}{ccc} P' & \xrightarrow{g} & P \\ \downarrow & & \downarrow \\ T' & \xrightarrow{f} & T \end{array} \tag{1.1}$$

where the columns are G-torsors and g is G-equivariant, we have a homomorphism $F(P \to T) \to F(P' \to T')$ that is linear with respect to the natural ring homomorphism $\mathcal{O}(T) \to \mathcal{O}(T')$. These data are required to satisfy the following conditions.

(i) Suppose that we are given a G-torsor $P \to T$. Then we get a presheaf of $\mathcal{O}_S$-modules $F_{P\to T}$ defined by sending a Zariski-open subscheme U to the $\mathcal{O}(U)$-module $F(P\mid_U \to U)$. We assume that this a quasi-coherent sheaf on S.

(ii) Suppose that we have a commutative diagram like (1.1). Then we get a homomorphism of quasi-coherent sheaves $F_{P\to T} \to f_* F_{P'\to T'}$, defined by the given homomorphism

$$\begin{array}{ccc} F_{P\to T}(U) & & F_{P'\to T'}\big(f^{-1}(U)\big) \\ \| & & \| \\ F(P\mid_U \to U) & \longrightarrow & f^{-1}(U)\big) \end{array}$$

for each open subscheme $U \subseteq T$. Then the corresponding homomorphism $f^* F_{P\to T} \to F_{P'\to T'}$ is supposed to be an isomorphism.

Suppose that $\phi\colon H \to G$ is a homomorphism of group schemes, there are two natural additive functors, the restriction functor

$$\phi^*\colon \mathrm{QCoh}^G(S) \longrightarrow \mathrm{QCoh}^H(S)$$

and the induction functor

$$\phi_*\colon \mathrm{QCoh}^H(S) \longrightarrow \mathrm{QCoh}^G(S).$$

2→ The first is evident.[2] The second, ϕ_* can be defined using functrial actions, or using Hopf algebras - $\phi_* F = F \otimes_{\mathbf{H}_H} \mathbf{H}_G$, but quite usefully

[2](Dan) Changed text

these may be thought of as follows: ϕ induces a morphism of algebraic stacks

$$\Phi\colon \mathcal{B}_S H \longrightarrow \mathcal{B}_S G$$

defined as usual by sending a principal H-bundle $Q \to T$ to the principal G-bundle $(Q \times_T G_T)/H_T$, with H_T acting "in the middle". Then ϕ^* is pullback of quasi-coherent sheaves along Φ, while ϕ_* is pushforward along Φ.

A few important points about these functors:[3] ←3

(1) The functor ϕ^* is always exact. Indeed, in terms of actions, $\phi^* F$ is the same sheaf F but with the G action replaced by the action of H through ϕ, and the action does not intervene in exactness.
(2) If H is a group subscheme of G, then Φ is finite, and in particular affine; hence ϕ_* is exact. In this case we denote it by Ind_H^G.[4] ←4
(3) If we think of the structure morphism $\pi\colon G \to S$ as a homomorphism to the trivial group scheme and F is a G-equivariant quasi-coherent sheaf on S, then we denote $\pi_* F$ by F^G. This quasi-coherent sheaf F^G is naturally embedded in F, and is called the *invariant subsheaf*.
(4) Suppose $\phi : H \to G$ is surjective, with kernel a flat group scheme K. For $F \in \mathrm{QCoh}^H(S)$ we have $\phi_* F = F^K$ with the induced action of G. On the other hand if $F \in \mathrm{QCoh}^G(S)$ then the adjunction morphism $F \to \phi_*\phi^* F$ is an isomorphism, since the action of H on $\phi^* F$ is trivial. In other words, we have a canonical isomorphism $\phi_* \circ \phi^* \simeq \mathrm{id}$.

1.2. **Linearly reductive group schemes.**

Definition 1.1. A group scheme $G \to S$ is *linearly reductive* if the functor $\mathrm{QCoh}^G(S) \to \mathrm{QCoh}(S)$ sending F to F^G is exact.

Definition 1.2. Assume that S is noetherian. Then G is linearly reductive if and only if the functor $\mathrm{Coh}^G(S) \to \mathrm{Coh}(S)$ defined as $F \mapsto F^G$ is exact.

Proof. This follows immediately from the fact that every quasi-coherent sheaf with an action of G is a direct limit of coherent sheaves with an action of G (see [?]).[5] ♠ ←5

[3](Dan) I added a few here.

[4](Dan) I'm pretty sure this is the right order - we induce from H up to G. I changed all along. This is Serre's notation.

[5](Dan) What's a reference?

In particular, if k is a field, the category of coherent sheaves with an action of G is equivalent to the category of finite-dimensional representations of G; hence a finite group scheme over a field is linearly reductive if and only if the functor $V \mapsto V^G$, from finite-dimensional representations of G to vector spaces, is exact.

Another, perhaps more customary, way to state this condition is to require that every finite-dimensional representation of G be a sum of
6→ irreducible representations.[6]

Proposition 1.3. *Let $S' \to S$ be a morphism of schemes, $G \to S$ a group scheme, $G' \stackrel{\text{def}}{=} S' \times_S G$.*

(a) *If $G \to S$ is linearly reductive, then $G' \to S'$ is linearly reductive.*
(b) *If $G' \to S'$ is linearly reductive and $S' \to S$ is flat and surjective, then $G \to S$ is linearly reductive.*

Proof. Let us prove part (b).

There is a cartesian diagram

$$\begin{array}{ccc} \mathcal{B}_{S'}G' & \xrightarrow{g} & \mathcal{B}_S G \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{f} & S \end{array}$$

from which we deduce that the two functors $f^*\pi_*$ and π'_*g^* are isomorphic. Since f is flat, g is flat as well; also π'_* is exact by assumption, so π'_*g^* is exact, hence $f^*\pi_*$ is exact. but since f is faithfully flat we have that π_* is exact, as required.

Now for part (a).

First assume that S' is an open subscheme of S. Then every exact sequence

$$0 \longrightarrow F'_1 \longrightarrow F'_2 \longrightarrow F'_3 \longrightarrow 0$$

of G-equivariant quasi-coherent sheaves on S' extends to an exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

of G-equivariant quasi-coherent sheaves on S. Since taking invariants commutes with restriction to open subschemes, the result follows.

Now, if $\{S_i\}$ is an open covering of S by affines, and each restriction G_{S_i} is linearly reductive over S_i is linearly reductive, then the disjoint union $\sqcup_i G_{S_i}$ is linearly reductive over $\sqcup_i S_i$; and we conclude from part (b) that G is linearly reductive over S. Hence being linearly reductive is a local property in the Zariski topology. So to prove part (a) we may assume that S and S' are both affine. In

[6](Angelo) Added this.

this case $g_*\colon \mathrm{QCoh}^{G'}(S') \to \mathrm{QCoh}^{G}_{S}$ has the property that a sequence $M_1' \to M_2' \to M_3'$ in $\mathrm{QCoh}^{G'}(S')$ is exact if and only if the induced sequence $g_*M_1' \to g_*M_2' \to g_*M_3'$ is. Furthermore

$$(g_*M')^G = \pi_* g_* M' = f_* \pi'_* M' = f_*\bigl(M'^{G'}\bigr).$$

The result follows easily.
[7] [8]

♠ ←7
←8

Proposition 1.4. *The class of linearly reductive group schemes is closed under taking*

(a) *subgroup schemes,*
(b) *quotients, and*
(c) *extensions.*

Proof. For part (a), consider a subgroup-scheme $G'' \subset G$ and the resulting commutative diagram

$$\begin{array}{ccc} \mathcal{B}_S G' & \xrightarrow{\ i\ } & \mathcal{B}_S G \\ & \searrow^{\pi_{G'}} & \downarrow{\scriptstyle \pi_G} \\ & & S. \end{array}$$

It is enough to observe that $i_* = \mathrm{Ind}_{G'}^{G}$ is exact as i is affine. Since π_{G*} is exact by assumption, and since $\pi_{G'*} \simeq \pi_{G*} \circ i_*$ [9] we have that $\pi_{G'*}$ ←9 is exact, as required.

