TWISTED STABLE MAPS TO TAME ARTIN STACKS

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Contents

1. Introduction

This paper is a continuation of [4], where the basic theory of tame Artin stacks is developed. Our main goal here is the construction of an appropriate analogue of Kontsevich's space of stable maps in the case where the target is a tame Artin stack. When the target is a tame Deligne–Mumford stack, the theory was developed in [1], and found a number of applications. The theory for arbitrary tame Artin stacks developed here is very similar, but it is necessary to overcome a number of technical hurdles and to generalize a few questions of foundation. However the method of construction we use here is very different from the ad-hoc method of [1], and more natural: we rely on the second author's general results in [29, 30, 35], as extended in the appendices.

Section 2 is devoted to a number of basic properties of twisted curves in arbitrary characteristic, where, unlike the situation of [1], they may fail to

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be Deligne–Mumford stacks. In particular we show in Proposition 2.3 that the notion of twisted curve is stable under small deformations and can be tested on geometric fibers.

In Section 3 we collect together some facts about relative moduli spaces which will be used in the following section.

In Section 4 we define twisted stable maps into a tame stack $\mathcal{X}$ and show in Theorem 4.1 that they form an artin stack, which is proper and quasifinite over the Kontsevich space of the coarse moduli space X of $\mathcal{X}$. In particular, when X is projective, the stacks of twisted stable maps of $\mathcal{X}$ admit projective coarse moduli spaces. The construction proceeds rather naturally from the following:

(1) the existence of the stack of twisted curves, which was shown in [29] for Deligne–Mumford twisted curves and proved in our case in Appendix A, and
(2) The existence and finiteness properties of Hom-stacks, proved in [30] and [35] in many cases and extended in our case in Appendix C.

Some extra care is needed for proving the quasi-finite claim. Properness relies on Lemma 4.4, a suitable generalization of the Purity Lemma of [1].

In section 5 we concentrate on the case where the target stack is $\mathcal{B}G$[1], the
1→ classifying stack of a finite flat linearly reductive group scheme G. The main result here, Theorem 5.1, is that the space of stable maps with target $\mathcal{B}G$ is finite and flat over the corresponding Deligne–Mumford space. This result is known when G is tame and étale (see [2]) and relatively straightforward when G is a diagonalizable group scheme (or even a twist of such). However, our argument in general takes some delicate twists and turns.

As an example for the behavior of these stacks, we consider in Section 6 two ways to compactify the moduli space $X(2)$ of elliptic curves with full level-2 structure. The first is as a component in $\mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)$, the stack of totally branched $\boldsymbol{\mu}_2$-covers of stable 4-pointed curves of genus 0. This provides an opportunity to consider the cyclotomic inertia stacks and evaluation maps. The second is as a component in $\mathcal{K}_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2)$ parametrizing elliptic curves with $\mathcal{B}\boldsymbol{\mu}_2^2$, where $X(2)$ meets other components in a geometrically appealing way. We also find the Katz-Mazur regular model of $X(2)$ as the closure of a component of the generic fiber.

As already mentioned, the paper contains three appendices, written by the second author. Appendix A generalizes the main results of [29]. Appendix B contains some prepatory results needed for Appendix C which generalizes some of the results from [30] and [35] to tame stacks. The logical order of Appendix A is after section 2, whereas the Appendix C only uses results from [4].

1.1. **Acknowledgements.** Thanks to Johan de Jong for helpful comments. Thanks to Shaul Abramovich for help and advice with visualization.

[1](Dan) changed classifying stack to $\mathcal{B}$ everywhere

section4

2. Twisted curves

A1 **Definition 2.1.** Let S be a scheme. An *n-marked twisted curve over S* is a collection of data $(f\colon \mathcal{C} \to S, \{\Sigma_i \subset \mathcal{C}\}_{i=1}^n)$ as follows:

(i) $\mathcal{C}$ is a proper tame stack over S whose geometric fibers are connected of dimension 1, and such that the moduli space C of $\mathcal{C}$ is a nodal curve over S.

(ii) The $\Sigma_i \subset \mathcal{C}$ are closed substacks which are fppf gerbes over S, and whose images in C are contained in the smooth locus of the morphism $C \to S$.

(iii) If $\mathcal{U} \subset \mathcal{C}$ denotes the complement of the Σ_i and the singular locus of $\mathcal{C} \to S$, then $\mathcal{U} \to C$ is an open immersion.

(iv) For any geometric point $\bar{p} \to C$[2] mapping to a smooth point of C, ←2 there exists an integer r such that

$$\mathrm{Spec}(\mathcal{O}_{C,\bar{p}}) \times_C \mathcal{C} \simeq [D^{\mathrm{sh}}/\mu_r],$$

where D^{sh} denotes the strict henselization of $D\colon = \mathrm{Spec}(\mathcal{O}_{S,f(\bar{p})}[z])$ at the point $(\mathfrak{m}_{S,f(\bar{p})}, z)$ and $\zeta \in \mu_r$ acts by $z \mapsto \zeta \cdot z$ (note that $r = 1$ unless $\bar{p}$ maps to a point in the image of some Σ_i). Here $\mathfrak{m}_{S,f(\bar{p})}$ denote the maximal ideal in the strict henselization $\mathcal{O}_{S,f(\bar{p})}$.

(v) If $\bar{p} \to C$ is a geometric point mapping to a node of C, then there exists an integer r and an element $t \in \mathfrak{m}_{S,f(\bar{p})}$ such that

$$\mathrm{Spec}(\mathcal{O}_{C,\bar{p}}) \times_C \mathcal{C} \simeq [D^{\mathrm{sh}}/\mu_r],$$

where D^{sh} denotes the strict henselization of

$$D\colon = \mathrm{Spec}(\mathcal{O}_{S,f(\bar{p})}[z,w]/(zw - t))$$

at the point $(\mathfrak{m}_{S,f(\bar{p})}, z, w)$ and $\zeta \in \mu_r$ acts by $x \mapsto \zeta \cdot z$ and $y \mapsto \zeta^{-1} \cdot y$.

Remark 2.2. If $\mathcal{C} \to S$ is a proper tame Artin stack which admits a collection of closed substacks $\Sigma_i \subset \mathcal{C}$ ($i = 1, \dots, n$ for some n) such that $(\mathcal{C}, \{\Sigma_i\}_{i=1}^n)$ is an n-marked twisted curve, we will refer to $\mathcal{C}$ as a *twisted curve*, without reference to markings.

The following proposition allows us to detect twisted curves on fibers:

Prop:2.3 **Proposition 2.3.** *Let S be a scheme, and let $(f\colon \mathcal{C} \to S, \{\Sigma_i\}_{i=1}^n)$ be a proper flat tame stack $f\colon \mathcal{C} \to S$ with a collection of closed substacks $\Sigma_i \subset \mathcal{C}$ which are S-gerbes. If for some geometric point $\bar{x} \to S$ the fiber $(\mathcal{C}_{\bar{x}}, \{\Sigma_{i,\bar{x}}\})$ is an n-marked twisted curve, then there exists an open neighborhood of $\bar{x}$ in S over which $(\mathcal{C} \to S, \{\Sigma_i\})$ is an n-marked twisted curve.*

Proof. Let $\pi\colon \mathcal{C} \to C$ be the coarse moduli space of S. Since $\mathcal{C}$ is a tame stack flat over S, the space C is flat over S, and for any morphism $S' \to S$ the base change $C \times_S S'$ is the coarse moduli space of $\mathcal{C} \times_S S'$, by [4, Corollary 3.3]. It follows that in some étale neighborhood of $\bar{x} \to S$ the space C/S

[2](Dan) DANGER!! replaced all $\bar{s}$ on a curve by $\bar{p}$, since it can confuse with the base S

is a nodal curve over S (see for example [11, §1]). Shrinking on S we may therefore assume that C is a nodal curve over S. After further shrinking on S, we can also assume that the images of the Σ_i in C are contained in the smooth locus of the morphism $C \to S$ and that (2.1 (iii)) holds. To prove the proposition it then suffices to verify that conditions (iv) and (v) hold for geometric points of $\bar{p} \to C$ over $\bar{x}$.

We may without loss of generality assume that S is the spectrum of a strictly henselian local ring whose closed point is $\bar{x}$. Also let $\mathcal{C}_{\bar{p}}$ denote the fiber product

$$\mathcal{C}_{\bar{p}} := \mathcal{C} \times_{C,\bar{p}} \operatorname{Spec}(\mathcal{O}_{C,\bar{p}}).$$

Consider first the case when $\bar{p} \to C$ has image in the smooth locus of C/S. Since the closed fiber $C_{\bar{x}}$ is a twisted curve, we can choose an isomorphism

$$\mathcal{C}_{\bar{p}} \times_S \operatorname{Spec}(k(\bar{x})) \simeq [\operatorname{Spec}(k(\bar{x})[z])^{\mathrm{sh}}/\mu_r],$$

for some integer $r \geq 1$, where the μ_r-action is as in (2.1 (iv)). Let

$$P_0 \to \mathcal{C}_{\bar{p}} \times_S \operatorname{Spec}(k(\bar{x}))$$

be the μ_r-torsor

$$\operatorname{Spec}(k(\bar{x})[z])^{\mathrm{sh}} \to [\operatorname{Spec}(k(\bar{x})[z])^{\mathrm{sh}}/\mu_r].$$

As in the proof of [4, Proposition 3.6] there exists a μ_r-torsor $P \to \mathcal{C}_{\bar{p}}$ whose reduction is P_0. Furthermore, P is an affine scheme over S. Since $\mathcal{C}_{\bar{p}}$ is flat over S, the scheme P is also flat over S. Since its closed fiber is smooth over S, it follows that P is a smooth S–scheme of relative dimension 1 with μ_r-action. Write $P = \operatorname{Spec}(A)$ and fix a μ_r-equivariant isomorphism

$$A \otimes_{\mathcal{O}_S} k(\bar{x}) \simeq (k(\bar{x})[z])^{\mathrm{sh}}.$$

Since μ_r is linearly reductive we can find an element $\tilde{z} \in A$ lifting z such that an element $\zeta \in \mu_r$ acts on $\tilde{z}$ by $\tilde{z} \mapsto \zeta \cdot \tilde{r}$. Since A is flat over $\mathcal{O}_S$ the induced map

$$(\mathcal{O}_S[\tilde{z}])^{\mathrm{sh}} \to A$$

is an isomorphism, which shows that (iv) holds.

The verification of (v) requires some deformation theory. First we give an explicit argument. We then provide an argument using the cotangent complex.

Let $\bar{p} \to C$ be a geometric point mapping to a node of C. Then we can write

$$\mathcal{C}_{\bar{p}} = [\operatorname{Spec}(A)/\mu_r],$$

for some integer r, and A is a flat $\mathcal{O}_{S,\bar{x}}$-algebra such that there exists an isomorphism

$$\iota\colon A \otimes_{\mathcal{O}_{S,\bar{x}}} k(\bar{x}) \simeq (k(\bar{x})[z,w]/(zw))^{\mathrm{sh}},$$

with μ_r-action as in (v). We claim that we can lift this isomorphism to a μ_r-equivariant isomorphism

localiso (2.3.1) $$A \simeq (\mathcal{O}_{S,\bar{x}}[\tilde{z},\tilde{w}]/(\tilde{z}\tilde{w} - t))^{\mathrm{sh}}$$

for some $t \in \mathcal{O}_{S,\bar{x}}$. For this note that A^{μ_r} is equal to $\mathcal{O}_{C,\bar{p}}$ and that A is a finite $\mathcal{O}_{C,\bar{p}}$-algebra. By a standard application of the Artin approximation theorem, in order to find 2.3.1 it suffices to prove the analogous statements for the map on completions (with respect to $\mathfrak{m}_{C,\bar{p}}$)

$$\widehat{\mathcal{O}}_{C,\bar{p}} \to \widehat{A}.$$

For this in turn we inductively find an element $t_q \in \mathcal{O}_{S,\bar{x}}/\mathfrak{m}^q_{S,\bar{x}}$ and a μ_r-equivariant isomorphism

$$\rho_q\colon ((\mathcal{O}_{S,\bar{x}}/\mathfrak{m}^q_{S,\bar{x}})[z,w]/(zw-t_q))^\wedge \to \widehat{A}/\mathfrak{m}^q_{S,\bar{x}}\widehat{A}.$$

For ρ_1 we take the isomorphism induced by ι and $t_1 = 0$. For the inductive step we assume (ρ_q, t_q) has been constructed and find (ρ_{q+1}, t_{q+1}). For this choose first any liftings $\tilde{z}, \tilde{w} \in \widehat{A}/\mathfrak{m}^{q+1}_{S,\bar{x}}\widehat{A}$ of z and w such that $\zeta \in \mu_r$ acts by

goodaction (2.3.2) $$\tilde{z} \mapsto \zeta \cdot \tilde{z}, \quad \tilde{w} \mapsto \zeta^{-1} \cdot \tilde{w}.$$

This is possible because μ_r is linearly reductive.

Since $\widehat{A}$ is flat over $\mathcal{O}_{S,\bar{x}}$ we have an exact sequence

$$0 \to \mathfrak{m}^q_{S,\bar{x}}/\mathfrak{m}^{q+1}_{S,\bar{x}} \otimes_{k(\bar{x})} \widehat{A}/\mathfrak{m}_{S,\bar{x}} \to \widehat{A}/\mathfrak{m}^{q+1}_{S,\bar{x}}\widehat{A} \to \widehat{A}/\mathfrak{m}^q_{S,\bar{x}}\widehat{A} \to 0.$$

Choosing any lifting $t_{q+1} \in \mathcal{O}_{S,\bar{x}}/\mathfrak{m}^{q+1}_{S,\bar{x}}$ of t_q and consider

$$\tilde{z}\tilde{w} - t_{q+1}.$$

This is a μ_r-invariant element of

$$\mathfrak{m}^q_{S,\bar{x}}/\mathfrak{m}^{q+1}_{S,\bar{x}} \otimes_{k(\bar{x})} \widehat{A}/\mathfrak{m}_{S,\bar{x}} \simeq \mathfrak{m}^q_{S,\bar{x}}/\mathfrak{m}^{q+1}_{S,\bar{x}} \otimes_{k(\bar{x})} ((\mathcal{O}_{S,\bar{x}}/\mathfrak{m}_{S,\bar{x}})[z,w]/(zw))^\wedge.$$

It follows that after possibly changing our choice of lifting t_{q+1} of t_q, we can write

$$\tilde{z}\tilde{w} = t_{q+1} + \tilde{z}^r g + \tilde{w}^r h,$$

where $t_{q+1} \in \mathcal{O}_{S,\bar{x}}/\mathfrak{m}^{q+1}_{S,\bar{x}}$ and reduces to t_q, and $g, h \in \mathfrak{m}^q_{S,\bar{x}}\widehat{A}/\mathfrak{m}^{q+1}_{S,\bar{x}}\widehat{A}$ are μ_r-invariant. Replacing $\tilde{z}$ by $\tilde{z} - \tilde{w}^{r-1}h$ and $\tilde{w}$ by $\tilde{w} - \tilde{z}^{r-1}g$ (note that with these new choices the action of μ_r is still as in 2.3.2) we obtain t_{q+1} and ρ_{q+1}. ♠

We give an alternative proof of (v) using a description of the cotangent complex of $\mathcal{C}$, which might be of interest on its own. Since $\mathcal{C}$ is an Artin stack a few words are in order. The cotangent complex $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ of a morphism of Artin stacks $\mathcal{X} \to \mathcal{Y}$ is defined in [24], Chapter 17 as an object in the derived category of quasi-coherent sheaves in the lisse-étale site of $\mathcal{X}$. Unfortunately, as was observed by Behrend and Gabber, this site is not functorial, which had the potential of rendering $\mathbb{L}_{\mathcal{X}}$ both incomputable and useless. However, in [28], see especially section 8, it is shown that the lisse-étale site has surrogate properties replacing functoriality which are in particular sufficient for dealing with $\mathbb{L}_{\mathcal{X}}$, as an object of a carefully constructed $D'_{qcoh}(\mathcal{X}_{\text{lis-ét}})$, and for deformation theory.

Two key properties are the following:

(1) given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks over a third $\mathcal{Z}$, there is a natural distinguished triangle
$$\mathbf{L}f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \xrightarrow{[1]} .$$

(2) Given a fiber diagram
$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f"] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$
with the horizontal maps flat, we have $f^*\mathbb{L}_{\mathcal{X}/\mathcal{Y}} = \mathbb{L}_{\mathcal{X}'/\mathcal{Y}'}$.

If the base scheme is S, consider $S \to \mathcal{B}\mathbb{G}_{m,S}$. Applying the above to the fiber square
$$\begin{tikzcd} \mathbb{G}_{m,S} \arrow[r] \arrow[d] & S \arrow[d] \\ S \arrow[r] & \mathcal{B}\mathbb{G}_{m,S} \end{tikzcd}$$
and the morphisms $S \to \mathcal{B}\mathbb{G}_{m,S} \to S$ it is easy to see that $\mathbb{L}_{\mathcal{B}\mathbb{G}_{m,S}/S} = \mathcal{O}_{\mathcal{B}\mathbb{G}_{m,S}}[-1]$. (In fact for a general smooth group scheme G we have $\mathbb{L}_{S/\mathcal{B}G} = Lie(G)^*$ and therefore $\mathbb{L}_{\mathcal{B}G/S} = Lie(G)^*[-1]$ considered as a G-equivariant object.)

Let now $S = \operatorname{Spec} k$ with k separably closed, and consider a twisted curve $\mathcal{C}/S$ with coarse moduli space $\pi\colon \mathcal{C} \to C$. Let $\bar{p} \to C$ be a geometric point mapping to a node, and fix an isomorphism
$$\mathcal{C}_{\bar{p}} := \mathcal{C} \times_C \bar{p} \simeq [D^{\mathrm{sh}}/\mu_r],$$
where $D := \operatorname{Spec}(k[z,w]/(zw))$. By a standard limit argument, we can thicken this isomorphism to a diagram
$$\begin{tikzcd} & \mathcal{C}_U \arrow[dl, "\alpha"'] \arrow[dd, "\pi_U"] \arrow[dr, "\beta"] & \\ \mathcal{C} \arrow[dd, "\pi"] & & {[D/\mu_r]} \arrow[dd] \\ & U \arrow[dl, "a"'] \arrow[dr, "b"] & \\ C & & D^{\mu_r}, \end{tikzcd}$$
where D^{μ_r} denote the coarse space of $[D/\mu_r]$ (so D^{μ_r} is equal to the spectrum of the invariants in $k[z,w]/(zw)$), a and b are étale, U is affine, and the

squares are cartesian. In this setting we can calculate a versal deformation of $\mathcal{C}_U$ as follows.

First of all we have the deformation

$$[\mathrm{Spec}(k[z,w,t]/(zw-t))/\mu_r]$$

of $[D/\mu_r]$. Since β is étale and representable and by the invariance of the étale site under infinitesimal thickenings, this also defines a formal deformation (i.e. compatible family of deformations over the reductions)

$$\widehat{\mathcal{C}}_{U,t} \to \mathrm{Spf}(k[[t]])$$

of $\mathcal{C}_U$. We claim that this deformation is versal. Since this deformation is nontrivial modulo t^2, to verify that $\widehat{\mathcal{C}}_{U,t}$ is versal it suffices to show that the deformation functor of $\mathcal{C}_U$ is unobstructed and that the tangent space is 1-dimensional. For this it suffices to show that

$$\mathrm{Ext}^2(\mathbb{L}_{\mathcal{C}_U}, \mathcal{O}_{\mathcal{C}}) = 0$$

and

$$\mathrm{Ext}^1(\mathbb{L}_{\mathcal{C}_U}, \mathcal{O}_{\mathcal{C}}) = k.$$

The map $\mathcal{C}_U \to [D/\mu_r]$ induces a morphism $\mathcal{C}_U \to \mathcal{B}\mu_r$. Consider the composite map $\mathcal{C}_U \to \mathcal{B}\mu_r \to \mathcal{B}\mathbb{G}_m$. Since $\mathbb{L}_{\mathcal{B}\mathbb{G}_m/k} \simeq \mathcal{O}_S[-1]$ we then obtain a distinguished triangle

$$\mathcal{O}_{\mathcal{C}}[-1] \longrightarrow \mathbb{L}_{\mathcal{C}_U} \longrightarrow \mathbb{L}_{\mathcal{C}_U/\mathcal{B}\mathbb{G}_m} \xrightarrow{[1]} .$$

Considering the associated long exact sequence of Ext-groups one sees that

$$\mathrm{Ext}^i(\mathbb{L}_{\mathcal{C}_U}, \mathcal{O}_{\mathcal{C}}) = \mathrm{Ext}^i(\mathbb{L}_{\mathcal{C}_U/\mathcal{B}\mathbb{G}_m}, \mathcal{O}_{\mathcal{C}})$$

for $i > 0$. Now to prove that

$$\mathrm{Ext}^2(\mathbb{L}_{\mathcal{C}_U/\mathcal{B}\mathbb{G}_m}, \mathcal{O}_{\mathcal{C}}) = 0, \quad \text{and} \quad \mathrm{Ext}^1(\mathbb{L}_{\mathcal{C}_U/\mathcal{B}\mathbb{G}_m}, \mathcal{O}_{\mathcal{C}}) = k,$$

it suffices to show that $\mathbb{L}_{\mathcal{C}/\mathcal{B}\mathbb{G}_m}$ can be represented by a two-term complex

$$F^{-1} \to F^0$$

with the F^i locally free sheaves of finite rank and the cokernel of the map

$$\mathcal{H}om(F^0, \mathcal{O}_{\mathcal{C}_U}) \to \mathcal{H}om(F^{-1}, \mathcal{O}_{\mathcal{C}_U})$$

isomorphic to the structure sheaf $\mathcal{O}_{\mathcal{Z}}$ of the singular locus $\mathcal{Z} \subset \mathcal{C}_U$ (with the reduced structure).

Now since β is étale, the complex $\mathbb{L}_{\mathcal{C}_U/\mathcal{B}\mathbb{G}_m}$ is isomorphic to $\beta^*\mathbb{L}_{[D/\mu_r]/\mathcal{B}\mathbb{G}_m}$. Therefore to construct such a presentation $F^{\cdot}$ of $\mathbb{L}_{\mathcal{C}_U/\mathcal{B}\mathbb{G}_m}$, it suffices to construct a corresponding presentation of $\mathbb{L}_{[D/\mu_r]/\mathcal{B}\mathbb{G}_m}$.