For parts (b) and (c), consider an exact sequence

$$1 \to G' \to G \to G'' \to 1$$

and the corresponding commutative diagram

$$\begin{array}{ccccc} \mathcal{B}_S G' & \xrightarrow{\ i\ } & \mathcal{B}_S G & \xrightarrow{\ j\ } & \mathcal{B}_S G'' \\ & \searrow^{\pi_{G'}} & \downarrow{\scriptstyle \pi_G} & \swarrow_{\pi_{G''}} & \\ & & S & & \end{array}$$

To prove part (b), suppose that G is linearly reductive, so π_{G*} is exact. Recall that j^* is exact and $j_* \circ j^*$ is isomorphic to the identity, so

$$\pi_{G''*} \simeq \pi_{G''*} \circ j_* \circ j^* \simeq \pi_{G*} \circ j^*$$

is exact, as required.

[7](Dan) I rewrote this proof - please check!!

[8](Angelo) I restored the old proof, as I don't think that the new one is correct (in general your V is not affine over S, not even quasi-compact).

[9](Angelo) I took out "(Frobenius reciprocity)", because this is not Frobenius reciprocity.

For part (c), we have by assumption that $\pi_{G'*}$ and $\pi_{G''*}$ are exact. Considering the cartesian diagram

$$\begin{array}{ccc} \mathcal{B}_S G' & \xrightarrow{\ i\ } & \mathcal{B}_S G \\ {\scriptstyle \pi_{G'}}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ S & \longrightarrow & \mathcal{B}_S G'', \end{array}$$

since $S \to \mathcal{B}_S G''$ is faithfully flat we have that j_* is exact (concretely, taking invariants of a G sheaf by G' is exact even if when consider the induced G''-action). So

$$\pi_{G*} = \pi_{G''*} \circ j_*$$

is exact, as required. ♠

1.3. **Classifying linearly reductive group schemes.** Recall that a finite group scheme $\Delta \to S$ is said to be *diagonalizable* if it is abelian and its Cartier dual is étale. A finite étale group scheme $H \to S$ is said to be *tame* if its degree is prime to all residue characteristics.

Definition 1.5. A group scheme $\pi\colon G \to S$ is *well-split* when it is an extension

$$1 \longrightarrow \Delta \longrightarrow G \longrightarrow H \longrightarrow 1$$

where Δ is diagonalizable and H is étale and tame.

It is *locally well-split* if there is an fpqc covering $S' \to S$, such that $S' \times_S G \to S'$ is well-split.

In caracteristic 0 every finite flat group scheme is étale and tame, hence it is well-split.

Proposition 1.6. *Every locally well-split group scheme is linearly reductive.*

10→ *Proof.* By Propositions 1.3 it suffices to consider well-split group schemes.[10]

Since Δ is diagonalizable, after an étale surjective base change $S' \to S$ we have $\Delta_{S'}$ the Cartier dual of a constant abelian group scheme, hence a product $\Delta_{S'} \simeq \prod_i \boldsymbol{\mu}_{r_i}$. Since the structure sheaf $\mathcal{O}_{\boldsymbol{\mu}_{r_i}} = \operatorname{Spec}_{\mathcal{O}_{S'}}[x]/(x^{r_i})$ decomposes as a direct sum of linear characters, every representation does, so $\boldsymbol{\mu}_{r_i}$ is linearly reductive, so by Proposition 1.4 $\Delta_{S'}$ is linearly reductive, and by Proposition 1.3 Δ is linearly reductive.

A tame étale group schemes is well known to be linearly reductive by Maschke's Lemma. Propositions 1.4 says G is linearly reductive, as required. ♠

[10](Dan) I changed the proofs here too

Over perfect fields the situation is simple.

Lemma 1.7. *Let k be a perfect field, $G \to \operatorname{Spec} k$ a group scheme. If G is well-split, then there exists a connected diagonalizable group scheme Δ_0 and a tame étale group scheme H, both over k, together with an action of H onto Δ_0, such that $G \simeq H \ltimes \Delta_0$.*[11] ←11

Proof. By definition, G contains a diagonalizable subgroup Δ such that G/Δ is étale; hence the connected component Δ_0 of the identity in G coincides with the connected component in Δ; hence it is digonalizable, and the quotient $H \overset{\text{def}}{=} G/\Delta_0$ is étale and tame.

To get a splitting, it is enough to notice that the reduced subscheme G_{red}, which is a group subscheme because k is perfect, maps isomorphically to H. ♠

Proposition 1.8. *Let k be a field, $G \to \operatorname{Spec} k$ a finite group scheme. Then G is linearly reductive if and only if it is locally well-split.*

Proof. Let $\overline{k}$ be the algebraic closure of k; then by Proposition 1.3 $G_{\overline{k}}$ is linearly reductive (or locally well-split) if and only if G is linearly reductive (respectively locally well-split); so we may assume that k is algebraically closed. We know that locally well-split groups are linearly reductive, so assume that G is linearly reductive. Call p the characteristic of k.

Let G_0 be the connected component of the identity in G. Then G/G_0 is a linearly reductive constant group. If it were not tame it would contain a subgroup of order p, which is not linearly reductive. So we may assume that G is connected, and show that it is diagonalizable.

The following lemma may be known to the experts, but we have not found a reference.

Lemma 1.9. *If a connected finite group scheme G contains a diagonalizable normal subgroup H, and $Q = G/H$ is again diagonalizable, then G is also diagonalizable.*

Proof. If we show that G is abelian, then it is diagonalizable: its [12] ←12 Cartier dual is an extention with an étale quotient and an étale subgroup, which is therefore an étale group-scheme.

The action by conjugation of G on H defines a homomorphism of group schemes $G \to \underline{\operatorname{Aut}}_{Gr-Sch/k}(H) = \underline{\operatorname{Aut}}_{Gr-Sch/k}(H^C)$, where H^C is the Cartier dual of H; but the domain is local, while the target is

[11](Angelo) I corrected this statement by adding the condition that k be perfect; the statement is false otherwise.

[12](Angelo) Corrected typo

constant, so this homomorphism is trivial. Equivalently, H is central in G.

Let A be a commutative k-algebra. The groups $H(A)$ and $G(A)/H(A)$ are commutative, hence, by "calculus of commutators" [1], Section 6, in particular Lemma 6.1 we have a bilinear map

$$\begin{aligned} G(A) \times G(A) &\longrightarrow H(A) \\ (x, y) &\mapsto [x, y] \end{aligned}$$

This is functorial in A, therefore the commutator gives a bilinear map $G \times G \to H$, and since H is central this gives a bilinear map $Q \times Q \to H$; in particular we get a map of sheaves $Q \to \underline{\mathrm{Hom}}_{Grp-Sch/k}(Q, H)$, where both source and target are representable. But again the domain is local and the target is étale, hence the Q is mapped to the trivial map, in other words the commutator $Q \times Q \to H$ maps to the identity in H. This means that the commutator is trivial, hence G abelian.