For this consider the fiber diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ S \arrow[r] & \mathcal{B}\mathbb{G}_m \end{tikzcd}$$

Here X is the surface $\operatorname{Spec}(k[z', w', u, u^{-1}]/(z'w'))$ endowed with the action of $\mathbb{G}_m$ via $t\cdot(z', w', u) = (tz', t^{-1}w', t^r u)$. Since X/S is a local complete intersection, the map $\mathcal{C} \to \mathcal{B}\mathbb{G}_m$ is also a representable local complete intersection morphism, and therefore the natural map $\mathbb{L}_{\mathcal{C}/\mathcal{B}\mathbb{G}_m} \to H_0(\mathbb{L}_{\mathcal{C}/\mathcal{B}\mathbb{G}_m}) = \Omega^1_{\mathcal{C}/\mathcal{B}\mathbb{G}_m}$ is a quasi-isomorphism. Concretely, the pullback of $\mathbb{L}_{\mathcal{C}/\mathcal{B}\mathbb{G}_m}$ is isomorphic to $\Omega^1_{X/S}$. Now we have a standard two-term resolution

$$0 \to \mathcal{O}_X \to \mathcal{O}_X^3 \to \Omega^1_{X/S} \to 0$$

where the first map is given by $(w', z', 0)$ and the second by $(dz', dw', du/u)$. Our desired presentation of $\mathbb{L}_{[D/\mu_r]/\mathcal{B}\mathbb{G}_m}$ is then obtained by noting that this resolution clearly descends to a locally free resolution on $\mathcal{C}$.

Remark 2.4. Let S be a scheme. A priori the collection of twisted n-pointed curves over S (with morphisms only isomorphisms) forms a 2-category. However, by the same argument proving [1, 4.2.2] the 2-category of twisted n-pointed curves over S is equivalent to a 1-category. In what follows we will therefore consider the category of twisted n-pointed curves, whose arrows are isomorphism classes of 1-arrows in the 2-category.

cohlemb **Proposition 2.5.** *Let $f\colon \mathcal{C} \to S$ be a twisted curve. Then the adjunction map $\mathcal{O}_S \to f_*\mathcal{O}_\mathcal{C}$ is an isomorphism, and for any quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $R^i f_*\mathcal{F} = 0$ for $i \geq 2$.*

Proof. Let $\pi\colon \mathcal{C} \to C$ be the coarse space, and let $\bar{f}\colon C \to S$ be the structure morphism. Since π_* is an exact functor on the category of quasi-coherent sheaves on $\mathcal{C}$, we have for any quasicoherent sheaf $\mathcal{F}$ on $\mathcal{C}$ an equality

$$R^i f_*\mathcal{F} = R^i \bar{f}_*(\pi_*\mathcal{F}).$$

Since $\bar{f}\colon C \to S$ is a nodal curve this implies that $R^i f_*\mathcal{F} = 0$ for $i \geq 2$, and the first statement follows from the fact that $\pi_*\mathcal{O}_\mathcal{C} = \mathcal{O}_C$. ♠

We conclude this section by recording some facts about the Picard functor of a twisted curve (these results will be used in section 5 below).

Let S be a scheme and $f\colon \mathcal{C} \to S$ a twisted curve over S. Let $\mathcal{P}ic_{\mathcal{C}/S}$ denote the stack over S which to any S-scheme T associates the groupoid of line bundles on the base change $\mathcal{C}_T := \mathcal{C} \times_S T$.

3→ **Proposition 2.6.** *The stack $\mathcal{P}ic_{\mathcal{C}/S}$ is a smooth locally finitely presented*[3] *algebraic stack over S, and for any object $\mathcal{L} \in \mathcal{P}ic_{\mathcal{C}/S}(T)$ (a line bundle on $\mathcal{C}_T$), the group scheme of automorphisms of $\mathcal{L}$ is canonically isomorphic to $\mathbb{G}_{m,T}$.*

Proof. That $\mathcal{P}ic_{\mathcal{C}/S}$ is algebraic and locally finitely presented is a standard application of Artin's criterion for verifying that a stack is algebraic [5]. This stack can also be constructed as a Hom-stack. See [6] for details.

To see that $\mathcal{P}ic_{\mathcal{C}/S}$ is smooth we apply the infinitesimal criterion. For this it suffices to consider the case where S affine and we have a closed

[3](Angelo) Added "locally finitely presented"

subscheme $S_0 \subset S$ with square-0 ideal I. Writing $\mathcal{C}_0$ for $\mathcal{C} \times_S S_0$, we have an exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}} \otimes I \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}_0} \to 0$$

giving an exact sequence of cohomology

$$\mathcal{P}ic_{\mathcal{C}/S}(S) \to \mathcal{P}ic_{\mathcal{C}/S}(S_0) \to R^2 f_* \mathcal{O}_{\mathcal{C}} \otimes I.$$

By Proposition 2.5 the term on the right vanishes, giving the existence of the desired lifting of $\mathcal{P}ic_{\mathcal{C}/S}(S) \to \mathcal{P}ic_{\mathcal{C}/S}(S_0)$.

The statement about the group of automorphisms follows from the fact that the map $\mathcal{O}_S \to f_*\mathcal{O}_{\mathcal{C}}$ is an isomorphism, and the same remains true after arbitrary base change $T \to S$. ♠

Let $\mathrm{Pic}_{\mathcal{C}/S}$ denote the rigidification of $\mathcal{P}ic_{\mathcal{C}/S}$ with respect to the group scheme $\mathbb{G}_m$ [4, Appendix A].

Let $\pi \colon \mathcal{C} \to C$ denote the coarse moduli space of $\mathcal{C}$ (so C is a nodal curve over S). Pulling back along π defines a fully faithful functor

$$\pi^* \colon \mathcal{P}ic_{C/S} \to \mathcal{P}ic_{\mathcal{C}/S}$$

which induces a morphism

Picmapb (2.6.1) $$\pi^* \colon \mathrm{Pic}_{C/S} \to \mathrm{Pic}_{\mathcal{C}/S}.$$

This morphism is a monomorphism of group schemes over S. Indeed suppose given two S-valued points $[L_1], [L_2] \in \mathrm{Pic}_{C/S}(S)$ defined by line bundles L_1 and L_2 on C such that $\pi^*[L_1] = \pi^*[L_2]$. Then after possibly replacing S by an étale cover, the two line bundles π^*L_1 and π^*L_2 on $\mathcal{C}$ are isomorphic. Since $L_i = \pi_*\pi^*L_i$ $(i = 1, 2)$ this implies that L_1 and L_2 are also isomorphic.

cokernel **Lemma 2.7.** *The cokernel W of the morphism π^* in (2.6.1) is an étale group scheme over S.*

Remark 2.8. Note that, in general, W is non-separated. Consider S the affine line over a field k with parameter t, and consider the blowup X of $\mathbb{P}^1 \times S$ at the origin. The zero fiber has a singularity with local equation $uv = t$. There is a natural action of $\boldsymbol{\mu}_r$ along the fiber, which near the singularity looks like $(u, v) \mapsto (\zeta_r u, \zeta_r^{-1} v)$. The quotient $\mathcal{C} = [X/\boldsymbol{\mu}_r]$ is a twisted curve, with twisted markings at the sections at 0 and ∞, and a twisted node. The coarse curve $C = X/\boldsymbol{\mu}_r$ has an A_{r-1} singularity $xy = t^r$, where $x = u^r$ and $y = v^r$. Let $E \subset \mathcal{C}$ be one component of the singular fiber. The invertible sheaf $O_{\mathcal{C}}(E)$ gives a generator of W, but it coincides with the trivial sheaf away from the singular fiber. In fact W is the group-scheme obtained by gluing r copies of S along the open subset $t \neq 0$.

Proof. The space W is the quotient of a smooth group scheme over S by the action of a smooth group scheme, hence is smooth over S. To show that W is étale we use the infinitesimal lifting criterion. If $T_0 \hookrightarrow T$ is a square zero nilpotent thickening defined by a sheaf of ideals $I \subset \mathcal{O}_T$, and if $\mathcal{L}$ is a line bundle on $\mathcal{C}_T$ whose reduction $\mathcal{L}_0$ on $\mathcal{C}_{T_0}$ is obtained by pullback from a line bundle on C_{T_0}, then we need to show that $\mathcal{L}$ is the pullback of a line bundle on C_T. Using the exponential sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_T^\times \longrightarrow \mathcal{O}_{T_0}^\times \longrightarrow 1$$

this amounts to the statement that the map

$$\mathrm{H}^i(C_{T_0}, I\mathcal{O}_{C_T}) \to \mathrm{H}^i(\mathcal{C}_{T_0}, I\mathcal{O}_{\mathcal{C}_T})$$

4→ is an isomorphism.[4] ♠

Lemma 2.9. *The cokernel W is quasi-finite over S.*

5→ *Proof.* [5] We may assume that S is affine, say $S = \operatorname{Spec} A$. Then the twisted curve $\mathcal{C}$ is defined over some finitely generated $\mathbb{Z}$-subalgebra A_0, and formation of W commutes with base change; hence we may assume that $A = A_0$. In particular S is noetherian.

We may also assume that S is reduced. After passing to a stratification of S and to étale coverings, we may assume that S is connected, that $C \to S$ is of constant topological type, that the nodes of the geometric fibers of $C \to S$ are supported along closed subschemes $S_1, \ldots, S_k$ of C which map isomorphically onto S, and the reduced inverse image Σ_i of S_i in $\mathcal{C}$ is isomorphic to the classfying stack $\mathcal{B}\boldsymbol{\mu}_{m_i}$ for certain positive integers $m_1, \ldots, m_k$. Given a line bundle $\mathcal{L}$ on $\mathcal{C}$, we can view the pullback of $\mathcal{L}$ to Σ_i as a line bundle L_i on S_i, with an action of $\boldsymbol{\mu}_{m_i}$. Such action is given by a character $\boldsymbol{\mu}_{m_i} \to \mathbb{G}_\mathrm{m}$ of the form $t \mapsto t^{a_i}$ for some $a_i \in \mathbb{Z}/(m_i)$. By sending L into the collection (a_i) we obtain a morphism $\operatorname{Pic}_{\mathcal{C}/S} \to \prod_i \mathbb{Z}/(m_i)$, whose kernel is easily seen to be $\operatorname{Pic}_{C/S}$. Therefore we have a categorically injective map $W \to \prod_i \mathbb{Z}/(m_i)$; since W and $\prod_i \mathbb{Z}/(m_i)$ are étale over S, this is an open embedding. Since $\prod_i \mathbb{Z}/(m_i)$ is noetherian and quasi-finite over S, so is W. ♠

If N denotes an integer annihilating W we obtain a map

$$\times N \colon \operatorname{Pic}_{\mathcal{C}/S} \to \operatorname{Pic}_{C/S}.$$

Def:Pic^o **Definition 2.10.** Let $\operatorname{Pic}^{\mathrm{o}}_{\mathcal{C}/S}$ denote the fiber product

$$\operatorname{Pic}^{\mathrm{o}}_{\mathcal{C}/S} = \operatorname{Pic}_{\mathcal{C}/S} \times_{\times N, \operatorname{Pic}_{C/S}} \operatorname{Pic}^{\mathrm{o}}_{C/S}.$$

The open and closed subfunctor $\operatorname{Pic}^{\mathrm{o}}_{\mathcal{C}/S} \subset \operatorname{Pic}_{\mathcal{C}/S}$ classifies degree 0-line bundles on $\mathcal{C}$. In particular, $\operatorname{Pic}^{\mathrm{o}}_{\mathcal{C}/S}$ is independent of the choice of N in the above construction. Note also that any torsion point of $\operatorname{Pic}_{\mathcal{C}/S}$ is automatically contained in $\operatorname{Pic}^{\mathrm{o}}_{\mathcal{C}/S}$ since the cokernel of

$$\operatorname{Pic}^{\mathrm{o}}_{C/S} \to \operatorname{Pic}_{C/S}$$

is torsion free.

Since $\operatorname{Pic}^{\mathrm{o}}_{C/S}$ is a semiabelian scheme over S, we obtain the following:

[4](Dan) Changed here a bit. Unless I misunderstood the exponential sequence, we need isomorphism of H^2.

[5](Angelo) changed the proof

3.7 **Corollary 2.11.** *The group scheme* $\mathrm{Pic}^{\mathrm{o}}_{\mathcal{C}/S}$ *is an extension of a quasi-finite étale group scheme over* S *by a semiabelian group scheme.*

3. Interlude: Relative moduli spaces

interlude

In this section we gather some results on relative moduli spaces which will be used in the verification of the valuative criterion for moduli stacks of twisted stable maps in the next section.

For a morphism of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$, denote by $\mathcal{I}(\mathcal{X}/\mathcal{Y}) = \mathrm{Ker}\,(\mathcal{I}(\mathcal{X}) \to f^*\mathcal{I}(\mathcal{Y})) = \mathcal{X} \times_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}} \mathcal{X}$ the relative inertia stack.

Now let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks, locally of finite presentation, and assume the relative inertia $\mathcal{I}(\mathcal{X}/\mathcal{Y}) \to \mathcal{X}$ is finite. Recall that when $\mathcal{Y}$ is an algebraic space we know that the morphism factors through the coarse moduli space: $\mathcal{X} \to X \to \mathcal{Y}$, see [21, 10]. We claim that there is a relative construction that generalizes to our situation:

Th:rel-cms **Theorem 3.1.** *There exists an algebraic stack* X*, and morphisms* $\mathcal{X} \xrightarrow{\pi} X \xrightarrow{\overline{f}} \mathcal{Y}$ *such that* $f = \overline{f} \circ \pi$*, satisfying the following properties:*

(1) $\overline{f}\colon X \to \mathcal{Y}$ *is representable,*
(2) $\mathcal{X} \xrightarrow{\pi} X \xrightarrow{\overline{f}} \mathcal{Y}$ *is initial for diagrams satisfying (1), namely if* $\mathcal{X} \xrightarrow{\pi'} X' \xrightarrow{\overline{f'}} \mathcal{Y}$ *has representable* $\overline{f'}$ *then there is a unique* $h\colon X \to X'$ *such that* $\pi' = h \circ \pi$ *and* $\overline{f} = \overline{f}' \circ h$*,*
(3) π *is proper and quasifinite,*
(4) $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ *is an isomorphism, and*
(5) if X' *is a stack and* $X' \to X$ *is a representable flat morphism, then* $\mathcal{X} \times_X X' \to X' \to \mathcal{Y}$ *satisfies properties (1)-(4) starting from* $\mathcal{X} \times_X X' \to \mathcal{Y}$*.*

Definition 3.2. We call $\overline{f} : X \to \mathcal{Y}$ the relative coarse moduli space of $f\colon \mathcal{X} \to \mathcal{Y}$.

Proof. Consider a smooth presentation $R \rightrightarrows U$ of $\mathcal{Y}$. Denote by X_R the coarse moduli space of $\mathcal{X}_R = \mathcal{X} \times_{\mathcal{Y}} R$ and X_U the coarse moduli space of $\mathcal{X}_U = \mathcal{X} \times_{\mathcal{Y}} U$, both exist by the assumption of finiteness of relative inertia.

Since the formation of coarse moduli spaces commutes with flat base change, we have that $X_R = X_U \times_U R$. It is straightforward to check that $X_R \rightrightarrows U_R$ is a smooth groupoid. Denote its quotient stack $X := [X_R \rightrightarrows U_R]$. It is again straightforward to construct the morphisms $\mathcal{X} \xrightarrow{\pi} X \xrightarrow{\overline{f}} \mathcal{Y}$ from the diagram of relevant groupoids.

Now $X \times_{\mathcal{Y}} U \to X_U$ is an isomorphism, $U \to \mathcal{Y}$ is surjective and $X_U \to U$ is representable, hence $X \to \mathcal{Y}$ is representable, giving (1). Similarly $\mathcal{X}_U \to X_U$ is proper and quasifinite hence $\mathcal{X} \to X$ is proper and quasifinite, giving (3). Also (4) follows by flat base-change to X_U. Part (5) follows again since formation of moduli spaces commutes with flat base change: the coarse moduli space of $\mathcal{X}_U \times_X X'$ is $X_U \times_X X'$ and similarly the coarse moduli space of $\mathcal{X}_R \times_X X'$ is $X_R \times_X X'$.

Now consider the situation in (2). Since $X' \to \mathcal{Y}$ is representable, it is presented by the groupoid $X' \times_{\mathcal{Y}} R \rightrightarrows X' \times_{\mathcal{Y}} U$, which, by the universal property of coarse moduli spaces, is the target of a canonical morphism of groupoids from $X_R \rightrightarrows U_R$, giving the desired morphism $h\colon X \to X'$. ♠

Now we consider the tame case. We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ with finite relative inertia is *tame* if for some algebraic space U and faithfully flat $U \to \mathcal{Y}$ we have that $\mathcal{X} \times_{\mathcal{Y}} U$ is a tame algebraic stack. This notion is independent of the choice of U by [4, Theorem 3.2]. Let $\mathcal{X} \to X \to \mathcal{Y}$ be the relative coarse moduli space.

Proposition 3.3. *Assume $f\colon \mathcal{X} \to \mathcal{Y}$ is tame. If X' is an algebraic stack and $X' \to X$ is a representable morphism of stacks, then $X' \to \mathcal{Y}$ is the relative coarse moduli space of $\mathcal{X} \times_X X' \to \mathcal{Y}$.*

This is proven as in part (5) of the theorem, using the fact that formation of coarse moduli spaces of tame stacks commutes with arbitrary base change.

Now we consider a special case which is relevant for our study of tame stacks.

Let V be a strictly henselian local scheme, with a finite linearly reductive group scheme Γ acting, fixing a geometric point $s \to V$. Let $\mathcal{X} = [V/\Gamma]$, and consider a morphism $f\colon \mathcal{X} \to \mathcal{Y}$, with $\mathcal{Y}$ also tame.

Let $K \subset \Gamma$ be the subgroup scheme fixing the composite object $s \to V \to \mathcal{Y}$ (so K is the kernel of the map $\Gamma \to \mathrm{Aut}_{\mathcal{Y}}(f(s))$). Consider the geometric quotient $U = V/K$, the quotient group scheme $Q = \Gamma/K$ and let $\overline{\mathcal{X}} = [U/Q]$. There is a natural projection $q\colon \mathcal{X} \to \overline{\mathcal{X}}$.

Prop:B3 **Proposition 3.4.** *There exists a unique morphism $g\colon \overline{\mathcal{X}} \to \mathcal{Y}$ such that $f = g \circ q$, and this factorization of f identifies $\overline{\mathcal{X}}$ as the relative coarse moduli space of $\mathcal{X} \to \mathcal{Y}$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f'"] \arrow[rr, bend left, "f"] \arrow[dr] & \mathcal{Y}_{\mathcal{X}} \arrow[r, "h"] \arrow[d] & \mathcal{Y} \arrow[d] \\
 & X \arrow[r, "\overline{f}"] & Y
\end{tikzcd}$$

where X, Y are the coarse moduli spaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively, the square is cartesian, and $\overline{f}$ is induced from f.

Since h is fully faithful we may replace $\mathcal{Y}$ by $\mathcal{Y}_{\mathcal{X}}$. As the formation of coarse moduli space of tame stacks commutes with arbitrary base change, X is the coarse moduli space of $\mathcal{Y}_{\mathcal{X}}$. We may therefore assume that $X = Y$.

It also suffices to prove the proposition after making a flat base change on X. We may therefore assume that the residue field of V is algebraically closed. We can write $\mathcal{Y} = [P/G]$, where P is a finite X-scheme and G is a finite flat split linearly reductive group scheme over X. We can assume P is local strictly henselian as well, and that the action of G on P fixes the closed point $p \in P$.

The map f induces, by looking at residual gerbes, a map $\mathcal{B}\rho_0 \colon \mathcal{B}\Gamma_0 \to \mathcal{B}G_0$ where Γ_0 and G_0 denote the corresponding reductions to the closed point of X. This is induced by a group homomorphism $\rho_0 \colon \Gamma_0 \to G_0$, uniquely defined up to conjugation. Since Γ and G are split, the homomorphism ρ_0 lifts to a homomorphism $\rho \colon \Gamma \to G$.

We obtain a 2-commutative diagram

$$\begin{tikzcd} \mathcal{B}\Gamma_0 \arrow[r, "\mathcal{B}\rho_0"] \arrow[d, "\bar{s}"'] & \mathcal{B}G_0 \arrow[d, "\bar{p}"] \\ {[V/\Gamma]} \arrow[r, "f"] & {[P/G].} \end{tikzcd}$$

Let $P' \to [V/\Gamma]$ be the pullback of the G-torsor $P \to [P/G]$

Lemma 3.5. *There is an isomorphism of G-torsors over $[V/\Gamma]$*

$$[V \times G/\Gamma] \to P'.$$

Proof. Note that $V \times_{[V/\Gamma]} \mathcal{B}\Gamma_0 = s$ and $P' \times_{[V/\Gamma]} \mathcal{B}\Gamma_0 = \mathcal{B}\Gamma_0 \times_{\mathcal{B}G_0} p$. Then the commutativity of the diagram

$$\begin{tikzcd} s \arrow[r] \arrow[d] & p \arrow[d] \\ \mathcal{B}\Gamma_0 \arrow[r] & \mathcal{B}G_0 \end{tikzcd}$$

implies that there is an isomorphism of G_0-torsors over $\mathcal{B}\Gamma_0$

$$([V \times G/\Gamma] \times_{[V/\Gamma]} \mathcal{B}\Gamma_0 \xrightarrow{\sim} P' \times_{[V/\Gamma]} \mathcal{B}\Gamma_0.$$

As in the proof of [4, 3.6], this lifts to an isomorphism as required. ♠

Now we have a morphism

$$\begin{tikzcd} V \arrow[r] \arrow[rrr, bend left, "\tilde{f}"] & {[V \times G/\Gamma]} \arrow[r] & P' \arrow[r] & P \end{tikzcd}$$

compatible with the actions of Γ and G. By the universal property of the quotient, the map $\tilde{f}$ factors through U. Passing to the quotient by the respective group actions, we get morphisms

$$[V/\Gamma] \to [U/Q] \to [P/G].$$

Let $\mathcal{X} \to \mathcal{Z} \to \mathcal{Y}$ be the relative coarse moduli space. Since Q injects in G, the morphism $[U/Q] \to [P/G]$ is representable, and therefore we have a morphism $\mathcal{Z} \to [U/Q]$ over $\mathcal{Y}$.