♠

So we may proceed by induction on the dimension of the vector space $\mathrm{H}^0(G, \mathcal{O}_G)$, and assume that G does not contain any proper normal subgroup scheme. In particular, the Frobenius kernel G_1 of G is a normal subgroup scheme of G, which does not coincide with the identity, unless G is trivial: so we have that $G = G_1$ - in [6], p. 139 one says that G has *height* 1. Connected group schemes of height 1 are classified by their p-Lie algebras (see, e.g., [6], p. 139).

Lemma 1.10 (Jacobson [3], Chapter 5, Exercise 14, p. 196)**.** *Let G be a non-abelian group scheme of height 1. Then G contains $\boldsymbol{\alpha}_p$, and hence is not linearly reductive.*

Proof. Considering the p-lie algebra $\mathfrak{g}$ of G, we need to find an element $w \in \mathfrak{g}$ such that $w^p = 0$. Since $\mathfrak{g}$ is finite dimensional, for each $v \in \mathfrak{g}$ there is a minimal n such that $\{v, v^p, v^{p^2}, \dots, v^{p^n}\}$ is linearly dependent, giving a monic p-polynomial

$$f_v(x) = x^{p^n} + a_{n-1}^{(v)} x^{p^{n-1}} + \cdots + a_0^{(v)} x$$

such that $f_v(v) = 0$.

Note that if $a_0^{(v)} = 0$ then the nonzero element

$$w = f_v^{1/p}(v) = v^{p^{n-1}} + (a_{n-1}^{(v)})^{1/p} v^{p^{n-2}} + \cdots + (a_1^{(v)})^{1/p} v$$

satisfies $w^p = 0$. So, arguing by contradiction, we may assume that $a_0^{(v)} \neq 0$ for all nonzero v, i.e. f_v is separable. Since the minimal polynomial of $\mathrm{ad}(v)$ divides f_v, we have that $\mathrm{ad}(v)$ is semisimple for every nonzero v.

Since $\mathfrak{g}$ is assumed non-commutative, there is v with $\mathrm{ad}(v) \neq 0$, hence it has a nonzero eigenvector v' with nonzero eigenvalue. But then the action of $\mathrm{ad}(v')$ on $\mathrm{Span}(v, v')$ is nonzero nilpotent, contradicting semisimplicity. ♠

Back to the proposition, we deduce G is abelian. Every subgroup scheme is normal, so G can not contain any proper subgroup scheme. But by Cartier duality the only local abelian group schemes with this property are $\boldsymbol{\alpha}_p$ and $\boldsymbol{\mu}_p$; and again $\boldsymbol{\alpha}_p$ is not linearly reductive. Hence $G = \boldsymbol{\mu}_p$, and we are done. ♠

Lemma 1.11. *Assume that there is a point $s = \operatorname{Spec} k(s) \in S$, such that the fiber $G_s \to \operatorname{Spec} k(s)$ is locally well-split. Then there exists a flat quasi-finite map $U \to S$ of finite presentation, whose image includes s, a diagonalizable group scheme $\Delta \to U$ and an étale tame group scheme $H \to U$ acting on Δ, such that $G_U \to U$ is isomorphic to the semi-direct product $H \ltimes \Delta$.*

In particular, let V be the image of U in S, which is open; then the restriction $G_V \to V$ is locally well-split.

Proof. By standard arguments, we may assume that S is connected, affine and of finite type over $\mathbb{Z}$. There is a finite extension k of $k(s)$ such that G_k is of the form $H \ltimes \Delta_0$, where Δ_0 is a connected diagonalizable group scheme and H is an étale group scheme, associated with a finite group Γ. After base change by a finite flat morphism over S, we may assume that $k(s) = k$. The group scheme Δ_0 extends uniquely to a diagonalizable group scheme Δ_0 on S, that we still denote by Δ_0. Also, we denote again by H the group scheme over S associated with H; the action H on Δ_0 that is defined over s extends uniquely to an action of H on Δ_0. Set $G' = H \ltimes \Delta_0$. We claim that G and G' become isomorphic after passing to a flat morphism of finite type $U \to S$, whose image includes s.

We present two methods of proof, one abstract and one more explicit. Both use deformation theory.[13] ←13

Method 1: rigidity using the cotangent complex. It suffices to show that the group scheme G_k has no nontrivial formal deformations as a group scheme, equivalently no nontrivial infinitesimal deformations, which is the same as saying that $\mathcal{B}G_k$ has no nontrivial infinitesimal deformations. According to [5], as corrected in [7], there is a contangent complex $\mathbf{L}_{\mathcal{B}G_k} \in D(Coh(\mathcal{B}G_k))$ satisfying the requirements for applying [2], therefore first order deformations of $\mathcal{B}G_k$ lie in $\mathrm{Ext}^1(\mathbf{L}_{\mathcal{B}G_k}, \mathcal{O}_{\mathcal{B}G_k})$. Since $\mathcal{B}G_k$ is the quotient of a smooth scheme by a

[13](Dan) added stuff

smooth group action, it is easy to see that $\mathbf{L}_{\mathcal{B}G_k} \in D^{[0,1]}(Coh(\mathcal{B}G_k))$, therefore $\mathcal{E}xt^\bullet(\mathbf{L}_{\mathcal{B}G_k}, \mathcal{O}_{\mathcal{B}G_k}) \in D^{[-1,0]}(Coh(\mathcal{B}G_k)$, and since π_* is exact, $\operatorname{Ext}^\bullet(\mathbf{L}_{\mathcal{B}G_k}, \mathcal{O}_{\mathcal{B}G_k}) \in D^{[-1,0]}(Coh(\mathcal{B}G_k)$, in particular $\operatorname{Ext}^1 = 0$.

Method 2: lifting using lie-algebra cohomology Denote by R the henselization of the local ring $\mathcal{O}_{S,s}$, by $\mathfrak{m}$ is maximal ideal, and set $R_n \overset{\text{def}}{=} R/\mathfrak{m}^{n+1}$, $S_n \overset{\text{def}}{=} \operatorname{Spec} R_n$, $G_n \overset{\text{def}}{=} G_{S_n}$ and $G'_n \overset{\text{def}}{=} G'_{S_n}$. Clearly $S_0 = s$.

Let us start from the tautological G_0-torsor $S_0 \to \mathcal{B}_{S_0}G_0$, which we think of as a G'-torsor. Our aim now is to construct a sequence of G'-torsors $P_n \to \mathcal{B}_{S_n}G_n$, such that the restriction of each P_n to S_{n-1} is isomorphic to the G'-torsor $P_{n-1} \to \mathcal{B}_{S_{n-1}}G_{n-1}$.

The Lie algebra $\mathfrak{g}$ of $G_0 = G'_0$ is a representation of G_0, corresponding to a coherent sheaf on $\mathcal{B}_{S_0}G_0$. It is well known that the obstruction to extending $P_{n-1} \to \mathcal{B}_{S_{n-1}}G_{n-1}$ to a G'-torsor lies in the sheaf cohomology $\mathrm{H}^2\big(\mathcal{B}_{S_0}G_0, (\mathfrak{m}^n/\mathfrak{m}^{n+1})\otimes\mathfrak{g}\big)$; and this coincides with the cohomology of G_0 in the representation $(\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes \mathfrak{g}$, which is 0, because G_0 is linearly reductive.

Each G'-torsor $P_n \to \mathcal{B}_{S_n}G_n$ yields a (G, G')-bitorsor

$$I_n \overset{\text{def}}{=} S_n \times_{\mathcal{B}_{S_n}G_n} P_n \longrightarrow S_n,$$

where the morphism $S_n \to \mathcal{B}_{S_n}G_n$ is the one given by the trivial torsor $G_n \to S_n$. So we obtain a sequence of (G, G')-bitorsors $I_n \to S_n$, such that the restriction of each I_n to S_{n-1} is isomorphic to I_{n-1}.

By Artin's approximation theory, this shows that there is a (G, G')-bitorsor $I_R \to \operatorname{Spec} R$; and, by standard limit arguments, this proves that there is an étale morphism $U \to S$ containing s in its image, together with a (G, G')-bitorsor $I_U \to U$. The pullbacks of G and G' to I_U are isomorphic, and the composite $I_U \to U \to S$ is flat and quasi-finite. This completes the proof. ♠

14→ Here is our main result on linearly reductive group schemes.[14]

Theorem 1.12. *Let $G \to S$ be a finite flat group scheme. The following conditions are equivalent.*

(a) *$G \to S$ is linearly reductive.*
(b) *$G \to S$ is locally well-split.*
(c) *The geometric fibers of $G \to S$ are well-split.*

Furthermore, if S is noetherian we can add the following two conditions.[15] 15→

[14](Dan) added stuff here

[15](Angelo) Added a noetherian condition. I think that there are non-noetherian schemes without any closed points (I may be wrong).

(d) *The closed fibers of G are linearly reductive.*
(e) *The geometric closed fibers of $G \to S$ are well-split.*

Proof. This follows from Proposition 1.8 and Lemma 1.11. ♠

1.4. **Étale local extensions of linearly reductive group schemes.** [16] ←16

We will need the following result.

Proposition 1.13. *Let S be a scheme, $p \in S$ a point, $G_0 \to p$ a linearly reductive group scheme. There there exists an étale morphism $U \to S$, with a point $q \in U$ mapping to p, and a linearly reductive group scheme $\Gamma \to U$ whose restriction $\Gamma_q \to q$ is isomorphic to the pullback of G_0 to q.*

Proof. Let us start with a Lemma. Let k be a field, $G \to \operatorname{Spec} k$ a well-split group scheme. Let Δ be the connected component of the identity of G, $H = G/\Delta$. Call $\underline{\mathrm{Aut}}_k(G)$ the group scheme representing the functor of isomorphisms of G as a group scheme: there a homomorphism $\Delta \to \underline{\mathrm{Aut}}_k(G)$ sending each section of Δ into the corresponding inner automorphism of G; this induces an embedding $\Delta/\Delta^H \subseteq \underline{\mathrm{Aut}}_k(G)$.

Lemma 1.14. *The connected component of the identity of $\underline{\mathrm{Aut}}_k(G)$ is Δ/Δ^H.*

Proof. [17] Since Δ is a characteristic subgroup scheme of G, each automorphism of $G_A \to \operatorname{Spec} A$, where A is a k-algebra, preserves Δ_A. ←17
Hence we get homomorphisms of group schemes $\underline{\mathrm{Aut}}_k(G) \to \underline{\mathrm{Aut}}_k(\Delta)$ and $\underline{\mathrm{Aut}}_k(G) \to \underline{\mathrm{Aut}}_k\big(H\big)$, inducing a homomorphism

$$\underline{\mathrm{Aut}}_k(G) \longrightarrow \underline{\mathrm{Aut}}_k(\Delta) \times \underline{\mathrm{Aut}}_k\big(H\big);$$

the kernel of this homomorphism contains Δ/Δ^H. Let us denote by E this kernel, since $\underline{\mathrm{Aut}}_k(\Delta) \times \underline{\mathrm{Aut}}_k\big(H\big)$ is étale over $\operatorname{Spec} k$, it is enough that to prove that E coincides with Δ/Δ^H.

To do this, we may pass to the algebraic closure of k, and assume that k is algebraically closed; then it is enough to prove that given a k-algebra A, for any element $\alpha \in E(A)$ there exists a faithfully flat extension $A \subseteq A'$ such that the image of α in $E(A')$ comes from $(\Delta/\Delta^H)(A')$.

By passing to a faithfully flat extension, we may assume that $G(B) \to H(B)$ is surjective for any A-algebra B (because H is constant), so we have an exact sequence

$$1 \longrightarrow \Delta(B) \longrightarrow G(B) \longrightarrow H(B) \longrightarrow 1.$$

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[16] (Angelo) Added this subsection

[17] (Angelo) I corrected the proof of this lemma.

Furthermore, again because H is constant, for any A-algebra B we have

$$\Delta^H(B) = \Delta(B)^{H(B)};$$

hence for any B we have an injective homomorphism

$$\Delta(B)/\Delta(B)^{H(B)} \longrightarrow (\Delta/\Delta^H)(B).$$

Let us show that α comes from $(\Delta/\Delta^H)(A)$.

Set $B = \Gamma(G, \mathcal{O})$, so that $G = \operatorname{Spec} B \to \operatorname{Spec} A$. Then it is easy to see that the natural restriction homomorphism $\operatorname{Aut}_A(G_A) \to \operatorname{Aut}\bigl(G(B)\bigr)$ is injective. The group $\Delta(B)$ has an order that is a power of the characteristic of k, while the order of $H(B)$ is prime to the characteristic; so $\mathrm{H}^1\bigl(H(B), \Delta(B)\bigr) = 0$, which implies that there exists an element δ_B of $\Delta(B)$ whose image in $\operatorname{Aut}\bigl(G(B)\bigr)$ concides with the image of α. Call $\overline{\delta}_B$ the image of δ_B in $(\Delta/\Delta^H)(B)$; then. I claim that $\overline{\delta}_B$ the image of an element $\overline{\delta}$ of $(\Delta/\Delta^H)(A)$; then the image of δ in $E(A)$ must be α, because $\operatorname{Aut}_A(G_A)$ injects into $\operatorname{Aut}\bigl(G(B)\bigr)$.

To prove this, since $(\Delta/\Delta^H)(A)$ is the equalizer of the two natural maps $(\Delta/\Delta^H)(B) \rightrightarrows (\Delta/\Delta^H)(B \otimes_A B)$, it is enough to show that the two images of $\overline{\delta}_B$ in $(\Delta/\Delta^H)(B \otimes_A B)$ coincide. The two images of δ_B in $\operatorname{Aut}\bigl(G(B \times_A B)\bigr)$ are equal; since $\Delta(B \times_A B)/\Delta(B \times_A B)^{H(B \times_A B)}$ injects into $\operatorname{Aut}\bigl(G(B \times_A B)\bigr)$, this implies that the two images of δ_B into $\Delta(B \times_A B)^{H(B \times_A B)}$ coincide. The images of these via the natural injective homomorphism

$$\Delta(B \times_A B)/\Delta(B \times_A B)^{H(B \times_A B)} \longrightarrow (\Delta/\Delta^H)(B \times_A B)$$

are the two images of $\overline{\delta}_B$, and this completes the proof. ♠

Let $\overline{k(p)}$ be the algebraic closure of $k(p)$; the pullback $G_{\overline{k(p)}}$ is well-split, that is, it is a semi-direct product $H_{\overline{k(p)}} \ltimes \Delta_{\overline{k(p)}}$, where $H_{\overline{k(p)}}$ is étale, hence a constant group, and $\Delta_{\overline{k(p)}}$ is connected and diagonalizable. This is the pullback of a group scheme $\Gamma = H \ltimes \Delta \to S$, where H is constant and Δ is diagonalizable; passing to a Zariski open neighborhood of the image of p in S, we may assume that H is tame, so Γ is well split. The group scheme G_0 is a twisted form of the fiber Γ_p. So we need to show that every twisted form of $\Gamma_p \to S$ on a point $p \in S$ extends to a étale neighborhood of p. This twisted form is classified by an element of the non-abelian cohomology group $\mathrm{H}^1_{\mathrm{fppf}}\bigl(p, \underline{\operatorname{Aut}}_k(\Gamma)\bigr)$. Let us set $\Delta' = \Delta/\Delta^H$. The quotient $\underline{\operatorname{Aut}}_k(\Gamma)/\Delta'_p$ is étale, by Lemma 1.14; hence the image of this element into