To check this is an isomorphism it suffices to check after base change along the flat morphism $P \to \mathcal{Y}$. In other words, we need to check that $[V/\Gamma] \times_{[P/G]} P \to [U/Q] \times_{[P/G]} P$ is the coarse moduli space. This map can be identified with $[V \times G/\Gamma] \to U \times^Q G$. This follows by noting that

$$(\mathcal{O}_V \otimes \mathcal{O}_G)^\Gamma = ((\mathcal{O}_V)^K \otimes \mathcal{O}_G)^Q = (\mathcal{O}_U \otimes \mathcal{O}_G)^Q.$$

♠

Section:3

4. Twisted stable maps

This section relies on the results of Appendix A, in which the stack of twisted curves is constructed.

Let $\mathcal{M}$ be a finitely presented tame algebraic stack with finite inertia proper over a scheme S. A *twisted stable map* from an n-marked twisted curve $(f : \mathcal{C} \to S, \{\Sigma_i \subset \mathcal{C}\}_{i=1}^n)$ over S to $\mathcal{M}$ is a representable morphism of S-stacks

$$g : \mathcal{C} \to \mathcal{M}$$

such that the induced morphism on coarse spaces

$$\overline{g} : C \to M$$

is a stable map (in the usual sense of Kontsevich) from the n-pointed curve $(C, \{p_1, \dots, p_n\}_{i=1}^n)$ to M (where the points p_i are the images of the Σ_i).

If M is projective over a field, the *target class* of g is the class of $\overline{g}_*[C]$ in the chow group of 1-dimensional cycles up to numerical equivalence of M.

Th:3.1

Theorem 4.1. *Let $\mathcal{M}$ be a finitely presented tame algebraic stack proper over a scheme S with finite inertia. Then the stack $\mathcal{K}_{g,n}(\mathcal{M})$ of twisted stable maps from n-pointed genus g curves into $\mathcal{M}$ is a locally finitely presented algebraic stack over S, which is proper and quasi-finite over the stack of stable maps into M. If M is projective over a field, the substack $\mathcal{K}_{g,n}(\mathcal{M}, \beta)$ of twisted stable maps from n-pointed genus g curves into $\mathcal{M}$ with target class β is proper, and is open and closed in $\mathcal{K}_{g,n}(\mathcal{M})$.*

Proof. The statement is local in the Zariski topology of S, therefore we may assume S affine. Since $\mathcal{M}$ is of finite presentation, it is obtained by base change from the spectrum of a noetherian ring, and replacing S by this spectrum we may assume $\mathcal{M}$ is of finite type over a noetherian scheme S.

As $\mathcal{M}$ is now of finite type, there is an integer N such that the degree of the automorphism group scheme of any geometric point of $\mathcal{M}$ is $\leq N$.

Consider the stack of twisted curves $\mathfrak{M}_{g,n}^{\mathrm{tw}}$ defined in A.6. It contains an open substack $\mathfrak{M}_{g,n}^{\mathrm{tw},\leq N} \subset \mathfrak{M}_{g,n}^{\mathrm{tw}}$ of twisted curves of index bounded by N by A.8. The natural morphism $\mathfrak{M}_{g,n}^{\mathrm{tw},\leq N} \to \mathfrak{M}_{g,n}$ to the stack of prestable curves is quasi-finite by A.8.

Consider $\mathfrak{B} = \mathfrak{M}_{g,n}^{\mathrm{tw},\leq N} \times_{\operatorname{Spec}\mathbb{Z}} S$. It is an algebraic stack of finite type over $B = \mathfrak{M}_{g,n} \times_{\operatorname{Spec}\mathbb{Z}} S$. Denote $\mathcal{M}_{\mathfrak{B}} = \mathcal{M} \times_S \mathfrak{B}$, $M_{\mathfrak{B}} = M \times_S \mathfrak{B}$ and similarly $M_B = M \times_S B$. These are all algebraic stacks.

Finally consider the universal curves $\mathfrak{C}^{\mathrm{tw}} \to \mathfrak{M}^{\mathrm{tw}}_{g,n}$ and $\mathfrak{C} \to \mathfrak{M}_{g,n}$, and their corresponding pullbacks $\mathfrak{C}^{\mathrm{tw}}_{\mathfrak{B}}$, $\mathfrak{C}_{\mathfrak{B}}$ and $\mathfrak{C}_B$.

As discussed in Appendix C, there is an algebraic stack

$$\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathfrak{B}}(\mathfrak{C}^{\mathrm{tw}}_{\mathfrak{B}}, \mathcal{M}_{\mathfrak{B}}),$$

locally of finite type over $\mathfrak{B}$, parametrizing representable morphisms of the universal twisted curve to $\mathcal{M}$. We also have the analogous $\underline{\mathrm{Hom}}_{\mathfrak{B}}(\mathfrak{C}_{\mathfrak{B}}, M_{\mathfrak{B}})$ and $\underline{\mathrm{Hom}}_B(\mathfrak{C}_B, M_B)$. We have the natural morphisms

compositemap (4.1.1) $$\underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathfrak{B}}(\mathfrak{C}^{\mathrm{tw}}_{\mathfrak{B}}, \mathcal{M}_{\mathfrak{B}}) \xrightarrow{a} \underline{\mathrm{Hom}}_{\mathfrak{B}}(\mathfrak{C}_{\mathfrak{B}}, M_{\mathfrak{B}}) \xrightarrow{b} \underline{\mathrm{Hom}}_B(\mathfrak{C}_B, M_B).$$

5.2 **Proposition 4.2.** *The morphism a is quasi-finite.*

Proof. It suffices to prove the following. Let k be an algebraically closed field with a morphism $\mathrm{Spec}(k) \to S$ and $\mathcal{C}/k$ a twisted curve with coarse moduli space C. Let $f\colon C \to M$ be a S-morphism, and let $\mathcal{G} \to \mathcal{C}$ denote the pullback of $\mathcal{M}$ to $\mathcal{C}$ along the composite

$$\mathcal{C} \longrightarrow C \xrightarrow{f} M.$$

We then need to show that the groupoid $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{C})(k)$ (notation as in C.17) has only finitely many isomorphism classes of objects.

We first reduce to the case where C is smooth. Let C' denote the normalization of C (so C' is a disjoint union of smooth curves), and let $\mathcal{C}' \to \mathcal{C}$ be the maximal reduced substack of the fiber product $\mathcal{C} \times_C C'$. For any node $x \in C$ let Σ_x denote maximal reduced substack of $\mathcal{C}_x$ (so Σ_x is isomorphic to $B\mu_r$ for some r), and let Σ denote the disjoint union of the Σ_x over the nodes x. Then there are two natural inclusions $j_1, j_2\colon \Sigma \hookrightarrow \mathcal{C}'$, and by [3, A.0.2] the functor

$$\begin{array}{c} \underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{M})(k) \\ \Big\downarrow \\ \underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{M})(k) \times_{j_1^* \times j_2^*, \underline{\mathrm{Hom}}(\Sigma,\mathcal{M})(k) \times \underline{\mathrm{Hom}}(\Sigma,\mathcal{M})(k), \Delta} \underline{\mathrm{Hom}}(\Sigma, \mathcal{M})(k) \end{array}$$

is an equivalence of categories. Since the stack $\underline{\mathrm{Hom}}(\Sigma, \mathcal{M})$ clearly has quasi-finite diagonal, it follows that the map

$$\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{C}) \to \underline{\mathrm{Sec}}(\mathcal{G} \times_{\mathcal{C}} \mathcal{C}'/\mathcal{C}')$$

is quasi-finite.

It follows that to prove 4.2 it suffices to consider the case when $\mathcal{C}$ is smooth.

Let $\mathcal{Y}$ denote the normalization of $\mathcal{G}$. Then any section $\mathcal{C} \to \mathcal{G}$ factors uniquely through $\mathcal{Y}$ so it suffices to show that the set

$$\underline{\mathrm{Sec}}(\mathcal{Y}/\mathcal{C})(k)$$

is finite. To prove this we may without loss of generality assume that there exists a section $s\colon \mathcal{C} \to \mathcal{Y}$.

For any smooth morphism $V \to \mathcal{C}$ the coarse space of the fiber product $\mathcal{G}_V := \mathcal{G} \times_{\mathcal{C}} V$ is equal to V by [4], Corollary 3.3 (a). For a smooth morphism $V \to \mathcal{C}$, let

$$\mathcal{Y}_V \xrightarrow{c} \overline{\mathcal{Y}}_V \xrightarrow{d} V$$

be the factorization of $\mathcal{Y}_V \to V$ provided by [35, 2.1] (rigidification of the generic stabilizer). The section s induces a section $V \to \overline{\mathcal{Y}}_V$, and hence by [30, 2.9 (ii)] the projection $d\colon \overline{\mathcal{Y}} \to V$ is in fact an isomorphism. It follows that $\mathcal{Y}$ is a gerbe over $\mathcal{C}$. If $\mathbb{G} \to \mathcal{C}$ denotes the automorphism group scheme of s, then in fact $\mathcal{Y} = \mathcal{B}_{\mathcal{C}}\mathbb{G}$.

Let $\Delta \subset \mathbb{G}$ denote the connected component of $\mathbb{G}$, and let $\mathbb{H}$ denote the étale part of $\mathbb{G}$, so we have a short exact sequence

$$1 \to \Delta \to \mathbb{G} \to \mathbb{H} \to 1$$

of group schemes over $\mathcal{C}$.

Let $\mathcal{C}' \to \mathcal{C}$ be a finite étale cover such that $\mathbb{H}$ and the Cartier dual of Δ constant. If $\mathcal{C}''$ denotes the product $\mathcal{C}' \times_{\mathcal{C}} \mathcal{C}'$ and $\mathcal{Y}'$ and $\mathcal{Y}''$ the pullbacks of $\mathcal{Y}$ to $\mathcal{C}'$ and $\mathcal{C}''$ respectively, then one sees using a similar argument to the one used in [30, §3] that it suffices to show that

$$\underline{\mathrm{Sec}}(\mathcal{Y}'/\mathcal{C}')(k) \quad \text{and} \quad \underline{\mathrm{Sec}}(\mathcal{Y}''/\mathcal{C}'')(k)$$

are finite. We may therefore assume that $\mathbb{H}$ and the Cartier dual of Δ are constant.

The map $\mathbb{G} \to \mathbb{H}$ induces a projection over $\mathcal{C}$

$$\mathcal{B}_{\mathcal{C}}\mathbb{G} \to \mathcal{B}_{\mathcal{C}}\mathbb{H}.$$

Giving a section $z\colon \mathcal{C} \to \mathcal{B}_{\mathcal{C}}\mathbb{H}$ is equivalent to giving an $\mathbb{H}$-torsor $P \to \mathcal{C}$. Since $\mathcal{C}$ is normal and $\mathbb{H}$ is étale, such a torsor is determined by its restriction to any dense open subspace of $\mathcal{C}$. Since $\mathcal{C}$ contains a dense open subscheme and the étale fundamental group of such a subscheme is finitely generated it follows that

$$\underline{\mathrm{Sec}}(\mathcal{B}_{\mathcal{C}}\mathbb{H}/\mathcal{C})(k)$$

is finite.

Fix a section $z\colon \mathcal{C} \to \mathcal{B}_{\mathcal{C}}\mathbb{H}$, and let $\mathcal{Q} \to \mathcal{C}$ be the fiber product

$$\mathcal{Q} := \mathcal{C} \times_{z, \mathcal{B}_{\mathcal{C}}\mathbb{H}} \mathcal{B}_{\mathcal{C}}\mathbb{G}.$$

Then $\mathcal{Q}$ is a gerbe over $\mathcal{C}$ banded by a twisted form of Δ. By replacing $\mathcal{C}$ by a finite étale covering as above, we may assume that in fact $\mathcal{Q}$ is a gerbe banded by Δ and hence isomorphic to $\mathcal{C} \times_{\mathrm{Spec}(k)} \mathcal{B}\Delta_0$ for some finite diagonalizable group scheme Δ_0/k. Writing Δ_0 as a product of μ_r's, we even reduce to the case when $\Delta_0 = \mu_r$. In this case, as explained in C.25, giving a section $w\colon \mathcal{C} \to \mathcal{B}_{\mathcal{C}}\Delta$ is equivalent to giving a pair $(\mathcal{L}, \iota)$, where $\mathcal{L}$ is a line bundle on $\mathcal{C}$ and $\iota\colon \mathcal{O}_{\mathcal{C}} \to \mathcal{L}^{\otimes r}$ is an isomorphism. We conclude that the set of sections $\mathcal{C} \to \mathcal{Q}$ is in bijection with the set

$$\underline{\mathrm{Pic}}_{\mathcal{C}/k}[r](k)$$

of r-torsion points in the Picard space $\underline{\mathrm{Pic}}_{\mathcal{C}/k}$. By 2.11 this is a finite group, and hence this completes the proof of 4.2. ♠

Consider again the diagram 4.1.1. The morphism a is quasi-finite by 4.2 and of finite type by C.7. The second morphism is quasi-finite and of finite type as it is obtained by base change. Therefore the composite 4.1.1 is also quasi-finite and of finite type.

The Kontsevich moduli space $\overline{\mathcal{M}}_{g,n}(M)$ is open in $\underline{\mathrm{Hom}}_B(\mathfrak{C}_B, M_B)$. We have

$$\mathcal{K}_{g,n}(\mathcal{M}) \quad \simeq \quad \underline{\mathrm{Hom}}^{\mathrm{rep}}_{\mathfrak{B}}(\mathfrak{C}^{\mathrm{tw}}_{\mathfrak{B}}, \mathcal{M}_{\mathfrak{B}}) \times_{\underline{\mathrm{Hom}}_B(\mathfrak{C}_B, M_B)} \overline{\mathcal{M}}_{g,n}(M),$$

hence $\mathcal{K}_{g,n}(\mathcal{M}) \to \overline{\mathcal{M}}_{g,n}(M)$ is quasi-finite and of finite type.

Properness now follows from the valuative criterion, which is the content of the following Proposition 4.3. ♠

Prop:valuative

Proposition 4.3. *Let R be a discrete valuation ring, with spectrum T having generic point η and special point t. Fix a morphism $T \to S$ and let $\mathcal{C}_\eta \to \mathcal{M}$ be an n-pointed twisted stable map over the generic point of T, and $C \to M$ a stable map over T extending the coarse map $C_\eta \to M$. Then there is a discrete valuation ring R_1 containing R as a local subring, and corresponding morphism of spectra $T_1 \to T$, and an n-pointed twisted stable map $\mathcal{C}_{T_1} \to \mathcal{M}$, such that the restriction $\mathcal{C}_{T_1}|_{\eta_1} \to \mathcal{M}$ is isomorphic to the base change $\mathcal{C}_\eta \times_T T_1 \to \mathcal{M}$, and the coarse map coincides with $C \times_T T_1 \to M$.*

Such an extension, when it exists, is unique up to a unique isomorphism.

Before proving this Proposition, we need to extend a few basic results known in case $\mathcal{M}$ is a Deligne–Mumford stack.

Lem:purity

Lemma 4.4 (Purity Lemma)**.** *Let $\mathcal{M}$ be a separated tame stack with coarse moduli space M. Let X be a locally noetherian*[6] *separated scheme of dimension 2 satisfying Serre's condition S2. Let $P \subset X$ be a finite subset consisting of closed points, $U = X \smallsetminus P$. Assume that the local fundamental groups of U around the points of P are trivial and that the local Picard groups of U around points of P are torsion free.* ←6

Let $f \colon X \to M$ be a morphism. Suppose there is a lifting $\widetilde{f}_U \colon U \to \mathcal{M}$:

$$\begin{tikzcd} & & \mathcal{M} \arrow[d] \\ U \arrow[r] \arrow[urr, "\widetilde{f}_U"] & X \arrow[r, "f"] & M \end{tikzcd}$$

Then the lifting extends to X:

$$\begin{tikzcd} & & \mathcal{M} \arrow[d] \\ U \arrow[r] \arrow[urr, "\widetilde{f}_U"] & X \arrow[r, "f"] \arrow[ur, dotted, "\widetilde{f}"] & M \end{tikzcd}$$

[6](Angelo) Added "locally noetherian".

The lifting $\widetilde{f}$ is unique up to a unique isomorphism.

Remark 4.5. We will use this lemma principally in the following cases:

(1) X is regular,
(2) X is normal crossings, locally $\operatorname{Spec}(R[x,y]/xy)^{\mathrm{sh}}$, with R regular, or
(3) $X = X_0 \times G$, with X_0 one of the first two cases and G a linearly reductive finite group scheme.

Proof. The question is local in the étale topology, so we may replace X by its strict henselization over some point $p \in P$, and correspondingly we may replace M and $\mathcal{M}$ by the strict henselization at $f(p)$.

By [4, Theorem 3.2 (d)]we can write $\mathcal{M} = [V/G]$, where $V \to M$ is a finite morphism and $G \to M$ a linearly reductive group scheme acting on V. By [4] Lemma 2.20 we have an exact sequence

$$1 \longrightarrow \Delta \longrightarrow G \longrightarrow H \longrightarrow 1$$

of group schemes over M, where Δ is diagonalizable and H is tame and étale.

The morphism $\widetilde{f}_U$ is equivalent to the data of a G-torsor $P_U \to U$ and a G-equivariant morphism $U \to V$. We first wish to extend P_U over X.

Consider the H-torsor $Q_U \stackrel{\text{def}}{=} P_U/\Delta \to U$. Since the local fundamental groups of U around P are trivial, this H torsor is trivial, and extends trivially to an H torsor $Q \to X$.

Now $P_U \to Q_U$ is a Δ-torsor. We claim that it extends uniquely to a Δ-torsor $P \to Q$. Since Δ is diagonalizable, it suffices to treat the case $\Delta = \boldsymbol{\mu}_r$. In this case $P_U \to Q_U$ corresponds to an r-torsion line bundle with a chosen trivialization of its r-th power. The line bundle extends to X by the assumption on the local Picard groups, and the trivialization extends by the S2 assumption.

Next we need to extend the principal action of G on P_U to P. The closure Γ of the graph of the morphism $G \times_M P_U \to P_U$ inside the scheme $G \times_M P \times_X P$ is finite over $G \times_M P$, and an isomorphism over $G \times_M P_U$. As $G \times_M P$ is finite and flat over the S2 scheme X, it is also S2. It follows that the morphism $\Gamma \to G \times_M P$ is an isomorphism, hence the action extends to a morphism $G \times_M P \to P$. It is easy to check that this is a principal action, as required.

The same argument shows that the equivariant morphism $P_U \to V$ extends to an equivariant morphism $P \to V$, as required. ♠

Proof of Proposition 4.3. Step 1: We extend $\mathcal{C}_\eta \to \mathcal{M}$ over the generic points of the special fiber C_s.

The question is local in the étale topology of C, therefore we can replace C by its strict henselization at one of the generic points of C_s. We may similarly replace M by its strict henselization at the image point, and hence assume it is of the form $[V/G]$, where $V \to M$ is finite and G a locally well split group scheme, see [4], Theorem 3.2 (d) and Lemma 2.20. We again write G as an extension

$$1 \longrightarrow \Delta \longrightarrow G \longrightarrow H \longrightarrow 1.$$

The morphism $\mathcal{C}_\eta = C_\eta \to \mathcal{M}$ is equivalent to the data of a G-torsor $P_\eta \to C_\eta$ and a G-equivariant morphism $P_\eta \to V$. By Abhyankar's lemma, the H-torsor $Q_\eta \stackrel{\text{def}}{=} P_\eta/\Delta \to C_\eta$ extends uniquely to $Q \to C$ after a base change on R. We want to extend the Δ-torsor $P_\eta \to Q_\eta$ to a Δ-torsor $P \to Q$. Factoring Δ, we may assume $\Delta = \boldsymbol{\mu}_r$. In this case $P_\eta \to Q_\eta$ corresponds to an r-torsion line bundle with a chosen trivialization of its r-th power. This line bundle is trivial since we have localized, hence it trivially extends to Q. The extension of the trivializing section may have a zero or pole along C_s, but after an r-th order base change on R this zero or pole has multiplicity $r \cdot l$ for some integer l. Twisting the line bundle by $\mathcal{O}_C(lC_s)$, the trivialization extends uniquely over C_s as required.

We need to lift the principal action of G on P. Write again Γ for the closure of the graph of $G \times_T P_\eta \to P_\eta$ inside $G \times_T P_\eta \times_C P_\eta$. This is a Δ-equivariant subscheme, with respect to the action $\delta\cdot(g, p_1, p_2) = (\delta g, p_1, \delta p_2)$. The action is free, as it is already free on the last factor. Therefore the projection $\Gamma \to \Gamma/\Delta$ is a Δ-torsor. The quotient Γ/Δ includes the subscheme $(\Gamma/\Delta)_\eta = Q_\eta$. We claim that this is scheme-theoretically dense in Γ/Δ, because any nonzero sheaf of ideals with support over s would pull back to a nonzero sheaf of ideals on Γ. But Γ_η is by definition scheme theoretically dense in Γ. Since Q is normal, it follows that $\Gamma/\Delta = Q$. Now as both Γ and P are Δ-torsors over Q and $\Gamma \to P$ is Δ-equivariant, this morphism is an isomorphism. Hence we have a morphism $G \times P \to P$, extending the action. It is again easy to see this is a principal action with quotient C. It is also clearly unique.

Exactly the same argument shows that the morphism $P_\eta \to V$ extends uniquely to a G-equivariant $P \to V$, as required.

STEP 2: We extend the twisted stable map $C \to \mathcal{M}$ over the general locus C_{gen}, simply by applying the purity lemma (Lemma 4.4). Uniqueness in the Purity Lemma implies that the extension is unique up to unique isomorphism.

STEP 3: We extend the twisted curve $\mathcal{C}$ and the twisted stable map $\mathcal{C} \to \mathcal{M}$ along the smooth locus of C.

Consider the index r_i of $\mathcal{C}_\eta$ over a marking Σ_i^C of C. Let $\mathcal{C}_{\text{sm}}$ be the stack obtained by taking r_i-th root of Σ_i^C on C for all i. Then $(\mathcal{C}_{\text{sm}})_\eta$ is uniquely isomorphic to $(\mathcal{C}_\eta)_{\text{sm}}$.

We need to construct a representable map $\mathcal{C}_{\text{sm}} \to \mathcal{M}$ lifting $C_{\text{sm}} \to M$. The problem is local in the étale topology of $\mathcal{C}_{\text{sm}}$, so we may present $\mathcal{C}_{\text{sm}}$ at a point p on $\Sigma_i^C \cap C_s$ as $[D/\boldsymbol{\mu}_{r_i}]$, where D is smooth over T, with unique fixed point q over p. Applying the purity lemma to the map $D \smallsetminus \{q\} \to \mathcal{M}$, we have a unique extension $D \to \mathcal{M}$. The uniqueness applies also for $\boldsymbol{\mu}_{r_i} \times D \to \mathcal{M}$, which implies that the object $D \to \mathcal{M}$ is $\boldsymbol{\mu}_{r_i}$-equivariant, giving a unique morphism $[D/\boldsymbol{\mu}_{r_i}] \to \mathcal{M}$, as needed.