$$\mathrm{H}^1_{\mathrm{fppf}}\bigl(p, \underline{\operatorname{Aut}}_k(\Gamma)/\Delta'_p\bigr) = \mathrm{H}^1_{\text{ét}}\bigl(p, \underline{\operatorname{Aut}}_k(\Gamma)/\Delta'_p\bigr)$$

is killed after passing to a finite separable extension of $k(p)$. Any such extension is of the form $k(q)$, where $U \to S$ is an étale map and q is a point on U mapping on p. We can substitute S with U, and assume that the image of our element of $\mathrm{H}_{\mathrm{fppf}}\bigl(p, \underline{\mathrm{Aut}}_k(\Gamma)\bigr)$ in $\mathrm{H}^1_{\mathrm{fppf}}\bigl(p, \underline{\mathrm{Aut}}_k(\Gamma)/\Delta'_p\bigr)$ is 0. We have an exact sequence of pointed sets

$$\mathrm{H}^1_{\mathrm{fppf}}\bigl(p, \Delta'_p\bigr) \to \mathrm{H}^1_{\mathrm{fppf}}\bigl(p, \underline{\mathrm{Aut}}_k(\Gamma)\bigr) \to \mathrm{H}^1_{\mathrm{fppf}}\bigl(p, \underline{\mathrm{Aut}}_k(\Gamma)/\Delta'_p\bigr);$$

so we may assume that our element comes from $\mathrm{H}^1_{\mathrm{fppf}}(p, \Delta'_p)$. Since Δ' is diagonalizable, it is enough to prove that every element of $\mathrm{H}^1_{\mathrm{fppf}}(p, \boldsymbol{\mu}_n)$ comes from $\mathrm{H}^1_{\mathrm{fppf}}(S, \boldsymbol{\mu}_n)$, after restricting S in the Zariski topology.

By Kummer theory, every $\boldsymbol{\mu}_n$-torsor over $k(p)$ is of the form

$$\operatorname{Spec} k(p)[t]/(t^n - a) \longrightarrow \operatorname{Spec} k(p)$$

for some $a \in k(p)^*$, with the obvious action of $\boldsymbol{\mu}_n$ on $\operatorname{Spec} k(p)[t]/(t^n - a)$. After passing to a Zariski neighborhood of $p \in S$, we may assume that a is the restriction of a section $f \in \mathcal{O}^*(S)$. Then the $\boldsymbol{\mu}_n$-torsor $\operatorname{Spec}_S \mathcal{O}_S[t]/(t^n - f) \to S$ restricts to $\operatorname{Spec} k(p)[t]/(t^n - a) \to \operatorname{Spec} k(p)$, and this completes the proof. ♠

2. Tame stacks

Let S be a scheme, $\mathcal{M} \to S$ a finitely presented algebraic stack over S. We denote by $\mathcal{I} \to \mathcal{M}$ the inertia group stack; we will always assume that $\mathcal{I} \to \mathcal{M}$ is finite (and we say that $\mathcal{M}$ has *finite inertia*). If $T \to S$ is a morphism, and ξ is an object of $\mathcal{M}(T)$, then the group scheme $\underline{\mathrm{Aut}}_T(\xi) \to T$ is the pullback of $\mathcal{I}$ along the morphism $T \to \mathcal{M}$ corresponding to ξ.

Under this hypothesis, because of [4], there exists a moduli space $\rho\colon \mathcal{M} \to M$; the morphism ρ is proper.[18] ←18

Definition 2.1. The stack $\mathcal{M}$ is *tame* if the functor $\rho_*\colon \operatorname{QCoh} \mathcal{M} \to \operatorname{QCoh} M$ is exact.

When $G \to S$ is a finite flat group scheme, then the moduli space of $\mathcal{B}_S G \to S$ is S itself; so $\mathcal{B}_S G$ is tame if and only if G is linearly reductive.

Theorem 2.2. *The following conditions are equivalent.*

(a) *$\mathcal{M}$ is tame.*
(b) *If k is an algebraically closed field with a morphism $\operatorname{Spec} k \to S$, and ξ is an object of $\mathcal{M}(\operatorname{Spec} k)$, then the automorphism group scheme $\underline{\mathrm{Aut}}_k(\xi) \to \operatorname{Spec} k$ is linearly reductive.*

[18](Angelo) added these last few words.

(c) *There exists an fppf covering $M' \to M$, a linearly reductive group scheme $G \to M'$ acting an a finite scheme $U \to M'$, together with isomorphisms*

$$\mathcal{M} \times_M M' \simeq [U/G]$$

19→ *of algebraic stacks over M'.*[19]

20→ (d) *Same as (c), but $M' \to M$ is assumed to be étale and surjective.* [20]

Corollary 2.3. *The formation of moduli space for a tame stack commutes with base change: that is, if $M' \to M$ is a morphism of algebraic spaces, then the moduli space of $M' \times_M \mathcal{M}$ is M'.*

Corollary 2.4. *If $\mathcal{M} \to S$ tame and $S' \to S$ is a morphism of schemes, then $S' \times_S \mathcal{M}$ is a tame stack over S'.*

Corollary 2.5. *The stack $\mathcal{M} \to S$ tame if and only if for any morphism* $\operatorname{Spec} k \to S$, *where k is an algebraically closed field, the geometric fiber* $\operatorname{Spec} k \times_S \mathcal{M}$ *is tame.*

21→ *Proof of Theorem 2.2.* [21] It is obvious that (d) implies (c). It is straightforward to see that (c) implies both (a) and (b).

Let us check that (a) implies (b). Let $\operatorname{Spec} k \to \mathcal{M}$ be the morphism corresponding to the object ξ of $\mathcal{M}(\operatorname{Spec} k)$; set $G = \underline{\mathrm{Aut}}_k(\xi)$. Call $\mathcal{M}_0$ the pullback $\operatorname{Spec} k \times_M \mathcal{M}$; this admits a section $\operatorname{Spec} k \to \mathcal{M}_0$, and the residual gerbe of this section, which is a closed substack of $\mathcal{M}_0$, is isomorphic to $\mathcal{B}_k G$. So we get a commutative (non cartesian) diagram

$$\begin{array}{ccc} \mathcal{B}_k G & \xrightarrow{g} & \mathcal{M} \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} \\ \operatorname{Spec} k & \xrightarrow{f} & M \end{array}$$

whose rows are affine. So we have that $g_*\colon \mathrm{QCoh}(\mathcal{B}_k G) \to \mathrm{QCoh}(\mathcal{M})$ is an exact functor, while $\phi\colon \mathrm{QCoh}(\mathcal{M}) \to \mathrm{QCoh}(M)$ is exact by definition. Also we have an equality of functors $f_*\rho'_* = \rho_* g_*$; hence, if

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

is an exact sequence of representations of G, considered as an exact sequence of quasi-coherent sheaves on $\mathcal{B}_k G$, we have that the sequence

$$0 \longrightarrow f_*\bigl(V_1^G\bigr) \longrightarrow f_*\bigl(V_2^G\bigr) \longrightarrow f_*\bigl(V_3^G\bigr) \longrightarrow 0$$

is exact; and this implies that

$$0 \longrightarrow V_1^G \longrightarrow V_2^G \longrightarrow V_3^G \longrightarrow 0$$

[19](Dan) changed the statement since has to be done locally on M, not S

[20](Angelo) added this

[21](Angelo) I changed this proof in various places

is exact. Hence G is linearly reductive, as claimed.

Now let us prove that (b) implies (d). In fact, we will prove a stronger version of this implication.

Proposition 2.6. *Let $\mathcal{M} \to S$ be an algebraic stack with finite inertia and moduli space $\rho\colon \mathcal{M} \to M$. Let k be a field with a morphism $\operatorname{Spec} k \to S$, and let ξ be an object of $\mathcal{M}(\operatorname{Spec} k)$; assume that the automorphism group scheme $\underline{\operatorname{Aut}}_k(\xi) \to \operatorname{Spec} k$ is linearly reductive. Denote by $p \in M$ the image of the composite $\operatorname{Spec} k \to \mathcal{M} \to M$. Then there exists an étale morphism $U \to M$ having p in its image, a linearly reductive group scheme $G \to U$ acting on a finite scheme $V \to U$ of finite presentation, and an isomorphism $[V/G] \simeq U \times_S \mathcal{M}$ of algebraic stacks over U.*

Thus, if $\mathcal{M}$ has an object over a field with linearly reductive automorphism group, then there is an open tame substack of $\mathcal{M}$ (the image of U) containing this object.

Proof. The proof is divided into three steps.

We may assume that M is affine and of finite type over $\mathbb{Z}$.

THE CASE $k = k(p)$. We start by assuming that the residue field $k(p)$ of $p \in M$ equals k. After passing to an étale morphism to M, we may also assume that $\underline{\operatorname{Aut}}_k(\xi)$ extends to a linearly reductive group scheme $G \to M$ (Proposition 1.13).

By standard limit arguments we may assume that M is the spectrum of a local henselian ring R with residue field k. The result will follows once we have shown that there is a representable morphism $\mathcal{M} \to \mathcal{B}_M G$ of algebraic stacks (or, equivalently, a G-torsor $P \to \mathcal{M}$ in which the total space is an algebraic space).

Let us denote by $\mathcal{M}_0$ the residual gerbe $\mathcal{B}_k \underline{\operatorname{Aut}}_k(\xi) = \mathcal{B}_k G_p$; this is a closed substack of $\mathcal{M}$, having $\operatorname{Spec} k$ as its moduli space. This closed substack gives a sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{M}}$; we denote by $\mathcal{M}_n$ the closed substack of $\mathcal{M}$ whose sheaf of ideals is $\mathcal{I}^{n+1}$. Denote by $\mathfrak{g}$ the Lie algebra of $\underline{\operatorname{Aut}}_k(\xi)$.

The obstruction to extending a G-torsor $P_{n-1} \to \mathcal{M}_{n-1}$ to a G-torsor $P_n \to \mathcal{M}_n$ lies in

$$\begin{aligned}\mathrm{H}^2\bigl(\mathcal{M}_0, (\mathcal{I}^n/\mathcal{I}^{n+1}) \otimes \mathfrak{g}\bigr) &= \mathrm{H}^2\bigl(G_p, (\mathcal{I}^n/\mathcal{I}^{n+1}) \otimes \mathfrak{g}\bigr)\\ &= 0;\end{aligned}$$

Alternatively, in terms of the cotangent complex, the obstruction lies in $\operatorname{Ext}^2(\mathbf{L}_{\mathcal{B}_k G_p}, \mathcal{I}^n/\mathcal{I}^{n+1}) = 0.$ [22] ←22

[22](Dan) added this sentence

Hence we can construct a sequence of G-torsors $P_n \to \mathcal{M}_n$, such that the restriction of P_n to P_{n-1} is isomorphic to P_{n-1}, and such that the torsor $P_0 \to \mathcal{M}_0$ has $\operatorname{Spec} k$ as its total space.

Let $\mathfrak{m}$ be the maximal ideal of R, and set $M_n = \operatorname{Spec} R/\mathfrak{m}^{n+1}$. The systems of ideals $\{\mathcal{I}^n\}$ and $\mathfrak{m}^n\mathcal{O}_{\mathcal{M}}$ are cofinal; hence we get a sequence of G-torsors $Q_n \to M_n \times_M \mathcal{M}$, such that the restriction of Q_n to Q_{n-1} is isomorphic to Q_{n-1}, and such that the restriction of Q_0 to $\mathcal{M}_0$ has $\operatorname{Spec} k$ as its total space. We can define a functor from R-algebras to sets that sends each R-algebra A to the set of isomorphism classes of G-torsors on the stack $\mathcal{M}_A$. This functor is easily checked to be limit-preserving (for example, by using a presentation of $\mathcal{M}$, and descent for G-torsors). So we can apply Artin's approximation theorem, and conclude that there exists a G-torsor on $\mathcal{M}$, whose restriction to $\mathcal{M}_0$ has $\operatorname{Spec} k$ as its total space.

The total space $\mathcal{P}$ is an algebraic stack with finite inertia; furthermore, the inverse image of $\mathcal{M}_0$ in $\mathcal{P}$ is isomorphic to $\operatorname{Spec} k$. The locus where the inertia stack $\mathcal{I}_{\mathcal{P}} \to \mathcal{P}$ has fiber of length larger than 1 is a closed substack of $\mathcal{P}$, whose image in $M = \operatorname{Spec} R$ is a closed subscheme that does not contain p; hence this locus is empty. So $\mathcal{P}$ is an algebraic space (in fact an affine scheme); and this concludes the proof of the first case.

Obtaining a flat morphism. Now we prove a weaker version of the Proposition, with the same statement, except that the morphism $U \to M$ is only supposed to be flat and finitely presented, instead of étale.

By passing to the algebraic closure of k we may assume that k is algebraically closed.

We claim that there exists a finite extension k' of the residue field $k(p)$ contained in k, such that the object ξ is defined over k'. In fact, it follows from the definition of moduli space that there exists an object η of $\operatorname{Spec} \overline{k(p)}$ whose pullback to $\operatorname{Spec} k$ is isomorphic to ξ. This η gives an object of the algebraic stack $\operatorname{Spec} k(p) \times_M \mathcal{M}$ over $\operatorname{Spec} \overline{k(p)}$, which is finitely presented over $\operatorname{Spec} k(p)$, and any such object is defined a finite extension k' of $k(p)$. Hence we may assume that k is a finite extension of $k(p)$.

There is a flat morphism of finite presentation $M' \to M$, with a point $q \in U$ mapping to p, such that $k(q) = k$; hence, by applying the first step to $M' \times_M \mathcal{M}$, there is an étale morphism $U \to M'$ containing q in its image, such that $U \times_M \mathcal{M}$ has the required quotient form.

The conclusion. The argument of the proof of the previous case shows that to conclude we only need the following fact.

Proposition 2.7. *Let $\mathcal{M} \to S$ be a tame stack with moduli space $\rho\colon \mathcal{M} \to M$, k a field. Given a morphism $\operatorname{Spec} k \to M$, there exists a finite separable extension $k \subseteq k'$ and a lifting $\operatorname{Spec} k' \to \mathcal{M}$ of the composite $\operatorname{Spec} k' \to \operatorname{Spec} k \to M$.*

Proof. We are going to need the following lemmas.