STEP 4: We extend the twisted curve $\mathcal{C}$ and the twisted stable map $\mathcal{C} \to \mathcal{M}$ over the closure of the singular locus of C_η.

This is similar to step 3.

STEP 5: Extension over isolated singular points.

Consider an isolated node p of C. Passing to an extension on R we may assume it is rational over the residue field k. Its strict henselization is isomorphic to $(\operatorname{Spec} R[u,v]/(uv - \pi^l))^{\mathrm{sh}}$, where π is a uniformizer of R, and by Remark A.9 there is a twisted curve $\mathcal{C}_l$ which is regular over p with index l. This twisted curve has a chart of type $[D/\boldsymbol{\mu}_l]$, where the strict henselization of D looks like $\operatorname{Spec} R[x,y]/(xy - \pi)$, where $\boldsymbol{\mu}_l$ acts via $(x,y) \mapsto (\zeta_l x, \zeta_l^{-1} y)$, and $u = x^l, v = y^l$. There is a unique point q of D over p.

To construct $\mathcal{C}_l \to \mathcal{M}$ it suffices to apply the purity lemma to the map $D \smallsetminus \{q\} \to \mathcal{M}$; the resulting extension $D \to \mathcal{M}$ is $\boldsymbol{\mu}_l$-equivariant by applying the Purity Lemma to $\boldsymbol{\mu}_l \times D \to \mathcal{M}$.

We need to replace the morphism $\mathcal{C}_l \to \mathcal{M}$, as it is not necessarily representable, and the construction of $\mathcal{C}_l$ does not commute with base change.

Consider the morphism $\mathcal{C}_l \to C \times \mathcal{M}$. This morphism is proper and quasi-finite. Let $\mathcal{C}_l \to \mathcal{C} \to C \times \mathcal{M}$ be the relative moduli space. By the local computation (Proposition 3.4), $\mathcal{C}$ is a twisted curve, hence $\mathcal{C} \to \mathcal{M}$ is a twisted stable map. Further, its formation commutes with further base change since the formation of moduli space does. ♠

5. REDUCTION OF SPACES OF GALOIS ADMISSIBLE COVERS

reductionsection

Let S be a scheme and G/S a locally well-split finite flat group scheme. We can then consider the stack $\mathcal{K}_{g,n}(\mathcal{B}G)$ of twisted stable maps from genus g twisted curves with n marked points to the classifying stack $\mathcal{B}G$. This is the stack which to any S-scheme T associates the groupoid of data

$$(\mathcal{C}, \{\Sigma_i\}_{i=1}^n, h\colon \mathcal{C} \to \mathcal{B}G),$$

where $(\mathcal{C}, \Sigma_i\}_{i=1}^n$ is an n-marked twisted curve such that the coarse space $(C, \{p_i\}_{i=1}^n)$ (with the marking induced by the Σ_i) is a stable n-pointed curve in the sense of Deligne-Mumford-Knudsen, and h is a representable morphism of stacks over S. As in [2], this is equivalent to the data

$$(\mathcal{C}, \{\Sigma_i\}_{i=1}^n, P \to \mathcal{C}),$$

where $P \to \mathcal{C}$ is a principal G-bundle, with P representable. When G is tame and étale, $P \to C$ is an admissible G-cover, and the stack $\mathcal{K}_{g,n}(\mathcal{B}G)$ has been studied extensively. See [2] for discussion and further references.[7] 7→

The main result of this section is the following:

flattheorem

Theorem 5.1. *The stack $\mathcal{K}_{g,n}(\mathcal{B}G)$ is flat over S, with local complete intersection fibers*[8]. 8→

[7] (Dan) added discussion

[8] (Angelo) I have added this. Correspondingly I have had to add several bits in the proof below.

Here we say that an algebraic stack $\mathcal{X}$ is local complete intersection if it admits a smooth surjective morphism from an local complete intersection algebraic space. This is equivalent to the condition that locally $\mathbb{L}_{\mathcal{X}}$ be a perfect complex supported in degrees $[-1,1]$.[9] ←9

In particular, it follows that $\mathcal{K}_{g,n}(\mathcal{B}G)$ is Cohen–Macaulay if S is Cohen–Macaulay, and Gorenstein if S is Gorenstein.

Given a scheme X of finite type over a field k and an extension k' of k, it is well known that X is a local complete intersection if and only if $X_{k'}$ is; hence the statement of Theorem 5.1 is local in the fpqc topology on S.[10] ←10

To prove this theorem we in fact prove a stronger result.

Let $\mathcal{M}^{\mathrm{tw}}_{g,n}(G)$ denote the stack over S which associates to any S-scheme T the groupoid of data

$$(\mathcal{C}, \{\Sigma_i\}_{i=1}^n, h\colon \mathcal{C} \to \mathcal{B}G),$$

where $(\mathcal{C}, \{\Sigma_i\}_{i=1}^n)$ is an n-marked twisted curve over T and h is a morphism of stacks over S (so $(C, \{p_i\}_{i=1}^n)$ is not required to be stable and h is not necessarily representable).

Lemma 5.2. *The natural inclusion $\mathcal{K}_{g,n}(\mathcal{B}G) \subset \mathcal{M}^{\mathrm{tw}}_{g,n}(G)$ is representable by open immersions.*

Proof. This is because the condition that an n-pointed nodal curve is stable is an open condition as is the condition that a morphism of stacks $\mathcal{C} \to \mathcal{B}G$ is representable, see Corollary C.7 and the discussion preceding it. ♠

To prove 5.1 it therefore suffices to show that $\mathcal{M}^{\mathrm{tw}}_{g,n}(G)$ is flat over S. For this we in fact prove an even stronger result. Let $\mathcal{M}^{\mathrm{tw}}_{g,n}$ denote the stack defined in A.6. Forgetting the map to $\mathcal{B}G$ defines a morphism stacks

flatmap (5.2.1) $$\mathcal{M}^{\mathrm{tw}}_{g,n}(G) \to \mathcal{M}^{\mathrm{tw}}_{g,n} \times_{\mathrm{Spec}(\mathbb{Z})} S.$$

Since $\mathcal{M}^{\mathrm{tw}}_{g,n}$ is smooth over $\mathbb{Z}$ by A.6 the following theorem implies 5.1.

flattheorembis **Theorem 5.3.** *The morphism 5.2.1 is a flat morphism of algebraic stacks, with local complete intersection fibers.*

Equivalently: let T be a scheme and $\mathcal{C} \to T$ a twisted curve. Then $\underline{Hom}_T(\mathcal{C}, \mathcal{B}G)$ is flat over T, with local complete intersection fibers.

The proof of 5.3 occupies the remainder of this section.

Note first of all that $\mathcal{M}^{\mathrm{tw}}_{g,n}(G)$ is an algebraic stack locally of finite presentation over $\mathcal{M}^{\mathrm{tw}}_{g,n}$. Indeed the stack $\mathcal{M}^{\mathrm{tw}}_{g,n}(G)$ is equal to the relative Hom-stack

$$\underline{\mathrm{Hom}}_{\mathcal{M}^{\mathrm{tw}}_{g,n}}(\mathcal{C}^{\mathrm{univ}}, \mathcal{B}G \times \mathcal{M}^{\mathrm{tw}}_{g,n}),$$

where $\mathcal{C}^{\mathrm{univ}} \to \mathcal{M}^{\mathrm{tw}}_{g,n}$ denotes the universal twisted curve. This stack is locally of finite type over $\mathcal{M}^{\mathrm{tw}}_{g,n}$ by [30, 1.1]. It is also shown there that a Hom stack is algebraic whenever the base is representable. This extents to

[9](Dan) added this discussion - please check!

[10](Dan) on S

the case where the base is an algebraic stack by the following well known result:

Lem:relative-algebraic

Lemma 5.4. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks, and assume the following.*

(1) The stack $\mathcal{Y}$ is algebraic.
(2) There is an algebraic space Z and a smooth surjective $Z \to \mathcal{Y}$ such that $\mathcal{X}_Z := Z \times_{\mathcal{Y}} \mathcal{X}$ is algebraic.

Then $\mathcal{X}$ is algebraic.

Proof. First we note that the relative diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable. Indeed, let U be an algebraic space and $U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ a morphism. Form the pullback $U_Z = U \times_{\mathcal{Y}} Z$. We have a morphism $U_Z \to \mathcal{X}_Z \times_Z \mathcal{X}_Z$, and by assumption $\mathcal{X}_Z \times_{\mathcal{X}_Z \times_Z \mathcal{X}_Z} U_Z$ is representable. Applying the same to $Z \times_{\mathcal{Y}} Z$ we obtain flat descent data for $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} U$, showing that the latter is representable. Therefore the relative diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

Now $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the pullback of the diagonal $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ via $\mathcal{X} \times \mathcal{X} \to \mathcal{Y} \times \mathcal{Y}$. Since the diagonal $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable we have that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.

Composing we get that the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable. It remains to produce a smooth presentation for $\mathcal{X}$.

There exists a scheme V with a smooth surjective $V \to \mathcal{X}_Z$. Since $\mathcal{X}_Z \to$
11→ $\mathcal{X}$ is smooth surjective, we have $V \to \mathcal{X}$ smooth surjective. [11] ♠

We prove 5.3 by first studying two special cases and then reducing the general case to these special cases.

Sec:G-tame-etale

5.5. **The case when G is a tame étale group scheme.** In this case we claim that 5.2.1 in fact is étale. Indeed to verify this it suffices to show that 5.2.1 is formally étale since it is a morphism locally of finite presentation. If $T_0 \hookrightarrow T$ is an infinitesimal thickening and $\mathcal{C}_T \to T$ is a twisted curve over T, then the reduction functor from G-torsors on $\mathcal{C}_T$ to G-torsors on $\mathcal{C}_{T_0}$ is an equivalence of categories since G is étale.

5.6. **The case when G is locally diagonalizable.** Let T be an S-scheme, and let $\mathcal{C} \to T$ be a twisted curve. The stack over T of morphisms $\mathcal{C} \to \mathcal{B}G$ is then equivalent to the stack $\mathrm{TORS}_{\mathcal{C}/T}(G)$ associating to any $T' \to T$ the groupoid of G-torsors on $\mathcal{C} \times_T T'$. Let X denote the Cartier dual $\mathrm{Hom}(G, \mathbb{G}_m)$ so that $G = \mathrm{Spec}_S \mathcal{O}_S[X]$.

We use the theory of Picard stacks - see [7, XVIII.1.4]. Let $\mathcal{P}ic_{\mathcal{C}/T}$ denote the Picard stack of line bundles on $\mathcal{C}$, and let $\mathrm{Pic}_{\mathcal{C}/T}$ denote the rigidification of $\mathcal{P}ic_{\mathcal{C}/T}$ with respect to $\mathbb{G}_m$, so that $\mathrm{Pic}_{\mathcal{C}/T}$ is the relative Picard functor of $\mathcal{C}/T$. By Lemma 2.7 we have that $\mathrm{Pic}_{\mathcal{C}/T}$ is an extension of a semi-abelian group scheme by an étale group scheme.

[11] (Dan) added this since Barbara asked where this is proven

Let $\mathrm{Pic}_{\mathcal{C}/T}[X]$ denote the group scheme of homomorphisms

$$X \to \mathrm{Pic}_{\mathcal{C}/T}.$$

Lemma 5.7. *The scheme* $\mathrm{Pic}_{\mathcal{C}/T}[X]$ *is flat over* T*, with local complete intersection fibers.*

Proof. The assertion is clearly local in the étale topology on S so we may assume that S is connected and G diagonalizable. We may write $G = \prod \boldsymbol{\mu}_{n_i}$ and $X = \prod \mathbb{Z}/(n_i)$. Then $\mathrm{Pic}_{\mathcal{C}/T}[X] = \prod_T \mathrm{Pic}_{\mathcal{C}/T}[\mathbb{Z}/(n_i)]$.

It then suffices to consider the case where $X = \mathbb{Z}/(n)$, in which case $\mathrm{Pic}_{\mathcal{C}/T}[X] = \mathrm{Pic}_{\mathcal{C}/T}[n]$. This is the fiber of the map $\mathrm{Pic}_{\mathcal{C}/T} \xrightarrow{\times n} \mathrm{Pic}_{\mathcal{C}/T}$ over the identity section. This is a map of smooth schemes of the same dimension. It has finite fibers since $\mathrm{Pic}_{\mathcal{C}/T}$ is an extension of a semi-abelian group scheme by an étale group scheme. Therefore this map is flat with local complete intersection fibers. Hence $\mathrm{Pic}_{\mathcal{C}/T}[n] \to T$ is flat with local complete intersection fibers as well.[12] ♠ ←12

Let $\mathcal{P}ic_{\mathcal{C}/T}[X]$ denote the Picard stack of morphisms of Picard stacks

$$X \to \mathcal{P}ic_{\mathcal{C}/T},$$

where X is viewed as a discrete stack. Then $\mathcal{P}ic_{\mathcal{C}/T}[X]$ is a G–gerbe over $\mathrm{Pic}_{\mathcal{C}/T}[X]$, hence flat over T.

The result in the locally diagonalizable case therefore follows from the following lemma:

Lem:tors-picx

Lemma 5.8. *There is an equivalence of categories*

$$\mathrm{TORS}_{\mathcal{C}/T}(G) \to \mathcal{P}ic_{\mathcal{C}/T}[X],$$

were $\mathrm{TORS}_{\mathcal{C}/T}(G)$ *denotes the stack associating to any* $T' \to T$ *the groupoid of* G*-torsors on* $\mathcal{C} \times_T T'$*.*

Proof. For any morphism $s\colon \mathcal{C} \to \mathcal{B}G$, the pushforward $s_*\mathcal{O}_{\mathcal{C}}$ on $\mathcal{C} \times \mathcal{B}G$ has a natural action of G and therefore decomposes as a direct sum $\oplus_{x\in X}\mathcal{L}_x$. Base changing to $\mathcal{C} \to \mathcal{C} \times \mathcal{B}G$ (the map defined by the trivial torsor) one sees that each $\mathcal{L}_x$ is a line bundle, and that the algebra structure on $s_*\mathcal{O}_{\mathcal{C}}$ defines isomorphisms $\mathcal{L}_x \otimes \mathcal{L}_{x'} \to \mathcal{L}_{x+x'}$ giving a morphism of Picard stacks

$$F\colon X \to \mathcal{P}ic_{\mathcal{C}/T}[X].$$

Conversely given such a morphism F, let $\mathcal{L}_x$ denote $F(x)$. The isomorphisms

$$F(x + x') \simeq F(x) + F(x')$$

define an algebra structure on

$$\oplus_{x\in X}\mathcal{L}_x.$$

The relative spectrum over $\mathcal{C} \times \mathcal{B}G$ maps isomorphically to $\mathcal{C}$ and therefore defines a section. ♠

[12] (Dan) a few words on lci

Remark 5.9. With a bit more work, one can prove a more general result which may be of interest: given a twisted curve $\mathcal{C} \to T$ and a G-gerbe $\mathcal{G} \to \mathcal{C}$, the stack $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C})$ is a G-gerbe over its rigidification $\mathrm{Sec}_T(\mathcal{G}/\mathcal{C})$, and the T-space $\mathrm{Sec}_T(\mathcal{G}/\mathcal{C})$ is a pseudo-torsor under the flat group-scheme $\mathrm{Pic}_{\mathcal{C}/T}[X]$. In particular $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C}) \to T$ is flat.

5.10. **Observations on fixed points.** Before we consider general G, we make some observations about schemes of fixed points of group actions on semi-abelian group schemes.

Let S be a scheme, and let $A \to S$ be a smooth abelian group scheme. Let A^{o} denote the connected component of the identity and assume we have an exact sequence of group schemes

Aseq (5.10.1) $$0 \longrightarrow A^{\mathrm{o}} \longrightarrow A \longrightarrow W \longrightarrow 0,$$

with W a finite étale group scheme over S. In what follows we will assume that W is a constant group scheme (this always hold after making an étale base change on S) and that A^{o} is a semiabelian group scheme.

Let H be a finite group of order invertible on S acting on A (by homomorphisms of group schemes). Let N denote the order of H. Since A^{o} is a semiabelian group scheme multiplication by N is surjective and étale on A^{o}. It follows that multiplication by N on A is also étale and that there is an exact sequence

$$A \xrightarrow{\times N} A \longrightarrow W/NW \longrightarrow 0.$$

In particular, the image of $\times N\colon A \to A$ is an open and closed subgroup scheme $A' \subset A$ preserved by the H-action.

smoothlem **Lemma 5.11.** *The scheme of fixed points A^H is a smooth group scheme over S.*

13→ *Proof.* [13] It is clear that A^H is a group scheme; we need to show that it is smooth. This follows from the following Lemma (which is well known, but for which we don't have a good reference). ♠

Lemma 5.12. *Let X be a smooth finitely presented scheme over a scheme S, and let G be a finite group whose order is invertible in S. Suppose that G acts on X via S-scheme automorphisms. Then the scheme of fixed points X^G is smooth and finitely presented over S.*

Proof. The problem is local on S, so we may assume that $S = \operatorname{Spec} R$ is affine. Then X, together with the action, will be obtained by base change from a smooth scheme over a finitely generated subring of R. So we may assume that R is noetherian; and then the scheme X^G, which is by definition a closed subscheme in the $|G|$-th product $X \times \ldots \times X$ is automatically finitely
14→ presented.[14]

[13](Angelo) Changed the proof
[14](Dan) $H \mapsto G$

To check smoothness we are going to use Grothendieck's infinitesimal lifting criterion. Each point of X has an affine invariant neighborhood; we may replace X by such neighborhood and assume that X is affine; say $X = \operatorname{Spec} \mathcal{O}$.[15] Then X^G is the spectrum of the largest invariant ring quotient of $\mathcal{O}$; a morphism $\operatorname{Spec} A \to X^G$ correspond to an invariant homomorphism $\mathcal{O} \to A$. ←15

Suppose that we are given a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\rho"] \arrow[d] & A \arrow[d] \\ \mathcal{O} \arrow[r, "\phi"] \arrow[ur, dotted, "\psi"] & B \end{tikzcd}$$

(whithout the dotted arrow), where $A \to B$ is a surjective homomorphism of artinian rings, whose kernel I is such that $I^2 = 0$, while ρ and ϕ are invariant ring homomorphisms. We need to find an invariant ring homomorphism $\psi\colon \mathcal{O} \to A$ that fits above. Since $\mathcal{O}$ is smooth over R, we can find a ring homomorphism ψ, which is not however necessarily invariant. For each $\sigma \in G$ the homomorphism $\mathcal{O} \to I$ defined by the rule $f \mapsto \psi(\sigma f) - \psi(f)$ is an R-linear derivation; hence it defines a function from G to the group $\operatorname{Der}_R(\mathcal{O}, I)$ of R-linear derivations from $\mathcal{O}$ to I. There is a natural right action of G on $\operatorname{Der}_R(\mathcal{O}, I)$, defined by the rule $D\sigma(f) = D(\sigma f)$; the function $G \to \operatorname{Der}_R(\mathcal{O}, I)$ above is a 1-cocycle with respect to such action. Since $\mathrm{H}^1\bigl(G, \operatorname{Der}_R(\mathcal{O}, I)\bigr) = 0$, we have that there exists $D \in \operatorname{Der}_R(\mathcal{O}, I)$ such that

$$\psi(\sigma f) - \psi(f) = D(\sigma f) - D(f)$$

for all $\sigma \in G$ and all $f \in \mathcal{O}$. The function $\mathcal{O} \to A$ defined by $f \mapsto \psi(f) - D(f)$ is the desired invariant ring homomorphism. ♠

Now let X be a finitely generated $\mathbb{Z}/p^n$–module with H-action, where p is prime to N. Let $A[X]$ denote the scheme of homomorphisms $X \to A$. The group H acts on $A[X]$ as follows. An element $h \in H$ sends a homomorphism $\rho\colon X \to A$ to the homomorphism

1.5.6 (5.12.1) $$X \xrightarrow{h^{-1}} X \xrightarrow{\rho} A \xrightarrow{h} A.$$

The group scheme of fixed points $(A[X])^H$ is defined to be the fiber product of the diagram

$$\begin{tikzcd} & A[X] \arrow[d, "\Delta"] \\ A[X] \arrow[r, "\prod_{h\in H}(h\text{-action})"] & \prod_{h\in H} A[X]. \end{tikzcd}$$

Note that the group scheme of fixed points $(A[X])^H$ is potentially quite unrelated to A^H.

[15](Dan) S is already the base scheme... also later fixed ϕ to ψ, g to σ and s to f

5.9 **Proposition 5.13.** *The group scheme of fixed points $(A[X])^H$ is flat over S, with local complete intersection fibers.*

Proof. Choose a presentation of X as an H-representation

shortseq (5.13.1)
$$0 \longrightarrow K \longrightarrow F \longrightarrow X \longrightarrow 0,$$

where the underlying groups of K and F are free $\mathbb{Z}$-modules of finite rank. Let $A[F]$ (resp. $A[K]$) denote the space of homomorphisms $F \to A$ (resp. $K \to A$). Applying $\mathrm{Hom}(F, -)$ (resp. $\mathrm{Hom}(K, -)$) to 5.10.1 we see that $A[F]$ (resp. $A[K]$) sits in a short exact sequence

$$0 \longrightarrow A^{\mathrm{o}}[F] \longrightarrow A[F] \longrightarrow \mathrm{Hom}(F, W) \longrightarrow 0$$

$$(\text{resp. } 0 \longrightarrow A^{\mathrm{o}}[K] \longrightarrow A[K] \longrightarrow \mathrm{Hom}(K, W) \longrightarrow 0),$$

where $A^{\mathrm{o}}[F]$ (resp. $A^{\mathrm{o}}[K]$) is a semiabelian group scheme. Note also that the relative dimension over S of $A[F]$ (resp. $A[K]$) is equal to $\dim(A)\cdot \mathrm{rk}(F)$ (resp. $\dim(A)\cdot\mathrm{rk}(K)$). Since X is a torsion module we have $\mathrm{rk}(F) = \mathrm{rk}(K)$, so $A[F]$ and $A[K]$ have the same dimension. The inclusion $K \hookrightarrow F$ induces a homomorphism

map1 (5.13.2)
$$A[F] \longrightarrow A[K]$$

whose kernel is $A[X]$.