Lemma 2.8. *Let G be a linearly reductive group scheme over a field k. The stack $\underline{\mathrm{Hom}}_k^{\mathrm{rep}}(\mathcal{B}_k G, \mathcal{B}_k G)$ is of finite type over k.*

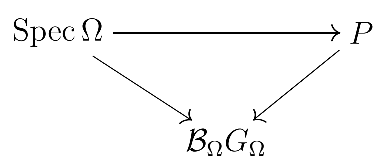
[23] ←23

Let k be a field, R be an artinian local k-algebra with residue field k, G a linearly reductive group scheme acting on R. Set $\mathcal{M} = [\operatorname{Spec} R/G]$, and assume that the moduli space of $\mathcal{M}$ is $\operatorname{Spec} k$ (this is equivalent to assuming that $R^G = k$). We have a natural embedding $\mathcal{B}_k G = [\operatorname{Spec} k/G] \subseteq [\operatorname{Spec} R/G] = \mathcal{M}$.

Lemma 2.9. *If T is a k-scheme, any representable morphism of k-stacks $\mathcal{B}_k G \times_{\operatorname{Spec} k} T \to \mathcal{M}$ factors through $\mathcal{B}_k G \subseteq \mathcal{M}$.*

Proof. Let P be the pullback of $\operatorname{Spec} R \to \mathcal{M}$ to $\mathcal{B}_k G \times_{\operatorname{Spec} k} T$; then P is an algebraic space with an action of G, such that the morphism $P \to \operatorname{Spec} R$ is G-equivariant. I claim that the composite $P \to \mathcal{B}_k G \times T \to T$ is an isomorphism. [24] ←24

Since it is finite and flat it enough to prove that is an isomorphism when pulled back to a geometric point $\operatorname{Spec} \Omega \to T$, were Ω is an algebraically closed field; so we may assume that $T = \operatorname{Spec} \Omega$. Choose a section $\operatorname{Spec} \Omega \to P$: since there is a unique morphism $\operatorname{Spec} \Omega \to \mathcal{B}_\Omega G_\Omega$ over Ω, we get a commutative diagram



Since both $\operatorname{Spec} \Omega \to \mathcal{B}_\Omega G_\Omega$ and $P \to \mathcal{B}_\Omega G_\Omega$ are G-torsors, the degrees of both over $\mathcal{B}_\Omega G_\Omega$ equal the order of G; hence $\operatorname{Spec} \Omega \to P$ is an isomorphism, and $P \to \operatorname{Spec} \Omega$ is its inverse.

Thus, since the composite $P \to \mathcal{B}_k G \times_{\operatorname{Spec} k} T \to T$ is G-equivariant, this means that the action of G on P is trivial. The morphism $P \to$

[23](Angelo) This would be a particular case of the boundedness result for tame stacks, right? If so, the proof would be in the appendix. I am omitting it for now.

[24](Angelo) This fact is also clear by looking at degrees, since the degree of P over $\mathcal{B}_k G \times_{\operatorname{Spec} k} T$ is $|G|$, while that of $\mathcal{B}_k G \times_{\operatorname{Spec} k} T$ over T is $1/|G|$. This is essentially Martin's argument. I am giving a slightly different proof.

$\operatorname{Spec} R$ corresponds to a ring homomorphism $R \to \mathcal{O}(P)$, which is G-equivariant, and the action of G on $\mathcal{O}(P)$ is trivial. But if $\mathfrak{m}$ is the maximal ideal of R, there is a splitting of G-modules $R \simeq \mathfrak{m} \oplus k$; and $\mathfrak{m}^G = 0$, because $R^G = k$. So $\mathfrak{m}$ is a sum of non-trivial irreducible representations, since G is linearly reductive, and any G-equivariant linear map $\mathfrak{m} \to \mathcal{O}(P)$ is trivial. So $P \to \operatorname{Spec} R$ factors through $\operatorname{Spec} k$, so $\mathcal{B}_k G \times_{\operatorname{Spec} k} T \to \mathcal{M}$ factors through $[\operatorname{Spec} k/G] = \mathcal{B}_k G$, as claimed. ♠

Let us prove Proposition 2.7. Since $\mathcal{M}$ is limit-preserving, it is sufficient to show that any morphism $\operatorname{Spec} k \to M$, where k is a separably closed field, lifts to $\operatorname{Spec} k \to \mathcal{M}$.

Let $U \to M$ be a flat finitely presented morphism with p in its image, such that $U \times_M \mathcal{M}$ is a quotient $[V/G]$, with $G \to U$ linearly reductive. The image of U in M is open, so we may replace M with this image and assume that $U \to M$ is surjective. Then for any morphism $T \to U$,
25→ the moduli space of $T \times_M \mathcal{M}$ is T[25]; and it follows easily that the same holds for morphisms $T \to M$.

By applying this to the morphism $\operatorname{Spec} k \to M$, we see that we may assume that $M = \operatorname{Spec} k$.

Let $k \subseteq k'$ be a finite field extension such that $\mathcal{M}(k')$ is non-empty. Pick an object $\xi \in \mathcal{M}(k')$, and set $G_{k'} = \underline{\operatorname{Aut}}_{k'}(\xi)$. After extending k', we may assume that $G_{k'}$ is of the form $H_{k'} \ltimes \Delta_{k'}$, where $\Delta_{k'}$ is a diagonalizable group scheme whose order is a power of the characteristic of k and $H_{k'}$ is a constant tame group scheme. There exist unique group schemes Δ and H, respectively diagonalizable and constant, whose pullbacks to $\operatorname{Spec} k'$ coincide with $\Delta_{k'}$ and $H_{k'}$; furthermore, the action of $H_{k'}$ on $\Delta_{k'}$ comes from a unique action of H on Δ. We set $G = H \ltimes \Delta$: this G is a group scheme on $\operatorname{Spec} k$ inducing $G_{k'}$ by base change.

Lemma 2.10. *The stack* $\underline{\operatorname{Hom}}_k^{\operatorname{rep}}(\mathcal{B}_k G, \mathcal{M})$ *is of finite type over* k.

Proof. It is enough to prove the result after base changing to k'; we can therefore assume that $\mathcal{M} = [\operatorname{Spec} R/G]$, where R is an artinian k-algebra with residue field k, because of the first part of the proof. Then by Lemma 2.9 the stack $\underline{\operatorname{Hom}}_k^{\operatorname{rep}}(\mathcal{B}_k G, \mathcal{M})$ is of isomorphic to $\underline{\operatorname{Hom}}_k^{\operatorname{rep}}(\mathcal{B}_k G, \mathcal{B}_k G)$, which is of finite type by Lemma 2.8. ♠

The morphism $t\colon \operatorname{Spec} k \to \mathcal{B}_k G$ corresponding to the trivial torsor induces a morphism

$$F\colon \underline{\operatorname{Hom}}_k^{\operatorname{rep}}(\mathcal{B}_k G, \mathcal{M}) \longrightarrow \mathcal{M}$$

by composition with t.

[25](Angelo) We'll have to prove this somewhere.

Consider the scheme-theoretic image $\overline{\mathcal{M}} \subseteq \mathcal{M}$ of the morphism F: this is the smallest closed substack of $\mathcal{M}$ with the property that $F^{-1}(\overline{\mathcal{M}}) = \underline{\mathrm{Hom}}_k^{\mathrm{rep}}(\mathcal{B}_k G, \mathcal{M})$. Its sheaf of ideals is the kernel of the homomorphism $\mathcal{O}_{\mathcal{M}} \to F_* \mathcal{O}_{\underline{\mathrm{Hom}}_k^{\mathrm{rep}}(\mathcal{B}_k G, \mathcal{M})}$.

Lemma 2.11. *We have*

$$\operatorname{Spec} k' \times_{\operatorname{Spec} k} \overline{\mathcal{M}} = \mathcal{B}_k G_{k'} \subseteq \mathcal{M}_{k'},$$

where $\mathcal{B}_{k'} G_{k'}$ is embedded in $\mathcal{M}_{k'}$ as the residual gerbe of ξ.

Proof. By the first part of the proof, we can write $\mathcal{M}_{k'}$ in the form $[\operatorname{Spec} R/G_{k'}]$, where R is an artinian k with residue field k'. Formation of scheme-theoretic images commutes with flat base change, hence we need to show that the scheme-theoretic image of the morphism

$$F_{k'} \colon \underline{\mathrm{Hom}}_{k'}^{\mathrm{rep}}(\mathcal{B}_{k'} G_{k'}, \mathcal{M}_{k'}) \longrightarrow \mathcal{M}_{k'}$$

is equal to $\mathcal{B}_{k'} G_{k'}$; or, equivalently, that for any morphism $g \colon T \to \underline{\mathrm{Hom}}_{k'}^{\mathrm{rep}}(\mathcal{B}_{k'} G_{k'}, \mathcal{M}_{k'})$, the composite $F_{k'} \circ g \colon T \to \mathcal{M}_{k'}$ factors through $[\operatorname{Spec} k'/G_{k'}]$. This follows from Lemma 2.9. ♠

Now we can replace $\mathcal{M}$ with $\overline{\mathcal{M}}$, and assume that $\mathcal{M}_{k'}$ is $\mathcal{B}_{k'} G_{k'}$.

Next we define an étale gerbe $\mathcal{N}$, with a morphism $\mathcal{G} \to \mathcal{N}$.[26] ←26

For any k-scheme T and any object $\xi \in \mathcal{M}(T)$, the automorphism group scheme $G_\xi \to T$ is linearly reductive; let

$$1 \longrightarrow \Delta_\xi \longrightarrow G_\xi \longrightarrow H_\xi \longrightarrow 1$$

be the connected étale sequence of G_ξ. More concretely, Δ_ξ is the subfunctor of G_ξ of automorphisms whose order is a power of the characteristic of k. If $f \colon T' \to T$ is a morphism of schemes, then $G_{f^*\xi} = T' \times_T G_\xi$ (this is a general property of fibered categories), and $\Delta_{f^*\xi} = T' \times_T \Delta_\xi$.

Given two objects $\xi \in \mathcal{M}(T)$ and $\xi' \in \mathcal{M}(T')$, there is a right action of $\Delta_\xi(T)$ and a left action of $\Delta_{\xi'}(T')$ on $\operatorname{Hom}_{\mathcal{M}}(\xi, \xi')$, and they commute. Furthermore, it is easy to see that

$$\operatorname{Hom}_{\mathcal{M}}(\xi, \xi')/\Delta_\xi(T) = \Delta_{\xi'}(T') \backslash \operatorname{Hom}_{\mathcal{M}}(\xi, \xi')/\Delta_\xi(T)$$

and that there is fibered category over the category of k-schemes, with the same objects as $\mathcal{M}$, such that the arrows $\xi \to \xi'$ are the elements of the quotient $\operatorname{Hom}_{\mathcal{M}}(\xi, \xi')/\Delta_\xi(T)$. We define $\mathcal{N}$ to be the stack associated with this fibered category. With a little work, one can show

[26](Angelo) I think this could be done with general results on non-abelian cohomology. I like Martin's construction, which is a more general form of rigidification than the ones that have been treated so far (the group depends on the object, and not only on the base scheme). The details will need to be expanded; it might even make sense to dedicate a separate section to a discussion of these more general rigidifications.

27→ that $\mathcal{N}$ is an étale gerbe over $\operatorname{Spec} k$[27]. There is a canonical morphism $\mathcal{M} \to \mathcal{N}$. The pullback $\mathcal{N}_{k'}$ is $\mathcal{B}_{K'}\Delta_{k'}$ (recall that $\Delta_{k'}$ is the connected component of the identity in $G_{k'}$).

Since $\mathcal{N}$ is an étale gerbe and k is separably closed, there is a k-morphism $\operatorname{Spec} k \to \mathcal{N}$. We can replace $\mathcal{M}$ by $\mathcal{M} \times_{\mathcal{N}} \operatorname{Spec} k$, so that $\mathcal{M}_{k'} = \mathcal{B}_{k'}\Delta_{k'}$.

28→ In this case, we claim that $\mathcal{M}$ is banded[28] by the diagonalizable group $\Delta \to \operatorname{Spec} k'$ (recall that we have defined this as the diagonalizable group scheme whose pullback to $\operatorname{Spec} k'$ is $\Delta_{k'}$). In fact, since $\mathcal{M}$ is a gerbe, and all of its objects have abelian automorphism groups, then the automorphism group schemes descend to a group scheme over
29→ $\operatorname{Spec} k$, whose pullback to $\operatorname{Spec} k'$ is $\Delta_{k'}$.[29] So this group scheme is a form of Δ in the fppf topology; but the automorphism group scheme of Δ is constant, so this form is in fact a form in the étale topology, and so it is trivial.

The class of the gerbe $\mathcal{M}$ banded by Δ is classified by the group $\mathrm{H}^2_{\mathrm{fppf}}(\operatorname{Spec} k, \Delta)$.

Lemma 2.12. *If Δ is a diagonalizable group scheme over a separably closed field k, we have $\mathrm{H}^2_{\mathrm{fppf}}(\operatorname{Spec} k, \Delta) = 0$.*

Proof. The group scheme Δ is a product of groups of the form $\boldsymbol{\mu}_n$, so it is enough to consider the case $\Delta = \boldsymbol{\mu}_n$. Then the result follows from the Kummer exact sequence of fppf sheaves

$$0 \longrightarrow \boldsymbol{\mu}_n \longrightarrow \mathbb{G}_{\mathrm{m}} \xrightarrow{\times n} \mathbb{G}_{\mathrm{m}} \longrightarrow 0$$

and the fact that $\mathrm{H}^i_{\mathrm{fppf}}(\operatorname{Spec} k, \mathbb{G}_{\mathrm{m}}) = \mathrm{H}^i_{\text{ét}}(\operatorname{Spec} k, \mathbb{G}_{\mathrm{m}}) = 0$ (see ????). ♠

This concludes the proofs of Propositions 2.7 and 2.6 and of Theorem 2.2. ♠

3. Twisted stable maps

Corollary 3.1. *Let $\mathcal{C} \to S$ be a separated finitely presented stack with markings by gerbes; assume that the geometric fibers of $\mathcal{C}$ over S are*
30→ *twisted curves. Then $\mathcal{C}$ is a twisted curve over S.* [30]

[27](Angelo) We don't need to go to $\operatorname{Spec} k'$, it is easy to verify this directly

[28](Angelo) I like to use "banded" instead of "bound", because the standard translation for "lien" is band.

[29](Angelo) this is standard and easy, I don't think that we need to recall the argument

[30](Dan) this needs to be expanded. I moved it here.

Theorem 3.2. *Let $\mathcal{M}$ be a proper finitely presented tame algebraic stack over a scheme S with finite inertia. Then the stack of stable twisted maps from n-pointed genus g curves into $\mathcal{M}$ is a finitely presented algebraic stack over S, which is proper over the stack of stable maps into M.*

4. Reduction of Spaces of Galois admissible covers

5. Example: reduction of $X(2)$ in characteristic 2

$\mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)$ vs. $\mathcal{K}_{1,0}((\mathcal{B}\boldsymbol{\mu}_2)^2)$.

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[31](Angelo) Added command \url from the package url, for typesetting web addresses.