Lemma 5.14. *The induced map*

$$A^{\mathrm{o}}[F] \longrightarrow A^{\mathrm{o}}[K]$$

is surjective.

Proof. The short exact sequence 5.13.1 induces an exact sequence of fppf-sheaves on S

$$A^{\mathrm{o}}[F] \longrightarrow A^{\mathrm{o}}[K] \longrightarrow \mathcal{E}xt^1(X, A^{\mathrm{o}}).$$

Since A^{o} is divisible we have $\mathcal{E}xt^1(X, A^{\mathrm{o}}) = 0$. The surjectivity of $A^{\mathrm{o}}[F] \to A^{\mathrm{o}}[K]$ follows. ♠

By 5.11 the induced morphism

kermap (5.14.1)
$$f\colon (A[F])^H \longrightarrow (A[K])^H$$

is a morphism of smooth group schemes with kernel the group scheme $(A[X])^H$. This is finite since $A[X]$ is a finite flat group scheme. Note that the connected component of the identity in $(A[F])^H$ (resp. $(A[K])^H$) is equal to the connected component of the identity in $A^{\mathrm{o}}[F]^H$ (resp. $A^{\mathrm{o}}[K]^H$).

Lemma 5.15. *The morphism*

$$A^{\mathrm{o}}[F]^H \longrightarrow A^{\mathrm{o}}[K]^H$$

induced by 5.14.1 is surjective.

Proof. Let k be a field and suppose given an H-invariant homomorphism $f\colon K \to A^{\mathrm{o}}$ over k. We wish to show that after possibly making a field extension of k we can find an extension $\tilde{f}\colon F \to A^{\mathrm{o}}$ of f which is H-invariant. Replacing S by $\mathrm{Spec}(k)$, for the rest of the proof we view everything as being over k. Consider the short exact sequence of abelian group schemes with H-action

$$0 \longrightarrow A^{\mathrm{o}}[X] \longrightarrow A^{\mathrm{o}}[F] \longrightarrow A^{\mathrm{o}}[K] \longrightarrow 0.$$

Viewing this sequence as an exact sequence of abelian sheaves on $\mathcal{B}H_{\mathrm{fppf}}$ we obtain by pushing forward to $\mathrm{Spec}(k)_{\mathrm{fppf}}$ an exact sequence

$$A^{\mathrm{o}}[F]^H(k) \longrightarrow A^{\mathrm{o}}[K]^H(k) \longrightarrow H^1(\mathcal{B}H_{\mathrm{fppf}}, A^{\mathrm{o}}[X]).$$

The lemma therefore follows from Lemma 5.16 below (which is stated in the generality needed later). ♠

cohlem **Lemma 5.16.** *Let $\mathcal{D}$ be an abelian sheaf on $\mathcal{B}H_{T,\mathrm{fppf}}$ such that every local section of $\mathcal{D}$ is torsion of order prime to N, the order of H. Let*

$$f\colon \mathcal{B}H_{T,\mathrm{fppf}} \longrightarrow T_{\mathrm{fppf}}$$

be the topos morphism defined by the projection. Then $R^i f_*\mathcal{D} = 0$ for all $i > 0$.

Proof. Let $T_\bullet \to \mathcal{B}H_T$ be the simplicial scheme over $\mathcal{B}H_T$ associated to the covering $T \to \mathcal{B}H_T$ defined by the trivial torsor (as in [24, 12.4]). For $p \geq 0$ let $f_p\colon T_{p,\mathrm{fppf}} \to T_{\mathrm{fppf}}$ be the projection. Note that T_p is a finite disjoint union of copies of T, and therefore for any abelian sheaf F on $T_{p,\mathrm{fppf}}$ we have $R^i f_{p*}F = 0$ for $i > 0$. From this it follows that $Rf_*\mathcal{D}$ is quasi-isomorphic in the derived category of abelian sheaves on T_{fppf} to the complex

$$f_{0*}\mathcal{D}|_{T_0} \longrightarrow f_{1*}\mathcal{D}|_{T_1} \longrightarrow \cdots .$$

Therefore the sheaf $R^i f_*\mathcal{D}$ is isomorphic to the sheaf associated to the presheaf which to any $T' \to T$ associates the group cohomology

cohgroups (5.16.1) $$H^i(H, \mathcal{D}(T')),$$

where $\mathcal{D}(T')$ denotes the H-module obtained by evaluating $\mathcal{D}$ on the object $T' \to \mathcal{B}H$ corresponding to the trivial torsor on T'. Since $\mathcal{D}(T')$ is a direct limit of groups of order prime to the order of H, it follows that the groups 5.16.1 are zero for all $i > 0$. ♠

Now we complete the proof of Proposition 5.13. It follows that the map on connected components of the identity

$$\big(A[F]^H\big)^{\mathrm{o}} \longrightarrow \big(A[K]^H\big)^{\mathrm{o}}$$

is a surjective homomorphism of smooth group schemes of the same dimension, and hence a flat morphism with local complete intersection fibers. Therefore the morphism

$$A[F]^H \longrightarrow A[K]^H$$

is also flat and hence its fiber over the identity section, namely $A[X]^H$, is flat over S, with local complete intersection fibers. ♠

5.17. **General G: setup.** We return to the proof of 5.3.

The assertion that 5.2.1 is flat is local in the fppf topology on S. We may therefore assume that G is well-split so that there is a split exact sequence

$$1 \longrightarrow D \longrightarrow G \longrightarrow H \longrightarrow 1$$

with H étale and tame and D diagonalizable. The map $G \to H$ induces a morphism over $\mathcal{M}^{\mathrm{tw}}_{g,n} \times S$

flatmap2 (5.17.1)
$$\mathcal{M}^{\mathrm{tw}}_{g,n}(G) \longrightarrow \mathcal{M}^{\mathrm{tw}}_{g,n}(H).$$

We may apply the case of étale group scheme (Section 5.5) to the group scheme H. Therefore we know that $\mathcal{M}^{\mathrm{tw}}_{g,n}(H)$ is étale over $\mathcal{M}^{\mathrm{tw}}_{g,n} \times S$, so it suffices to show that the morphism 5.17.1 is flat with local complete intersection fibers.

For this in turn it suffices to show that for any morphism $t\colon T \to \mathcal{M}^{\mathrm{tw}}_{g,n}(H)$, with T the spectrum of an artinian local ring, the induced map

relflat (5.17.2)
$$\mathcal{M}^{\mathrm{tw}}_{g,n}(G) \times_{\mathcal{M}^{\mathrm{tw}}_{g,n}(H)} T \longrightarrow T$$

is flat with local complete intersection fibers.

Let $(\mathcal{C} \to T, F\colon \mathcal{C} \to \mathcal{B}H)$ be the n-pointed genus g twisted curve together with the morphism $F\colon \mathcal{C} \to \mathcal{B}H$ corresponding to t, and let $\mathcal{C}' \to \mathcal{C}$ denote the H-torsor obtained by pulling back the tautological H-torsor over $\mathcal{B}H$. So we have a diagram with cartesian square

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ T \arrow[r] & \mathcal{B}H_T \arrow[d, "f"] \\ & T. \end{tikzcd}$$

For the rest of the argument we may replace S by T, pulling back all the objects over S to T.

The action of H on D defines a group scheme $\mathcal{D}$ over $\mathcal{B}H_T$ whose pullback along the projection $T \to \mathcal{B}H_T$ is the group scheme D with descent data defined by the action. Let $\mathbb{D}$ denote the pullback of this group scheme to $\mathcal{C}$. The pullback of $\mathbb{D}$ to $\mathcal{C}'$ is equal to $D \times_T \mathcal{C}'$, but $\mathbb{D}$ is a twisted form of D over $\mathcal{C}$. Similarly, the character group X of D has an action of the group H, giving a group scheme $\mathfrak{X}$ over $\mathcal{B}H_T$ which is Cartier dual to $\mathcal{D}$.

Let $\mathcal{G} \to \mathcal{C}$ denote the stack $\mathcal{C} \times_{\mathcal{B}H} \mathcal{B}G$, and let $\mathcal{G}' \to \mathcal{C}'$ denote the pullback to $\mathcal{C}'$.The stack $\mathcal{G}$ is a gerbe over $\mathcal{C}$ banded by the sheaf of groups $\mathbb{D}$. To prove that the morphism 5.17.2 is flat with local complete intersection fibers we need to show that the stack $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C})$ is flat with local complete intersection fibers over T. As T is local artinian, we may assume that $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C}) \to T$

is set theoretically surjective, and replacing T by a finite flat cover we may also assume that $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C}) \to T$ has a section over the reduction of T.

Note that the gerbe $\mathcal{G}' \to \mathcal{C}'$ is trivial: there is an isomorphism $\mathcal{G}' \simeq \mathcal{C}' \times BD$. This is because $T \to \mathcal{B}H$ corresponds to the trivial torsor, and any trivialization gives an isomorphism $T \times_{\mathcal{B}H} \mathcal{B}G \simeq BD$. It follows that there is an isomorphism of T-schemes $\underline{\mathrm{Sec}}_T(\mathcal{G}'/\mathcal{C}') \simeq \mathrm{TORS}_{\mathcal{C}'/T}(D)$. However this isomorphism is not canonical.

5.18. **Reduction to $\mathcal{C}'$ connected.** We wish to apply previous results such as 5.8 to $\mathcal{C}' \to T$, but our discussion applied only when this is connected. We reduce to the connected case as follows: suppose the H-bundle $\mathcal{C}' \to \mathcal{C}$ has disconnected fiber over the residue field of T, then since T is artinian we can choose a connected component $\mathcal{C}'' \subset \mathcal{C}'$. Let $H'' \subset H$ be the subgroup sending $\mathcal{C}''$ to itself. Then $\mathcal{C}'' \to \mathcal{C}$ is an H''-bundle. If we denote $G'' = G \times_H H''$, there is an equivalence of categories between G-bundles $\mathcal{E} \to \mathcal{C}$ lifting the given H-bundle $\mathcal{C}'$ and G''-bundles $\mathcal{E}'' \to \mathcal{C}$ lifting the H'' bundle $\mathcal{C}''$: one direction is by restricting $\mathcal{E}$ to $\mathcal{C}'' \subset \mathcal{C}'$, the other direction by inducing $\mathcal{E}'' \to \mathcal{C}$ from G'' to G. It therefore suffices to consider the case when $\mathcal{C}' \to T$ has connected fibers.

5.19. **General G: strategy.** Our approach goes as follows:

(1) By Lemma 5.8 we have a precise structure, as a gerbe over a group scheme, of the stack $\underline{\mathrm{Sec}}_T(\mathcal{G}'/\mathcal{C}')$.
(2) This provides an analogous precise stucture on $\underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ and its rigidification $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$: we have that $\underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ is a $\mathcal{D}$-gerbe over $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$. However $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ is not a group-scheme but a torsor (see below).
(3) The stack
$$\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C}) = \underline{\mathrm{Sec}}_T\left(\underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})\,/\,\mathcal{B}H_T\right)$$
can be thought of as a stack theoretic version of push-forward in the fppf topology from $\mathcal{B}H_T$ to T. Indeed, for a representable morphism $U \to \mathcal{B}H_T$, the fppf sheaf associated to the space $\underline{\mathrm{Sec}}_T(U/\mathcal{B}H_T)$ coincides with $f_*(U)_{\mathrm{fppf}}$. We analyze the structure of the push-forward of its building blocks, namely the rigidification $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ and the group-schemes underlying the torsor and gerbe structures.

5.20. **Structure of $\underline{\mathrm{Sec}}_T(\mathcal{G}'/\mathcal{C}')$ and $\underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$.** Let us explicitly state the structure in (1) above:

- $\underline{\mathrm{Sec}}_T(\mathcal{G}'/\mathcal{C}') \to \mathrm{Sec}_T(\mathcal{G}'/\mathcal{C}')$ is a gerbe banded by the group-scheme D, and
- the rigidification $\mathrm{Sec}_T(\mathcal{G}'/\mathcal{C}')$ of $\underline{\mathrm{Sec}}_T(\mathcal{G}'/\mathcal{C}')$ is isomorphic to the T-group scheme $\mathrm{Pic}_{\mathcal{C}'/T}[X]$.

We proceed with the structure in (2).

Lem:sec-BH-gerbe

Lemma 5.21. *The automorphism group of any object of* $\underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ *over* $Z \to \mathcal{B}H$ *is canonically isomorphic to* $\mathcal{D}(Z)$

Proof. This follows from the facts that $\mathcal{G}$ is a $\mathcal{D}$-gerbe and $\mathcal{C} \to \mathcal{B}H$ connected. Indeed if $s\colon \mathcal{C} \to \mathcal{G}$ is a section, then an automorphism of s is given by a map $\mathcal{C} \to \mathcal{D}$. This necessarily factors through $\mathcal{B}H_T$ since by connectedness $F_*\mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{B}H_T}$ (where $F\colon \mathcal{C} \to \mathcal{B}H_T$ is the structure morphism) and $\mathcal{D}$ is affine over $\mathcal{B}H_T$. ♠

The stack $\underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ is algebraic by Lemma 5.4. We can define $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ to be the rigidification $\underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C}) /\!\!/ \mathcal{D}$. This is also the relative coarse moduli space since $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C}) \to \mathcal{B}H$ is representable. We have that $\underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C}) \to \mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ is a gerbe banded by $\mathcal{D}$.

We also have naturally that $\underline{\mathrm{Sec}}_T(\mathcal{G}'/\mathcal{C}') = T \times_{\mathcal{B}H_T} \underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$. The lemma above and the base change property of rigidification gives

$$\mathrm{Sec}_T(\mathcal{G}'/\mathcal{C}') = T \times_{\mathcal{B}H_T} \mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C}).$$

Let us look at the structure underlying $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$.

The group H acts on $\mathrm{Pic}_{\mathcal{C}'/T}$ and on X. We therefore also obtain a left action of H on

$$\mathrm{Pic}_{\mathcal{C}'/T}[X] = \mathrm{Hom}(X, \mathrm{Pic}_{\mathcal{C}'/T}),$$

where $h \in H$ sends a homomorphism $\rho\colon X \to \mathrm{Pic}_{\mathcal{C}'/T}$ to the homomorphism

$$X \xrightarrow{h^{-1}} X \xrightarrow{\rho} \mathrm{Pic}_{\mathcal{C}'/T} \xrightarrow{h} \mathrm{Pic}_{\mathcal{C}'/T}. \tag{5.21.1}$$

This defines an fppf sheaf on $\mathcal{B}H_T$, which is easily seen to be represented by $\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathcal{X}]$.

Note that in general the identification $\mathrm{Sec}_T(\mathcal{G}'/\mathcal{C}') = T \times_{\mathcal{B}H_T} \mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ does not give an isomorphism of $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ with $\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathcal{X}]$, because the descent data may in general be different. However, we have the following:

Lemma 5.22. *There is a natural action of* $\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathcal{X}]$ *on* $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$*, which pulls back to the natural action of* $\mathrm{Pic}_{\mathcal{C}'/T}[X]$ *on itself by translation. In particular* $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ *is a torsor under* $\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathcal{X}]$.

Proof. Let $\Phi\colon X \to \mathcal{P}ic_{\mathcal{C}/\mathcal{B}H_T}$ be an object of $\mathcal{P}ic_{\mathcal{C}/\mathcal{B}H_T}[\mathcal{X}]$ over some arrow $\xi\colon Z \to \mathcal{B}H_T$ and $s\colon \mathcal{C} \to \mathcal{G}$ a section, again over Z. Then we can define a new section $s^\Phi\colon \mathcal{C} \to \mathcal{G}$ as follows. Let P^Φ denote the $\mathcal{D}$-torsor corresponding to Φ (see Lemma 5.8). The section s defines an isomorphism (which we denote by the same letter)

$$s\colon \mathcal{C} \times \mathcal{B}\mathcal{D} \to \mathcal{G},$$

and we let s^Φ denote the composite

$$\mathcal{C} \xrightarrow{1 \times P^\Phi} \mathcal{C} \times \mathcal{B}\mathcal{D} \xrightarrow{s} \mathcal{G}.$$

By the universal property of rigidification this defines an action of the group scheme $\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathfrak{X}]$ on $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$. The fact that its pullback is identified with the action of $\mathrm{Pic}_{\mathcal{C}'/T}[X]$ on itself by translation is routine.[16] ♠ ←16

5.23. **The pushforward of $\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathfrak{X}]$ and its torsor $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$.** We write $(\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathfrak{X}])_{\mathrm{fppf}}$ for the sheaf in the fppf topology on $\mathcal{B}H_T$ represented by $\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathfrak{X}]$. The fppf pushforward $f_*((\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathfrak{X}])_{\mathrm{fppf}})$ is represented by the scheme of fixed points $\mathrm{Pic}_{\mathcal{C}'/T}[X]^H$. We have:

1.5.1 **Lemma 5.24.** *The scheme of fixed points $\mathrm{Pic}_{\mathcal{C}'/T}[X]^H$ is finite and flat over T, with local complete intersection fibers.*

Proof. Note that in the notation of Section 2.11, since X is a torsion group we have

$$\mathrm{Pic}_{\mathcal{C}'/T}[X] = \mathrm{Pic}^{\mathrm{o}}_{\mathcal{C}'/T}[X]$$

where $\mathrm{Pic}^{\mathrm{o}}_{\mathcal{C}'/T}$ is the group scheme of degree-0 line bundles defined in 2.10. Now by 2.11, we have that $\mathrm{Pic}^{\mathrm{o}}_{\mathcal{C}'/T}$ is an extension of a semi-abelian group scheme by an étale group scheme. The latter étale group scheme is finite since T is assumed Artinian local. The lemma therefore follows from Proposition 5.13. ♠

Now consider the torsor $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ and the pushforward under f of the corresponding sheaf $(\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C}))_{\mathrm{fppf}}$. We have the following.

torsorlem **Lemma 5.25.** *Let $\mathcal{E}$ be an abelian sheaf on $\mathcal{B}H_{T,\mathrm{fppf}}$ such that every local section is torsion of order prime to N, the order of H, and let $P \to \mathcal{B}H_T$ be a $\mathcal{E}$–torsor. Then the sheaf of sets f_*P on T_{fppf} is a torsor under the sheaf $f_*\mathcal{E}$.*

Proof. The fact that P is a torsor under $\mathcal{E}$ immediately implies that f_*P is a pseudo-torsor under $f_*\mathcal{E}$. So the only issue is to show that fppf locally on T the sheaf f_*P has a section. Equivalently, we need to show that after making a flat surjective base change $T' \to T$ the torsor P itself becomes trivial. The class of the torsor is a class in $H^1(\mathcal{B}H_T, \mathcal{E})$. By 5.16 and the Leray spectral sequence this lies in $H^1(T, f_*\mathcal{E})$. Any class in this group can be killed by making a flat surjective base change $T' \to T$. ♠

In our situation, with $\mathcal{E} = (\mathrm{Pic}_{\mathcal{C}/\mathcal{B}H_T}[\mathfrak{X}])_{\mathrm{fppf}}$ and $P = \mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})_{\mathrm{fppf}}$, we obtain that the sheaf $f_*(\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C}))_{\mathrm{fppf}}$ is represented by a torsor under $\mathrm{Pic}_{\mathcal{C}'/T}[X]^H$. In particular the space representing this sheaf is flat over T with local complete intersection fibers.

We denote this T-space by the shorthand notation Sec^H - a complete notation would look like $\underline{\mathrm{Sec}}_T(\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})/\mathcal{B}H_T)$.

We turn our view to $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C})$.

Lemma 5.26. *The automorphism group-scheme of an object of $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C})$ ober a T-scheme B is the group scheme D^H representing $f_*\mathcal{D}$.*

[16](Dan) A little nasty here

Proof. Let $s\colon \mathcal{C} \to \mathcal{G}$ be a section over some T-scheme B. Since $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C}) = \underline{\mathrm{Sec}}_T(\underline{\mathrm{Sec}}_{\mathcal{B}H}(\mathcal{G}/\mathcal{C}) \,/\, \mathcal{B}H)$, an automorphism of s is a section over $B \times \mathcal{B}H$ of the automorphism group-scheme of s viewed as an object of $\underline{\mathrm{Sec}}_{\mathcal{B}H}(\mathcal{G}/\mathcal{C})$ over $B \times \mathcal{B}H$. By Lemma 5.21 this is a section over $B \times \mathcal{B}H$ of the group scheme $\mathcal{D}$, namely a section of $f_*\mathcal{D}$, as required. ♠

We can define $\mathrm{Sec}_T(\mathcal{G}/\mathcal{C})$ to be the rigidification. We have that the morphism $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C}) \to \mathrm{Sec}_T(\mathcal{G}/\mathcal{C})$ is a gerbe banded by the group scheme D^H representing $f_*\mathcal{D}$. This is automatically flat. It has local complete intersection fibers by the following:[17]

Lemma 5.27. *Let Δ be a diagonalizable group scheme over an algebraically closed field. Then $\mathcal{B}\Delta$ is local complete intersection.*

Proof. Factoring $\Delta = \prod \boldsymbol{\mu}_{r_i}$ we have $\mathcal{B}\Delta = \prod \mathcal{B}\boldsymbol{\mu}_{r_i}$, and it suffices to consider each factor. But $\mathcal{B}\boldsymbol{\mu}_{r_i} = [\mathbb{G}_m/\mathbb{G}_m]$, with the action though the r_i-th power map, in other words we have a smooth morphism $\mathbb{G}_m \to \mathcal{B}\boldsymbol{\mu}_{r_i}$ with smooth source, hence $\mathcal{B}\boldsymbol{\mu}_{r_i}$ is local complete intersection. ♠

It therefore suffices to show that $\mathrm{Sec}_T(\mathcal{G}/\mathcal{C})$ is flat with local complete intersection fibers. There is a natural map $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C}) \to \mathrm{Sec}^H$ inducing $\mathrm{Sec}_T(\mathcal{G}/\mathcal{C}) \to \mathrm{Sec}^H$. The following clearly suffices:

Prop:sec-T-gerbe

Proposition 5.28. *The morphism $\underline{\mathrm{Sec}}_T(\mathcal{G}/\mathcal{C}) \to Sec^H$ is a gerbe banded by the group scheme D^H representing $f_*\mathcal{D}$, and hence $\mathrm{Sec}_T(\mathcal{G}/\mathcal{C}) \to Sec^H$ is an isomorphism.*

Consider a morphism $Z \to \mathrm{Sec}^H$ and the fiber product

$$\mathcal{S} \colon = Z \times_{\mathrm{Sec}^H} \underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{C}).$$

Lemma 5.29. *Locally in the fppf topology on Z there exists a section $Z \to \mathcal{S}$.*

Proof. Let $\bar{s} \in f_*(\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})\,(Z)$ denote the section defined by $Z \to \mathrm{Sec}^H$. By definition it corresponds to a morphism $\mathcal{B}H_Z \to \mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ over the natural morphism $\mathcal{B}H_Z \to \mathcal{B}H_T$. Since $\underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ is a $\mathcal{D}$-gerbe over $\mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$, the obstruction to finding a section of $\mathcal{S}$ over $\bar{s}$ is equal to the class of the $\mathcal{D}$-gerbe of liftings of $\bar{s} : \mathcal{B}H_Z \to \mathrm{Sec}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ to a morphism $\mathcal{B}H_Z \to \underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$. This class lies in $H^2_{\mathrm{fppf}}(\mathcal{B}H_Z, \mathcal{D})$. Since by Lemma 5.16 we have $R^i f_*\mathcal{D} = 0$ for $i > 0$, the spectral sequence puts this class in $H^2_{\mathrm{fppf}}(Z, f_*\mathcal{D})$. This vanishes when pulled back to an fppf cover, and the lemma follows. ♠

If $s, s' \in \mathcal{S}(Z)$ are two sections, then we claim that after making a flat base change on Z the sections s and s' are isomorphic. Indeed let I be the sheaf over $\mathcal{B}H_{Z,\mathrm{fppf}}$ of isomorphisms between the two morphisms $\mathcal{B}H_Z \to \underline{\mathrm{Sec}}_{\mathcal{B}H_T}(\mathcal{G}/\mathcal{C})$ corresponding to s and s'. Then I is a $\mathcal{D}$-torsor over $\mathcal{B}H_{Z,\mathrm{fppf}}$, and since $R^1 f_*\mathcal{D} = 0$ it follows that fppf locally on Z this torsor is trivial. It

[17](Dan) added this, check!!

follows that $\mathcal{S}$ is a gerbe banded by $f_*\mathcal{D}$. This completes the proof of 5.28, implying 5.3. ♠

6. Example: reduction of $\mathcal{X}(2)$ in characteristic 2

examplesection

6.1. $\mathcal{X}(2)$ **as a distinguished component in** $\mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)$**.** Let $\mathcal{X}(2)$ denote the stack over $\mathbb{Q}$ associating to any scheme T the groupoid of pairs (E, ι), where E/T is a generalized elliptic curve with a full level 2-structure $\iota : (\mathbb{Z}/2)^2 \simeq E[2]$ in the sense of Deligne-Rapoport [12].

Order the elements of $(\mathbb{Z}/2)^2$ in some way. For any pair $(E, \iota) \in \mathcal{X}(2)(T)$, the involution $P \mapsto -P$ has fixed points the 2-torsion points $E[2] \subset E$, and hence the stack-theoretic quotient $[E/\pm 1]$ of E by this involution comes equipped with an ordered set of gerbes $\Sigma_i \subset [E/\pm 1]$ in the smooth locus. Furthermore, one checks by direct calculation on the geometric fibers of E that the coarse space of $[E/\pm 1]$ with the resulting four marked points is a stable genus 0 curve with four marked points. In this way we obtain a morphism of stacks

$$\mathcal{X}(2) \to \mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)_{\mathbb{Q}}$$

sending

$$(E, \iota) \mapsto (E \to [E/\pm 1], \{\Sigma_i \subset [E/\pm 1]\}).$$

One verifies immediately that this functor is fully faithful, and identifies $\mathcal{X}(2)$ with the closed substack $\mathcal{K}_{\mathbb{Q}} \subset \mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)_{\mathbb{Q}}$ classifying data $(B \to \mathcal{P}, \{\Sigma_i\})$ where $B \to \mathcal{P}$ is a $\boldsymbol{\mu}_2$-torsor such that the resulting map $B \to P$ to the coarse space of $\mathcal{P}$ is ramified over each of the marked points of P.

We have seen that $\mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)$ is flat over $\operatorname{Spec}(\mathbb{Z})$. We want to have a nice description of the closure $\mathcal{K}$ of $\mathcal{K}_{\mathbb{Q}}$, preferably as a naturally defined moduli stack flat over $\mathbb{Z}$. It turns out that this can be done in the best possible way: $\mathcal{K}$ is an open and closed substack of $\mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)$, defined naturally in terms of the behavior of the $\boldsymbol{\mu}_2$-cover at the marked points. One can describe this directly, but we find that this gives us a good pretext for introducing the rigidified cyclotomic inertia stack and evaluation maps.

6.2. **Cyclotomic inertia.** Recall that in [3, §3] one defined the *cyclotomic inertia stacks*: for fixed positive integer $r \geq 1$ we denote by $\mathcal{I}_{\boldsymbol{\mu}_r}(\mathcal{X}) \to \mathcal{X}$ the stack whose objects are pairs (ξ, ϕ) where ξ is an object of $\mathcal{X}$ and $\phi : \boldsymbol{\mu}_r \to \operatorname{Aut}(\xi)$ is a monomorphism, and whose arrows are commutative diagrams as usual; and

$$\mathcal{I}_{\boldsymbol{\mu}}(\mathcal{X}) = \coprod_{r=1}^{\infty} \mathcal{I}_{\boldsymbol{\mu}_r}(\mathcal{X}).$$

For a noetherian stack with finite diagonal, The argument of [3, Proposition 3.1.2] shows that the morphism $\mathcal{I}_{\boldsymbol{\mu}}(\mathcal{X}) \to \mathcal{X}$ is representable by quasiprojective schemes. Indeed, the fiber over an object $\xi \in \mathcal{X}(T)$ is the scheme $\coprod_r Hom_{gr-sch/T}(\boldsymbol{\mu}_r, \operatorname{Aut}_T(\xi))$, which is quasiprojective as the relevant r is bounded.

Proposition 6.3. *If $\mathcal{X}$ is tame, the morphism $\mathcal{I}_{\boldsymbol{\mu}}(\mathcal{X}) \to \mathcal{X}$ is finite and finitely presented.*

Proof. As the problem is local in the fppf topology of the coarse moduli space X of $\mathcal{X}$, we may assume $\mathcal{X} = [U/G]$ with G a linearly reductive group-scheme [4, Theorem 3.2 (c)]. The pull-back $\mathcal{I}_{\mathcal{X}} \times_{\mathcal{X}} U$ is a closed subgroup scheme of the finite flat linearly reductive group scheme G_U. It therefore suffices to show the following Lemma. ♠

grouphom **Lemma 6.4.** *Let G_1 and G_2 be two finite flat finitely presented linearly reductive group schemes over a scheme U. Then the scheme $\underline{\mathrm{Hom}}_{\mathrm{gs}/U}(G_1, G_2)$ of homomorphisms of group schemes is finite, flat and finitely presented over U.*

18

18→

Proof. This is a local statement in the fppf topology; hence we may assume that G_1 and G_2 are of the form $H_i \ltimes \Delta_i$, where the Δ_i are diagonalizable group stacks whose orders are powers of the same prime p, and H_i are constant tame group stacks of orders not divisible by p. Let us also assume that U is connected. If V is a scheme over U, every homomorphism $G_{1,V} \to G_{2,V}$ sends $\Delta_{1,V}$ into $\Delta_{2,V}$; hence we have an induced morphism of U-schemes

$$\underline{\mathrm{Hom}}_{\mathrm{gs}/U}(G_1, G_2) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{gs}/U}(\Delta_1, \Delta_2) \times_U \underline{\mathrm{Hom}}_{\mathrm{gs}/U}(H_1, H_2).$$

Since $\underline{\mathrm{Hom}}_{\mathrm{gs}/U}(\Delta_1, \Delta_2)$ and $\underline{\mathrm{Hom}}_{\mathrm{gs}/U}(H_1, H_2)$ are both unions of a finite number of copies of U, we may fix two homomorphisms of group schemes
19→ $\chi\colon \Delta_1 \to \Delta_2$[19] and $g\colon H_1 \to H_2$, and show that the open and closed subscheme $\underline{\mathrm{Hom}}_{g,\chi}(G_1, G_2)$ of $\underline{\mathrm{Hom}}_{\mathrm{gs}/U}(G_1, G_2)$ inducing the homomorphisms g and χ is finite, flat and finitely presented.

If the homomorphism $\chi\colon \Delta_1 \to \Delta_2$ is not H_1-equivariant, where H_1 acts on Δ_2 through the homomorphism $f\colon H_1 \to H_2$, then $\underline{\mathrm{Hom}}_{g,\chi}(G_1, G_2)$ is empty; so we may assume that χ is H_1-equivariant. Then $\underline{\mathrm{Hom}}_{g,\chi}(G_1, G_2)$ is not empty: in fact, $g \times \chi\colon H_1 \ltimes \Delta_1 \to H_2 \ltimes \Delta_2$ is in $\underline{\mathrm{Hom}}_{g,\chi}(G_1, G_2)(U)$. If V is a U-scheme, an element f of $\underline{\mathrm{Hom}}_{g,\chi}(G_1, G_2)$ is determined by a morphism of schemes $\gamma\colon H_1 \to \Delta_2$ so that $f(ax) = g(a)\phi(a)\chi(x)$. By a straightforward calculation, the condition that f be a homomorphism translates into the condition

$$\phi(ab) = \phi(a)^b b,$$

where we have denoted by $(x, b) \mapsto x^b$ the action of H_1 on $\Delta_2(V)$ acted upon H_1. That is, ϕ should be a 1-cocycle. Since the order of H_1 is prime to the exponent of $\Delta_2(V)$, every such cocyle is coboundary, that is, it is of the form $\phi(a) = d^a d^{-1}$ for some $d \in \Delta_2(V)$. The coboundary ϕ determines

[18] (Angelo) Made some changes to the statement and to the proof.

[19] (Dan) Changed h to χ because confused with H

d up to an element of $\Delta_2(V)^{H_1}$; hence we obtain a bijective correspondence between $\Delta_2(V)/\Delta_2(V)^{H_1}$ and $\underline{\mathrm{Hom}}_{g,\chi}(G_1, G_2)(V)$. Now, it is easy to see that the invariant subsheaf $\Delta_2^{H_1}$ is in fact a diagonalizable group scheme, and so is the quotient $\Delta_2/\Delta_2^{H_1}$, and we have just produced an isomorphism of functors of $\Delta_2/\Delta_2^{H_1}$ with $\underline{\mathrm{Hom}}_{g,\chi}(G_1, G_2)$. Since $\Delta_2/\Delta_2^{H_1}$ is finite, flat and finitely presented, this finishes the proof. ♠

6.5. **Rigidified cyclotomic inertia.** Each object of $\mathcal{I}_{\boldsymbol{\mu}_r}(\mathcal{X})$ has $\boldsymbol{\mu}_r$ sitting in the center of its automorphisms. We can thus rigidify it and obtain a stack $\overline{\mathcal{I}_{\boldsymbol{\mu}_r}}(\mathcal{X})$ with a canonical morphism $\mathcal{I}_{\boldsymbol{\mu}_r}(\mathcal{X}) \to \overline{\mathcal{I}_{\boldsymbol{\mu}_r}}(\mathcal{X})$. Taking the disjoint union over $r \geq 1$ we obtain $\mathcal{I}_{\boldsymbol{\mu}}(\mathcal{X}) \to \overline{\mathcal{I}_{\boldsymbol{\mu}}}(\mathcal{X})$. The stack $\overline{\mathcal{I}_{\boldsymbol{\mu}}}(\mathcal{X})$ is called the *rigidified cyclotomic inertia stack* of $\mathcal{X}$.

The stacks $\overline{\mathcal{I}_{\boldsymbol{\mu}_r}}(\mathcal{X})$ and $\overline{\mathcal{I}_{\boldsymbol{\mu}}}(\mathcal{X})$ have another interpretation, discussed in [3, Section 3.3]. Consider the 2-category whose objects consist of morphisms of stacks $\alpha : \mathcal{G} \to \mathcal{X}$, where $\mathcal{G}$ is a gerbe banded by $\boldsymbol{\mu}_r$ and α is a *representable* morphism; morphisms and 2-arrows are defined in the usual way. Since α is representable, this 2-category is equivalent to a category. This is isomorphic to the stack $\overline{\mathcal{I}_{\boldsymbol{\mu}_r}}(\mathcal{X})$. This gives the stack $\overline{\mathcal{I}_{\boldsymbol{\mu}}}(\mathcal{X})$ the interpretation as the stack of cyclotomic gerbes in $\mathcal{X}$, and $\mathcal{I}_{\boldsymbol{\mu}}(\mathcal{X}) \to \overline{\mathcal{I}_{\boldsymbol{\mu}}}(\mathcal{X})$ together with $\mathcal{I}_{\boldsymbol{\mu}}(\mathcal{X}) \to \mathcal{X}$ is the universal gerbe in $\mathcal{X}$. The non-identity components of $\overline{\mathcal{I}_{\boldsymbol{\mu}}}(\mathcal{X})$ are known as *twisted sectors*.

6.6. **Evaluation maps.** Consider now the stack of twisted stable maps $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$. This comes with a diagram

$$\begin{tikzcd} \Sigma_i \arrow[r] \arrow[dr] & \mathcal{C}_{g,n}(\mathcal{X},\beta) \arrow[r] \arrow[d] & \mathcal{X} \\ & \mathcal{K}_{g,n}(\mathcal{X},\beta). & \end{tikzcd}$$

where $\mathcal{C}_{g,n}(\mathcal{X}, \beta) \to \mathcal{X}$ is the universal twisted stable map and Σ_i are the n markings. For each i, the resulting diagram

$$\begin{tikzcd} \Sigma_i \arrow[r] \arrow[d] & \mathcal{X} \\ \mathcal{K}_{g,n}(\mathcal{X},\beta). & \end{tikzcd}$$

is a cyclotomic gerbe in $\mathcal{X}$ parametrized by $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$. This gives rise to n *evaluation maps*

$$e_i : \mathcal{K}_{g,n}(\mathcal{X}, \beta) \longrightarrow \overline{\mathcal{I}_{\boldsymbol{\mu}}}(\mathcal{X}).$$

Evaluation maps (and their sisters, twisted evaluation maps) are of central importance in Gromov–Witten theory of stacks, because the natural gluing maps

$$\mathcal{K}_{g_1,n_1+1}(\mathcal{X}, \beta_1) \times_{\overline{\mathcal{I}_{\boldsymbol{\mu}}}(\mathcal{X})} \mathcal{K}_{g_2,1+n_2}(\mathcal{X}, \beta_2) \longrightarrow \mathcal{K}_{g_1+g_2,n_1+n_2}(\mathcal{X}, \beta_2)$$

have the fibered product based on the evaluation map on the right and the *twisted evaluation map*, namely the evaluation map composed with inversion on the band, on the left.

6.7. **Back to $\mathcal{K} \subset \mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)$.** The rigidified cyclotomic inertia stack of $\mathcal{B}\boldsymbol{\mu}_2$ has two components - the identity component is $\mathcal{B}\boldsymbol{\mu}_2$ itself, and the non-identity component - the twisted sector - is a copy of the base scheme $\operatorname{Spec}\mathbb{Z}$.

Since $\mathcal{K}_{\mathbb{Q}}$ corresponds to totally ramified maps, it is precisely the locus in $\mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)$ where all four evaluation maps land in the twisted sector. Since $\mathcal{I}_{\boldsymbol{\mu}}(\mathcal{B}\boldsymbol{\mu}_2)$ is finite unramified, the inverse image of each component is open and closed. Therefore $\mathcal{K}$, the closure of $\mathcal{K}_{\mathbb{Q}}$ in $\mathcal{K}_{0,4}(\mathcal{B}\boldsymbol{\mu}_2)$ is open and closed. In particular it is flat over $\mathbb{Z}$. Also, since the generic fiber is irreducible, $\mathcal{K}$ is irreducible. It is simply the gerbe banded by $\boldsymbol{\mu}_2$ over the λ line $\overline{\mathcal{M}}_{0,4} \simeq \mathbb{P}^1$, associated to the class in $H^2_{\text{fppf}}(\mathbb{P}^1, \boldsymbol{\mu}_2)$ associated to $c_1(\mathcal{O}_{\mathbb{P}^1}(1)$.

6.8. **What does $\mathcal{K}_{\mathbb{F}_2}$ parametrize?** There is one little problem with the closure $\mathcal{K}$ of $X(2)_{\mathbb{Q}}$: in characteristic 2 it has nothing to do with elliptic curves.

Given a smooth rational curves with 4 marked points, say at $0, 1, \infty$ and λ, there is a unique $\boldsymbol{\mu}_2$-bundle over the twisted curve which is degenerate over the coarse curve at all the markings, namely the scheme given in affine coordinates by the equation

$$y^2 = x(x-1)(x-\lambda).$$

In characteristic 2, this is a cuspidal curve of geometric genus 0, with its cusp given by $x^2 = \lambda$. This well known example is possibly the simplest example of geometric interest of a reduced principal bundle over a smooth scheme with singular total space.

It is not difficult to see that a similar situation holds at the node of a singular curve of genus 0: the singularity of the $\boldsymbol{\mu}_2$-bundle is a tacnode.

6.9. $\mathcal{X}(2)$ **as a distinguished component in $\mathcal{K}_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2)$.** Let

$$\mathcal{K}^\circ_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2) \subset \mathcal{K}_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2)$$

be the open substack classifying $\boldsymbol{\mu}_2^2$-torsors over smooth elliptic curves. For any object $(E/T, P \to E) \in \mathcal{K}^\circ_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2)$ over some scheme T, there is a natural action of $\boldsymbol{\mu}_2^2$ on this object given by the action on P. Let $\overline{\mathcal{K}}^\circ_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2)$ be the rigidification.

A simple, but not concrete, description of $\overline{\mathcal{K}}^\circ_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2)$ was given in Lemma 5.8: let $E/\mathcal{M}_{1,1}$ be the universal elliptic curve. Then as discussed before, $\overline{\mathcal{K}}^\circ_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2) = \operatorname{Pic}_{E/\mathcal{M}_{1,1}}[X]$ where $X = (\mathbb{Z}/(2))^2$. It follows that

$$\overline{\mathcal{K}}^\circ_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2) = \operatorname{Pic}_{E/\mathcal{M}_{1,1}}[2] \times_{\mathcal{M}_{1,1}} \operatorname{Pic}_{E/\mathcal{M}_{1,1}}[2].$$

The compactification over the boundary of $\overline{\mathcal{M}}_{1,1}$ is not hard to describe as well. But we wish to describe this stack in terms of more classical objects.

The stack $\overline{\mathcal{K}}^{\circ}_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2)$ can be reinterpreted as follows.

Given an object $(E/T, \pi : P \to E) \in \mathcal{K}^{\circ}_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2)$ over a scheme T, the sheaf $\pi_*\mathcal{O}_P$ on E has a $\boldsymbol{\mu}_2^2$-action given by the action on P, and therefore decomposes as a direct sum

$$\pi_*\mathcal{O}_P = \oplus_{\chi \in (\mathbb{Z}/2)^2} L_\chi.$$

Furthermore, since P is a torsor over E each L_χ is a locally free sheaf of rank 1 on E. The sheaves L_χ therefore define a homomorphism

$$\begin{array}{rcl} \phi_P : (\mathbb{Z}/2)^2 & \to & \underline{\mathrm{Pic}}^0(E) \simeq E, \\ \chi & \mapsto & [L_\chi]. \end{array}$$

Let $\mathcal{H}(2)$ denote the stack over $\mathbb{Z}$ associating to any scheme T the groupoid of pairs (E, ϕ), where E/T is an elliptic curve and $\phi : (\mathbb{Z}/2)^2 \to E[2]$ is a homomorphism of group schemes over T. The above construction gives a functor

$$\mathcal{K}^{\circ}_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2) \to \mathcal{H}(2)$$

and a straightforward verification shows that this functor induces an isomorphism

$$\overline{\mathcal{K}}^{\circ}_{1,1}(\mathcal{B}\boldsymbol{\mu}_2^2) \simeq \mathcal{H}(2).$$

For any quotient of finite abelian groups $\pi : (\mathbb{Z}/2)^2 \to A$, let $\mathcal{H}^A(2)$ denote the stack associating to any scheme T the groupoid of pairs (E, ϕ_A), where E/T is an elliptic curve and $\phi_A : A \to E[2]$ is an A-structure in the sense of Katz-Mazur [20, 1.5.1]. Sending such a pair (E, ϕ_A) to

$$(E,\ (\mathbb{Z}/2)^2 \xrightarrow{\pi} A \xrightarrow{\phi_A} E[2]\ )$$

defines a morphism of stacks

KMinclusion (6.9.1) $$\mathcal{H}^A(2) \to \mathcal{H}(2).$$

Using [20, 1.6.2] one sees that this in fact is a closed immersion.

We can describe the above in more classical notation as follows. Define stacks $\mathcal{Y}(2)$, $\mathcal{Y}_1(2)$, and $\mathcal{Y}(1)$ by associating to any scheme T the following groupoids:

- $\mathcal{Y}(2)$: The groupoid of pairs (E, ϕ), where E/T is an elliptic curve and $\phi : (\mathbb{Z}/2)^2 \to E[2]$ is a $(\mathbb{Z}/2)^2$-generator;
- $\mathcal{Y}_1(2)$: The groupoid of pairs (E, ϕ), where E/T is an elliptic curve and $\phi : (\mathbb{Z}/2) \to E[2]$ is a $(\mathbb{Z}/2)$-generator;
- $\mathcal{Y}(1)$: The groupoid of elliptic curves over T.

Then 6.9.1 gives a closed immersion

$$i^{(2)} : \mathcal{Y}(2) \hookrightarrow \mathcal{H}(2)$$

corresponding to the identity map $(\mathbb{Z}/2)^2 \to (\mathbb{Z}/2)^2$, three closed immersions

$$i^{(1)}_K : \mathcal{Y}_1(2) \hookrightarrow \mathcal{H}(2), \quad j = 1, 2, 3,$$

corresponding to the three surjections $(\mathbb{Z}/2)^2 \to \mathbb{Z}/2$ (indexed by index 2 subgroups $K \subset (\mathbb{Z}/2)^2$), and a closed immersion

$$i^{(0)} : \mathcal{Y}(1) \hookrightarrow \mathcal{H}(2)$$

corresponding to the unique surjection $(\mathbb{Z}/2)^2 \to 0$. The resulting map

coprod

$$\mathcal{Y}(2) \sqcup \Big(\mathcal{Y}_1(2) \sqcup \mathcal{Y}_1(2) \sqcup \mathcal{Y}_1(2)\Big) \sqcup \mathcal{Y}(1) \longrightarrow \mathcal{H}(2) \tag{6.9.2}$$

is then a proper surjection, which over $\mathbb{Z}[1/2]$ is an isomorphism. Note in particular that the forgetful map $\mathcal{H}(2) \to \mathcal{Y}(1)$ sending (E, ϕ) to E has degree 16, as $\mathcal{H}(2) \to \mathcal{Y}(1)$ is flat by 5.3 and over $\mathbb{Z}[1/2]$ the map clearly has degree 16.

Over $\mathbb{F}_2$, however, the map 6.9.2 joins together the various components in an interesting way. If E/k is an ordinary elliptic curve over a field k of characteristic 2, then any homomorphism

fibermap

$$\phi : (\mathbb{Z}/2)^2 \to E[2] \simeq \mathbb{Z}/(2) \times \mu_2 \tag{6.9.3}$$

has a nontrivial kernel, and the irreducible components of $\mathcal{Y}(2)_{\mathbb{F}_2}$ are indexed by these kernels. Since a $Z/(2)^2$-structure is surjective on geometric points, the kernel cannot be the whole group. Therefore $\mathcal{Y}(2)_{\mathbb{F}_2}$ has 3 irreducible components: for a subgroup $K \subset (\mathbb{Z}/2)^2$ of index 2 we get an irreducible component $\mathcal{Y}(2)^K_{\mathbb{F}_2}$ corresponding to pairs (E, ϕ), where the map 6.9.3 has kernel K. Note also that for any ordinary elliptic curve E/k over a field k, the fiber product of the diagram

$$\begin{array}{ccc} & & \mathcal{Y}(2)^K_{\mathbb{F}_2} \\ & & \big\downarrow \\ \mathrm{Spec}(k) & \xrightarrow{E} & \mathcal{Y}(1)_{\mathbb{F}_2} \end{array}$$

has length 2 over k.

Similarly the reduction $\mathcal{Y}_1(2)_{\mathbb{F}_2}$ has two irreducible components

$$\mathcal{Y}_1(2)_{\mathbb{F}_2} = \mathcal{Y}_1(2)' \cup \mathcal{Y}_1(2)^{\emptyset},$$

where $\mathcal{Y}_1(2)'$ (resp. $\mathcal{Y}_1(2)^{\emptyset}$) classifies pairs $(E, \phi : \mathbb{Z}/2 \to E[2])$ where the map ϕ is injective (resp. the zero map). This time only the fiber of $\mathcal{Y}_1(2)'$ over a k-point of $\mathcal{Y}(1)_{\mathbb{F}_2}$ has length 2.

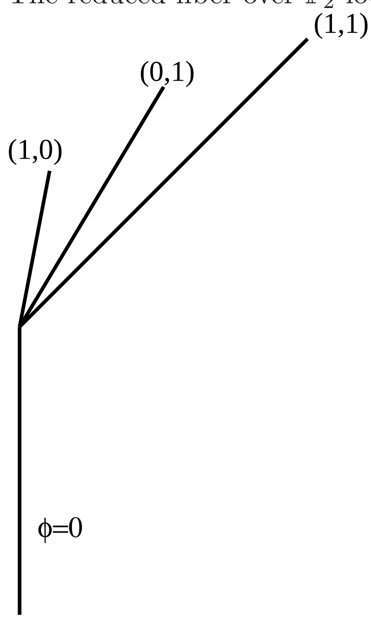
The closed fiber $\mathcal{H}(2)_{\mathbb{F}_2}$ then has four irreducible components. For a subgroup $K \subset (\mathbb{Z}/2)^2$ of index 2, we have an irreducible component Z_K which is set-theoretically identified with $\mathcal{Y}(2)^K_{\mathbb{F}_2}$ as well as the component $\mathcal{Y}_1(2)'$ given by the inclusion $i^{(1)}_K$. The fourth component $Z_{\emptyset}$ is set-theoretically identified with $\mathcal{Y}(1)_{\mathbb{F}_2}$ via $i^{(0)}$, and with all the components $\mathcal{Y}_1(2)^{\emptyset}$ via the inclusions $i^{(1)}_K$.

Over the point of $\mathcal{Y}(1)(\overline{\mathbb{F}}_2)$ given by the supersingular elliptic curve E, there is only one homomorphism (the zero map)

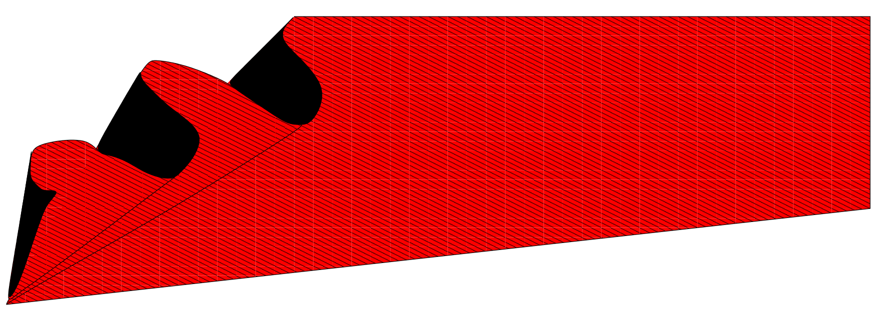
$$\phi : (\mathbb{Z}/2)^2 \to E[2],$$

and hence the four irreducible components of $\mathcal{H}(2)_{\mathbb{F}_2}$ all intersect at this point (and nowhere else). Note also that over the ordinary locus of $\mathcal{Y}(1)$, each of the components Z_K has length 4 over $\mathcal{Y}(1)_{\mathbb{F}_2}$ as does the component $Z_\emptyset$ (so $\mathcal{H}(2)_{\mathbb{F}_2}$ has length 16 over $\mathcal{Y}(1)_{\mathbb{F}_2}$, as we already knew).

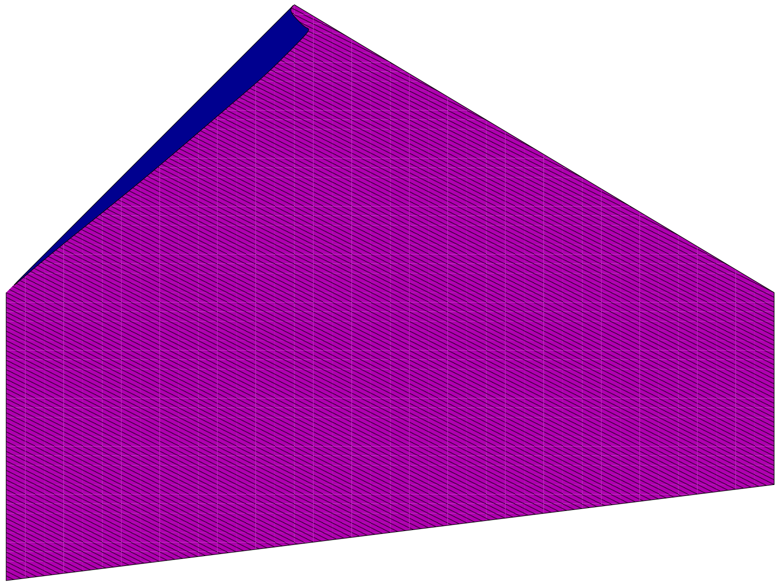
The reduced fiber over $\mathbb{F}_2$ looks roughly like this:



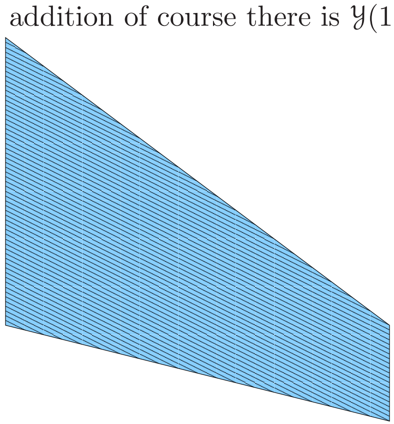
The following figure schematically describes the main component $\mathcal{Y}(2)$:



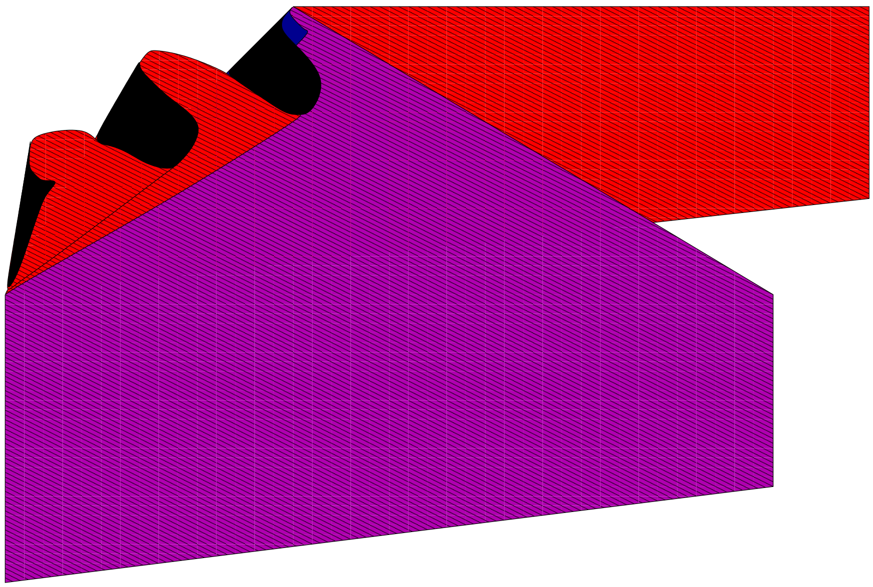
Here is the component $\mathcal{Y}_1(2)^K$ for K generated by $(1,1)$:



There are of course two more corresponding to the two other K of order 2. In addition of course there is $\mathcal{Y}(1)$:

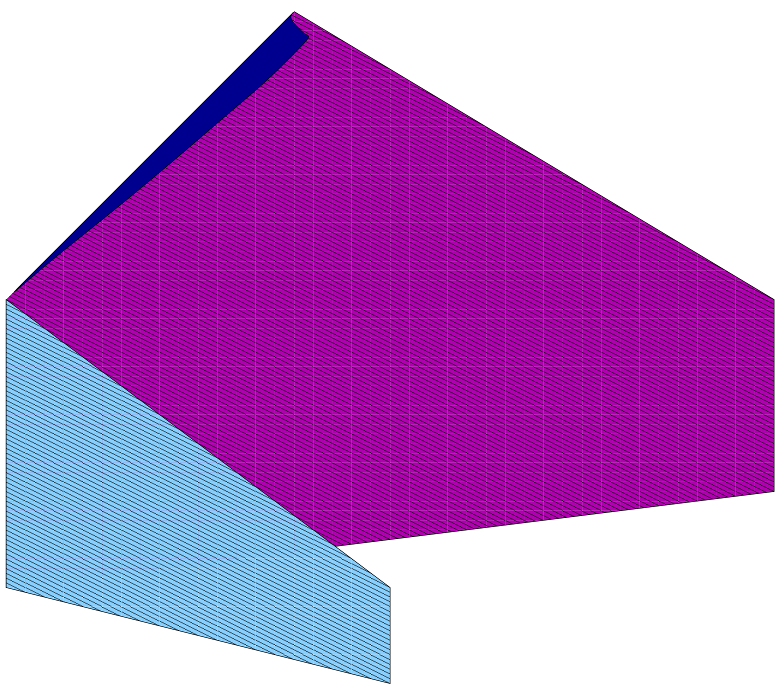


The way the main component $\mathcal{Y}(2)$ meets $\mathcal{Y}_1(2)^K$ is described as follows:

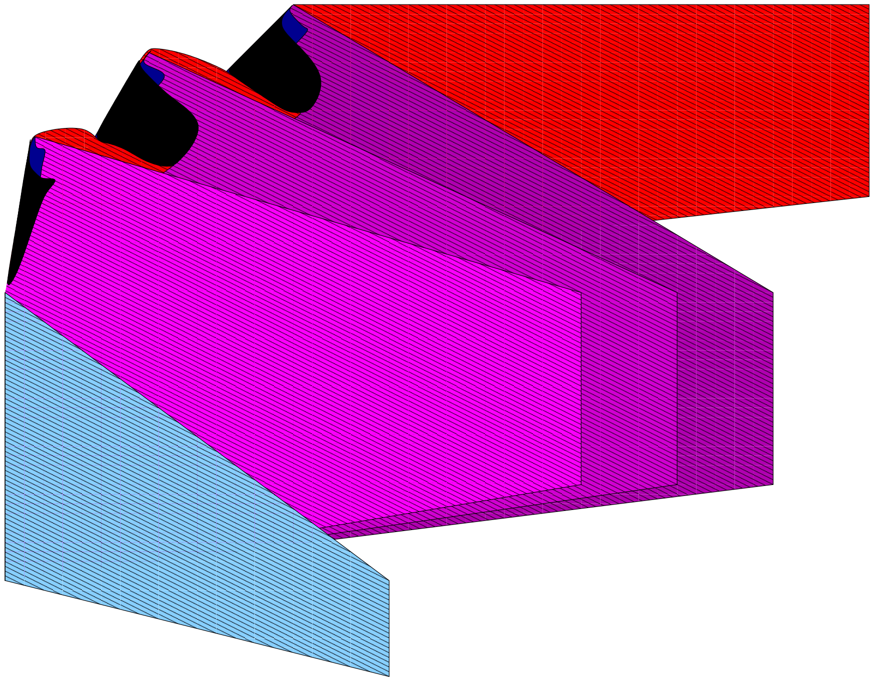


and the other two are similar, of course along the corresponding component $\mathcal{Y}(2)^K_{\mathbb{F}_2}$.

The main component $\mathcal{Y}(2)$ meets $\mathcal{Y}(1)$ only at the supersingular point, but $\mathcal{Y}_1(2)^K$ meets $\mathcal{Y}(1)$ along the component corresponding to $\phi = 0$:



The complete picture is something like this:



An important question is that of describing explicitly the "most important" part of $\overline{\mathcal{K}}_{g,n}(\mathcal{B}\boldsymbol{\mu}_m^{2g})$, in this example $\mathfrak{X}(2)$. In genus 1, Katz and Mazur gave an ideal solution: the moduli stack of generalized elliptic curves with $(\mathbb{Z}/m)^2$ structure, which is a closed substack of our $\overline{\mathcal{K}}_{1,1}(\mathcal{B}\boldsymbol{\mu}_m^2)$, is regular and flat over $\overline{\mathcal{M}}_{1,1}$. But for higher genus a satisfactory conclusion is missing: in [20, Section 1.9] Katz and Mazur proposed a solution using norms, but it was shown to be ill behaved in [9, Appendix A]. It still may be of interest to study the closure of the $\mathbb{Q}$-stack $\overline{\mathcal{M}}_g^{(m)}$ of level m curves in $\overline{\mathcal{K}}_g(\mathcal{B}\boldsymbol{\mu}_m^{2g})$.

Appendix A. Twisted curves and log curves by Martin Olsson

AppendixA

In [29] we introduced a notion of log twisted curve, and proved that the category of log twisted curves (over some scheme S) is naturally equivalent to the category of twisted Deligne-Mumford curves over S. The definition of a log twisted curve in loc. cit. included various assumptions about certain integers being invertible on the base scheme. In this appendix we show that if we remove these assumptions in the definition of log twisted curve, we obtain a category equivalent to the category of twisted curves.

locallyfree

Definition A.1 ([25, 3.1])**.** Let X be an Artin stack.

(i) A fine log structure $\mathcal{M}$ on X is called *locally free* if for every geometric point $\bar{x} \to X$ the monoid $\overline{\mathcal{M}}_{\bar{x}} := \mathcal{M}_{\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$ is isomorphic to $\mathbb{N}^r$ for some r.

(ii) A morphism $\mathcal{M} \to \mathcal{N}$ of locally free log structures on X is called *simple* if for every geometric point $\bar{x} \to X$ the monoids $\overline{\mathcal{M}}_{\bar{x}}$ and $\overline{\mathcal{N}}_{\bar{x}}$ have the same rank, the morphism $\varphi : \overline{\mathcal{M}}_x \to \overline{\mathcal{N}}_{\bar{x}}$ is injective, and for every irreducible element $f \in \overline{\mathcal{N}}_{\bar{x}}$ there exists an irreducible element $g \in \overline{\mathcal{M}}_{\bar{x}}$ and a positive integer n such that $\varphi(g) = nf$.

Remark A.2. This differs from [29, 1.5] where the integer n in (A.1 (ii)) was assumed invertible in $k(\bar{x})$.

Let $f : C \to S$ be a nodal curve over a scheme S. As discussed in [29, §3], there exist canonical log structures $\mathcal{M}_C$ and $\mathcal{M}_S$ on C and S respectively, and an extension of f to a log smooth morphism

$$(C, \widetilde{\mathcal{M}}_C) \to (S, \mathcal{M}_S).$$

1.5 **Definition A.3.** A *n-pointed log twisted curve* over a scheme S is a collection of data

$$(C/S, \{\sigma_i, a_i\}_{i=1}^n, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S),$$

where C/S is a nodal curve, $\sigma_i : S \to C$ are sections, the a_i are positive integer-valued locally constant functions on S, and $\ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S$ is a simple morphism of log structures on S, where $\mathcal{M}_S$ denotes the canonical log structure on S mentioned above.

Let $(C/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$ be a log twisted curve over S, and let

logmap (A.3.1) $$(C, \mathcal{M}_C) \longrightarrow (S, \mathcal{M}_S)$$

be the morphism of log schemes obtained from the pointed curve

$$(C/S, \{\sigma_i\})$$

as in [29, 3.10] (note that $\mathcal{M}_C$ is not equal to $\widetilde{\mathcal{M}}_C$ mentioned above, as $\mathcal{M}_C$ also takes into account the marked points). We construct a twisted n–pointed curve $(\mathcal{C}, \{\Sigma_i\})/S$ from the log twisted curve $(C/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$ as in [29, §4].

Define $\mathcal{C}$ to be the fibered category over S which to any $h : T \to S$ associated the groupoid of data consisting of a morphism $s : T \to C$ over h together with a commutative diagram of locally free log structures on T

3.15.2 (A.3.2)
$$\begin{array}{ccc} h^*\mathcal{M}_S & \xrightarrow{\ \ell\ } & h^*\mathcal{M}'_S \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ s^*\mathcal{M}_C & \xrightarrow{\ k\ } & \mathcal{M}'_C, \end{array}$$

where:
(1.3 (i)) The map k is a simple, and for every geometric point $\bar{t} \to T$, the map $\overline{\mathcal{M}}'_{S,\bar{t}} \to \overline{\mathcal{M}}'_{C,\bar{t}}$ is either an isomorphism, or of the form $\mathbb{N}^r \to \mathbb{N}^{r+1}$ sending e_i to e_i for $i < r$ and e_r to either e_r or $e_r + e_{r+1}$.
(1.3 (ii)) For every i and geometric point $\bar{t} \to T$ with image under s in $\sigma_i(S) \subset C$, the group

3.14.3 (A.3.3) $$\mathrm{Coker}(\overline{\mathcal{M}}'^{\mathrm{gp}}_{S,\bar{t}} \oplus \overline{\mathcal{M}}^{\mathrm{gp}}_{C,\bar{t}} \longrightarrow \overline{\mathcal{M}}'^{\mathrm{gp}}_{C,\bar{t}})$$

is a cyclic group of order $a_i(\bar{t})$.

For every i, define $\Sigma_i \subset \mathcal{C}$ to be the substack classifying morphisms $s : T \to C$ which factor through $\sigma_i(S) \subset C$ and diagrams (A.3.2) such that for every geometric point $\bar{t} \to T$ the image of

$$(\mathcal{M}'_{C,\bar{t}} - \tau(h^*\mathcal{M}'_{S,\bar{t}})) \to \mathcal{O}_{T,\bar{t}}$$

is zero.

Proposition A.4. *The data $(\mathcal{C}, \{\Sigma_i\}_{i=1}^n)$ is a twisted n-pointed curve.*

Proof. This follows from the argument given in [29, §4]. ♠

The main result of this appendix is the following:

Bthm **Theorem A.5.** *Let S be a scheme. The functor*

thefunctor (A.5.1) $\quad$ (n-pointed log twisted curves) $\to$ (twisted n-pointed curves)

sending $(C/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$ to the twisted n-pointed curve $(\mathcal{C}, \{\Sigma_i\}_{i=1}^n)$ constructed above is an equivalence of categories. Moreover, this equivalence of categories is compatible with base change $S' \to S$.

We prove this theorem below. Before giving the proof, however, let us record the main consequences of this result that we will use.

Fix integers g and n, and let $\mathcal{S}_{g,n}$ denote the fibered category over $\mathbb{Z}$ which to any scheme S associates the groupoid of all (not necessarily stable) n–pointed genus g nodal curves C/S. The stack $\mathcal{S}_{g,n}$ is algebraic, and as explained in section [29, §5] the substack $\mathcal{S}^0_{g,n} \subset \mathcal{S}_{g,n}$ classifying smooth curves defines a log structure $\mathcal{M}_{\mathcal{S}_{g,n}}$ on $\mathcal{S}_{g,n}$. The same argument used in [29, §5] yields the following theorem:

3.19 **Theorem A.6.** *Let $\mathcal{M}^{\mathrm{tw}}_{g,n}$ denote the fibered category over $\mathbb{Z}$ which to any scheme T associates the groupoid of n–marked genus g twisted curves $(\mathcal{C}, \{\Sigma_i\})$ over T. Then $\mathcal{M}^{\mathrm{tw}}_{g,n}$ is a smooth Artin stack, and the natural map*

3.19.1 (A.6.1) $$\pi : \mathcal{M}^{\mathrm{tw}}_{g,n} \longrightarrow \mathcal{S}_{g,n}$$

sending $(\mathcal{C}, \{\Sigma_i\})$ to its coarse moduli space with the marked points induced by the Σ_i is representable by tame stacks. Moreover, there is a natural locally free log structure $\mathcal{M}_{\mathcal{M}^{\mathrm{tw}}_{g,n}}$ on $\mathcal{M}^{\mathrm{tw}}_{g,n}$.

Remark A.7. Consider a field k and an object $(\mathcal{C}, \{\Sigma_i\}) \in \mathcal{M}^{\mathrm{tw}}_{g,n}(k)$. Let $(C, \{\sigma_i\})$ be the coarse moduli space, and let R be a versal deformation space for the object $(C, \{\sigma_i\}) \in \mathcal{S}_{g,n}(k)$. Let $q_1, \ldots, q_m \in C$ be the nodes and let r_i be the order of the stabilizer group of a point of $\mathcal{C}$ lying above q_i. As in [11, 1.5], there is a smooth divisor $D_i \subset \mathrm{Spec}(R)$ classifying deformations where q_i remains a node. In other words, if $t_i \in R$ is an element defining D_i then in an étale neighborhood of q_i the versal deformation $\widetilde{C} \to \mathrm{Spec}(R)$ of $(C, \{\sigma_i\})$ is isomorphic to

$$\mathrm{Spec}(R[x,y]/(xy - t_i)).$$

It follows from the argument in [29, §5] that the fiber product

$$\mathcal{M}^{\mathrm{tw}}_{g,n} \times_{\mathcal{S}_{g,n}} \mathrm{Spec}(R)$$

is isomorphic to the stack-theoretic quotient of

$$\mathrm{Spec}(R[z_1, \ldots, z_m]/(z_1^{r_1} - t_1, \ldots, z_m^{r_m} - t_m))$$

by the action of $\mu_{r_1} \times \cdots \mu_{r_m}$ for which $(\zeta_1, \ldots, \zeta_r) \in \mu_{r_1} \times \cdots \mu_{r_m}$ sends z_i to $\zeta_i z_i$. In particular $\mathcal{M}^{\mathrm{tw}}_{g,n}$ is flat over $\mathcal{S}_{g,n}$, and hence also flat over $\mathbb{Z}$.

We also get a generalization of [29, 1.11]:

1.7 **Corollary A.8.** *For any integer $N > 0$, let $\mathcal{M}^{\mathrm{tw},\leq N}_{g,n}$ denote the substack of $\mathcal{M}^{\mathrm{tw}}_{g,n}$ classifying n–pointed genus g twisted curves such that the order of the stabilizer group at every point is less than or equal to N. Then $\mathcal{M}^{\mathrm{tw},\leq N}_{g,n}$ is an open substack of $\mathcal{M}^{\mathrm{tw}}_{g,n}$ and the map $\mathcal{M}^{\mathrm{tw},\leq N}_{g,n} \to \mathcal{S}_{g,n}$ is of finite type and quasi-finite.*

curveconstruct **Remark A.9.** Let R be a discrete valuation ring with uniformizer π and separably closed residue field, and let C/R be a nodal curve. Let $p_1, \ldots, p_r$ be the nodes of C in the closed fiber. Then we have a log smooth morphism

$$(C, \mathcal{M}_C) \to (\mathrm{Spec}(R), \mathcal{M}_R),$$

where the log structure $\mathcal{M}_R \to R$ admits a chart $\mathbb{N}^r \to R$ such that the image of the i-th standard generator is equal to π^{l_i}, where $l_i \in \mathbb{N} \cup \{\infty\}$ is an element such that in an étale neighborhood of p_i the curve C_i is isomorphic to

$$\mathrm{Spec}(R[x, y]/(xy - \pi^{l_i}),$$

where by convention if $l_i = \infty$ we set $\pi^{l_i} = 0$.

Now assume some l_i is finite. Then we obtain a twisted curve by taking the stack corresponding to the morphism of log structures $\mathcal{M}_R \to \mathcal{M}'_R$, where $\mathcal{M}'_R$ is the log structure associated to the map $\mathbb{N}^r \to R$ sending e_j to π^{l_j} for $j \neq i$, and e_i to π.

proofofBthm

Proof of A.5. The proof of A.5 follows the same outline as the proof of [29, 1.9]. We review the argument here indicating the necessary changes for this more general setting.

esssems **Definition A.10** ([29, 3.3])**.** A log smooth morphism of fine log schemes $f : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is *essentially semi-stable* if for each geometric point $\bar{x} \to X$ the monoids $(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}}$ and $\overline{\mathcal{M}}_{X,\bar{x}}$ are free monoids, and if for suitable isomorphisms $(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}} \simeq \mathbb{N}^r$ and $\overline{\mathcal{M}}_{X,\bar{x}} \simeq \mathbb{N}^{r+s}$ the map

$$(f^{-1}\overline{\mathcal{M}}_S)_{\bar{x}} \to \overline{\mathcal{M}}_{X,\bar{x}}$$

is of the form

chartformula (A.10.1)
$$e_i \mapsto \begin{cases} e_i & \text{if } i \neq r \\ e_r + e_{r+1} + \cdots + e_{r+s} & \text{if } i = r, \end{cases}$$

where e_i denotes the i-th standard generator of $\mathbb{N}^r$.

With notation as in A.10, if $U \to X$ is a smooth surjection of schemes and $\mathcal{M}_U$ denotes the pullback of $\mathcal{M}_X$, then it follows immediately from the definition that $(X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is essentially semistable if and only if the morphism $(U, \mathcal{M}_U) \to (S, \mathcal{M}_S)$ is essentially semistable (the property of being semistable is local in the smooth topology on X). It follows that the notion of a morphism being essentially semistable extends to Artin stacks: If $(\mathcal{X}, \mathcal{M}_\mathcal{X})$ is an Artin stack with a fine log structure, and $f : (\mathcal{X}, \mathcal{M}_\mathcal{X}) \to (S, \mathcal{M}_S)$ is a morphism to a fine log scheme, then f is *essentially semistable* if for some smooth surjection $U \to \mathcal{X}$ with U a scheme the induced morphism of log schemes

testmap (A.10.2) $$(U, \mathcal{M}_\mathcal{X}|_U) \to (S, \mathcal{M}_S)$$

is essentially semistable. As usual if for some smooth surjection $U \to \mathcal{X}$ the morphism A.10.2 is essentially semistable then for any smooth surjection $V \to \mathcal{X}$ the induced morphism $(V, \mathcal{M}_\mathcal{X}|_V) \to (S, \mathcal{M}_S)$ is essentially semistable.

As explained in [29, §3], if $(X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is an essentially semistable morphism of log schemes, then for any geometric point $\bar{s} \to S$ there is a canonical induced map

$$s_{X_{\bar{s}}} \colon \{\text{singular points of } X_{\bar{s}}\} \to \mathrm{Irr}(\overline{\mathcal{M}}_{S,\bar{s}}),$$

where $\mathrm{Irr}(\overline{\mathcal{M}}_{S,\bar{s}})$ denotes the set of irreducible elements in the monoid $\overline{\mathcal{M}}_{S,\bar{s}}$.

If $\mathcal{Y}$ is an Artin stack over a field k, let $\pi_0(\mathcal{Y}_{\mathrm{sing}})$ denote the set of connected components of the complement of the maximal open substack $\mathcal{U} \subset \mathcal{Y}$ which is smooth over k. Note that if $\mathcal{Y}' \to \mathcal{Y}$ is a smooth morphism of k-stacks, then there is an induced map

$$\pi_0(\mathcal{Y}'_{\mathrm{sing}}) \to \pi_0(\mathcal{Y}_{\mathrm{sing}}).$$

If $\mathcal{Y}' \to \mathcal{Y}$ is also surjective then this map is also surjective.

Definition A.11. Let $f : (\mathcal{X}, \mathcal{M}_\mathcal{X}) \to (S, \mathcal{M}_S)$ be an essentially semistable morphism from a log Artin stack to a fine log scheme, and let $\bar{s} \to S$ be a geometric point. We say that f is *special at* $\bar{s}$ if for any smooth surjection $U \to \mathcal{X}$ with U a scheme the map

$$s_{U_{\bar{s}}} \colon \{\text{singular points of } U_{\bar{s}}\} \to \mathrm{Irr}(\overline{\mathcal{M}}_{S,\bar{s}})$$

factors through the composite (which is surjective)

$$\{\text{singular points of } U_{\bar{s}}\} \longrightarrow \pi_0(U_{\bar{s},\mathrm{sing}}) \longrightarrow \pi_0(\mathcal{X}_{\bar{s},\mathrm{sing}})$$

to give an isomorphism

$$\pi_0(\mathcal{X}_{\bar{s},\mathrm{sing}}) \to \mathrm{Irr}(\overline{\mathcal{M}}_{S,\bar{s}}).$$

canlogthm **Theorem A.12** (Generalization of [29, 3.6])**.** *Let $(f : \mathcal{C} \to S, \{\Sigma_i\})$ be an n-marked twisted curve. Then there exist log structures $\widetilde{\mathcal{M}}_\mathcal{C}$ and $\mathcal{M}'_S$ on $\mathcal{C}$ and S respectively, and a special morphism*

$$(f, f^b) : (\mathcal{C}, \widetilde{\mathcal{M}}_\mathcal{C}) \longrightarrow (S, \mathcal{M}'_S).$$

Moreover, the datum $(\widetilde{\mathcal{M}}_{\mathcal{C}}, \mathcal{M}'_S, f^b)$ *is unique up to unique isomorphism.*

Proof. The proof will be in several steps (ending in paragraph following A.17).

The uniqueness statement follows as in [29, 3.6] from the argument proving the uniqueness in [31, 2.7].

Given the uniqueness, to prove existence we may work étale locally on S, and by a limit argument as in the proof of [29, 3.6] may even assume that S is the spectrum of a strictly henselian local ring. Let $\bar{s} \in S$ be the closed point.

Let $p_1, \dots, p_n \in C$ be the nodes of the closed fiber, and choose for each $i = 1, \dots, n$ an affine open set $U_i \subset C$ containing p_i and no other nodes. Let $\mathcal{U}_i \subset \mathcal{C}$ denote the inverse image of U_i.

A14 **Lemma A.13.** *For any quasi-coherent sheaf* $\mathcal{F}$ *on* $\mathcal{U}_i$*, we have*

$$H^j(\mathcal{U}_i, \mathcal{F}) = 0$$

for $j > 0$*.*

Proof. This follows from the same argument proving 2.5. $\spadesuit$

Let $t_i \in \mathcal{O}_S$ be an element such that the fiber product

$$\mathcal{C} \times_C \operatorname{Spec}(\mathcal{O}_{C,\bar{p}_i})$$

is isomorphic to

$$[\operatorname{Spec}(\mathcal{O}_S[z, \widetilde{w]/(zw} - t_i))/\mu_n]$$

as in (2.1 (v)). Let $\mathcal{M}_S^i$ be the log structure on S associated to the morphism $\mathbb{N} \to \mathcal{O}_S$ sending 1 to t_i

Definition A.14. Let $f : X \to S$ be a morphism of schemes. A *t_i-semistable log structure on X* is a pair $(\mathcal{M}_X, f^b)$, where $\mathcal{M}_X$ is a locally free log structure on X and $f^b : f^*\mathcal{M}_S^i \to \mathcal{X}$ is a morphism of log structures, such that the following hold:

(i) The morphism of log schemes

$$(f, f^b) : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S^i)$$

is log smooth;

(ii) For every geometric point $\bar{x} \to X$ the induced map of free monoids

$$\mathbb{N} \to \overline{\mathcal{M}}^i_{S,f(\bar{x})} \to \overline{\mathcal{M}}_{X,\bar{x}}$$

is the diagonal map.

smoothdes **Remark A.15.** By [29, 3.4], if $X \to S$ admits a t_i-semistable log structure, then étale locally on X there exists a smooth morphism

$$X \to \operatorname{Spec}(\mathcal{O}_S[z, w]/(zw - t_i)).$$

To prove A.12, it suffices by the same argument used in [29, proof of 3.6] to show that there exists a t_i-semistable log structure on each $\mathcal{U}_i$.

localexist **Lemma A.16.** *Let $X \to \mathcal{U}_i$ be a smooth morphism with X a scheme. Then étale locally on X there exists a t_i-semistable log structure.*

Proof. It suffices to prove the existence in some étale neighborhood of a geometric point $\bar{x} \to X$ mapping to the node p_i of C. Making an étale base change on C, it therefore suffices to show that if

$$g : X \to [\mathrm{Spec}(\mathcal{O}_S[z,w]/(zw - t_i)/\mu_n]$$

is a smooth morphism then there exists a t_i-semistable log structure on X. For this note that there is a log structure $\mathcal{M}$ on $[\mathrm{Spec}(\mathcal{O}_S[z,w]/(zw-t_i)/\mu_n]$ and a morphism of log stacks

stackmap (A.16.1) $$([\mathrm{Spec}(\mathcal{O}_S[z,w]/(zw - t_i)/\mu_n], \mathcal{M}) \to (S, \mathcal{M}_S)$$

induced by the commutative diagram

$$\begin{array}{ccc} \mathbb{N}^2 & \xrightarrow{\beta} & \mathcal{O}_S[z,w]/(zw - t_i) \\ {\scriptstyle\Delta}\uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{1 \mapsto t_i} & \mathcal{O}_S \end{array}$$

where β sends $(1,0)$ to z and $(0,1)$ to w. By [32, 5.23], the morphism A.16.1 is log étale. It follows that the pullback $g^*\mathcal{M}$ on X is a t_i-semistable log structure on X. ♠

Let SS_{t_i} denote the presheaf on the lisse-étale site Lis-Et$(\mathcal{U}_i)$ of $\mathcal{U}_i$ which to any smooth morphism $X \to \mathcal{U}_i$ associates the set of isomorphism classes of t_i-semistable log structures on X. As in the proof of [31, 3.18] a t_i-semistable log structure admits no nontrivial automorphisms, and hence SS_{t_i} is in fact a sheaf. In fact, SS_{t_i} is a torsor under a certain sheaf of abelian groups which we now describe.

For any smooth morphism $X \to \mathcal{U}_i$, there exists by A.16 and A.15 étale locally on X a smooth morphism

$$\rho : X \to \mathrm{Spec}(\mathcal{O}_S[z,w]/(zw - t_i)).$$

As explained in [31, 3.12] the ideal $J := (z,w)\mathcal{O}_X \subset \mathcal{O}_X$ is independent of the choice of the smooth morphism ρ. It follows that these locally defined sheaves of ideals descend to a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{\mathcal{U}_i}$. Let $\mathcal{D} \subset \mathcal{U}_i$ be the closed substack defined by this sheaf of ideals. The local description (2.1 (v)) of the stack $\mathcal{U}_i$ implies that there is an isomorphism

$$\mathcal{D} \simeq \mathcal{B}\mu_n \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathcal{O}_S/(t_i)).$$

Let $K_{t_i} \subset \mathcal{O}_S$ be the kernel of multiplication by t_i on $\mathcal{O}_S$, and let $K_{t_i}^{\mathcal{U}_i}$ denote the kernel of multiplication by t_i on $\mathcal{O}_{\mathcal{U}_i}$. Note that since $\mathcal{U}_i$ is flat over S the sheaf $K_{t_i}^{\mathcal{U}_i}$ is equal to the pullback of K_{t_i}. Let $\mathcal{Z} \subset \mathcal{U}$ be the closed substack defined by $K_{t_i}^{\mathcal{U}_i} \cdot \mathcal{J}$. Define

$$G := \mathrm{Ker}(\mathcal{O}_{\mathcal{U}_i}^* \to \mathcal{O}_{\mathcal{Z}}^*),$$

and let $G_2 \subset \mathcal{O}^*_{\mathcal{U}_i}$ denote the subsheaf of units u such that $ut_i = t_i$. There is a natural inclusion $G \subset G_2$, and as explained in the proof of [31, 3.18] the sheaf SS_{t_i} is naturally a torsor under G_2/G. To prove the existence of a t_i-semistable log structure on $\mathcal{U}_i$ we therefore must show that the class of this torsor

$$o \in H^1(\mathcal{U}_i, G_2/G)$$

is zero.

$\boxed{\texttt{A18}}$ **Lemma A.17.** *The map $H^1(\mathcal{U}_i, G_2/G) \to H^1(\mathcal{D}, \mathcal{O}^*_{\mathcal{D}})$ induced by the composite*

$$G_2 \subset \mathcal{O}^*_{\mathcal{U}_i} \to \mathcal{O}^*_{\mathcal{D}}$$

is injective.

Proof. This follows from the same argument proving [29, 3.7] (and using A.13). ♠

The image of the class o in $H^1(\mathcal{D}, \mathcal{O}^*_{\mathcal{D}})$ corresponds to an invertible sheaf $\mathcal{L}$ on $\mathcal{D}$. As in [31, proof of 3.16] this invertible sheaf $\mathcal{L}$ can be described as follows. Consider the inclusion

$$\mathcal{B}\mu_n \hookrightarrow [\mathrm{Spec}(\mathcal{O}_S[z,\widetilde{w]/(zw} - t_i))/\mu_n] \simeq \mathcal{U}_i \times_{U_i} \mathrm{Spec}(\mathcal{O}_{C,\bar{p}_i}).$$

Then $\mathcal{L}$ corresponds to the representation of μ_N with basis $z\dot{w}$. By the assumptions on the μ_n-action in (2.1 (v)) it follows that $\mathcal{L}$ is trivial, and hence $o = 0$. ♠

As in [29, 3.10], the log structure $\widetilde{\mathcal{M}}_{\mathcal{C}}$ is not the "right" log structure on $\mathcal{C}$ as it does not take into account the markings. Exactly as in loc. cit., for each $i = 1, \ldots, n$ the ideal sheaf $\mathcal{I}_i \subset \mathcal{O}_{\mathcal{C}}$ defining Σ_i defines a log structure $\mathcal{N}_i$ on $\mathcal{C}$. Set

$$\mathcal{M}_{\mathcal{C}} := \widetilde{\mathcal{M}}_{\mathcal{C}} \oplus_{\mathcal{O}^*_{\mathcal{C}}} (\oplus^n_{i=1,\mathcal{O}^*_{\mathcal{C}}} \mathcal{N}_i).$$

The map $f^*\mathcal{M}'_S \to \widetilde{\mathcal{M}}_{\mathcal{C}}$ then induces a log smooth morphism

$$(\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \to (S, \mathcal{M}'_S).$$

If $g : (C, \sigma_1, \ldots, \sigma_n) \to S$ is the coarse moduli space of $\mathcal{C}$ with its n sections defined by the p_i, then the above construction applied to $(C, \sigma_1, \ldots, \sigma_n)$ yields log structures $\mathcal{M}_C$ and $\mathcal{M}_S$ on C and S respectively and a special morphism

$$(C, \mathcal{M}_C) \to (S, \mathcal{M}_S).$$

Proposition A.18 (Generalization of [29, 4.7])**.** *Let $\pi : \mathcal{C} \to C$ be the projection. There exists canonical morphisms of log structures $\pi^b : \pi^*\mathcal{M}_C \to \mathcal{M}_{\mathcal{C}}$ and $\ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S$ such that the diagram of log stacks*

$$\begin{tikzcd}
(\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \arrow[r, "{(\pi,\pi^b)}"] \arrow[d, "f"'] & (C, \mathcal{M}_C) \arrow[d, "g"] \\
(S, \mathcal{M}'_S) \arrow[r, "{(id,\ell)}"] & (S, \mathcal{M}_S)
\end{tikzcd}$$

commutes. Moreover, the map $\ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S$ is a simple extension.

Proof. This follows from the same argument proving [29, 4.7]. The key point is the local description (2.1 (iv) and (v)) of a twisted curve which enables one to rewrite [29, 4.6] verbatim in the present situation. ♠

Finally note that for any geometric point $\bar{s} \to S$ the gerbe $\Sigma_{i,\bar{s}}$ is necessarily trivial and hence isomorphic to $\mathcal{B}\mu_{a_i(\bar{s})}$ for some integer $a_i(\bar{s})$. One verifies immediately that the a_i are positive integer valued locally constant functions on S.

The association

$$(\mathcal{C}, \{\Sigma_i\}) \mapsto (C/S, \{\sigma_i, a_i\}, \ell : \mathcal{M}_S \hookrightarrow \mathcal{M}'_S)$$

therefore defines a functor

inversefunctor (A.18.1) (twisted n-pointed curves) $\to$ (n-pointed log twisted curves)

As in [29, 4.8] one sees that A.5.1 and A.18.1 are inverse functors thereby proving A.5. ♠

Appendix B. Remarks on Ext-groups and base change
By Martin Olsson

AppendixC

In this appendix we gather together some fairly standard results about Ext-groups and base change, which will be used in the following appendix C. The results B.2 and B.10 are well-known to experts, but we include them here due to lack of a suitable reference.

B.1. **Boundedness of cohomology.**

cohfinite **Theorem B.2.** *Let $f : X \to Y$ be a separated, representable, finite type morphism between noetherian algebraic stacks. Then there exists an integer n_0 such that for any quasi-coherent sheaf $\mathcal{F}$ on X and $i \geq n_0$ we have $R^i f_* \mathcal{F} = 0$.*

Proof. The assertion is fppf local on Y, so we may assume that Y is a noetherian affine scheme, and that X is an algebraic space. In this case we must show that there exists an integer n_0 such that for any quasi-coherent sheaf $\mathcal{F}$ on X and $i \geq n_0$ we have $H^i(X, \mathcal{F}) = 0$.

nillem **Lemma B.3.** *Let $Z_0 \hookrightarrow Z$ be a closed immersion of noetherian algebraic spaces over Y defined by an nilpotent ideal $\mathcal{J} \subset \mathcal{O}_Z$. Assume that B.2 holds for $Z_0 \to Y$ and let n_{Z_0} be an integer such that for any quasi-coherent sheaf $\mathcal{F}_0$ on Z_0 we have $H^i(Z_0, \mathcal{F}_0) = 0$ for $i \geq n_{Z_0}$. Then for any quasi-coherent sheaf $\mathcal{F}$ on Z we have $H^i(Z, \mathcal{F}) = 0$ for $i \geq n_{Z_0}$.*

Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on Z. Let n be an integer such that $\mathcal{J}^n = 0$, and set $\mathcal{F}_k = \mathcal{J}^k \mathcal{F}$ so that we have an increasing sequence of quasi-coherent sheaves

$$0 = \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \cdots \mathcal{F}_0 = \mathcal{F}$$

with successive quotients $\mathcal{F}_k/\mathcal{F}_{k+1}$ supported on Z_0. The result then follows by descending induction on k and consideration of the short exact sequences

$$0 \to \mathcal{F}_{k+1} \to \mathcal{F}_k \to \mathcal{F}_k/\mathcal{F}_{k+1} \to 0$$

which gives rise to exact sequences

$$H^i(Z, F_{k+1}) \to H^i(Z, F_k) \to H^i(Z_0, \mathcal{F}_k/\mathcal{F}_{k+1}),$$

where the last term is zero by assumption. ♠

Now we prove B.2 by noetherian induction.

By [23, II.6.7] there exists a dense open subspace $j : U \subset X$ with U an affine scheme. Let $Z \subset X$ be the complement (with the reduced structure). Let n_Z be an integer such that for any quasi-coherent sheaf $\mathcal{F}_Z$ on Z we have $H^i(Z, \mathcal{F}_Z) = 0$ for $i \geq n_Z$.

limlem **Lemma B.4.** *Let $\mathcal{G}$ be a quasi-coherent sheaf on X whose restriction to U is the zero sheaf. Then $H^i(X, \mathcal{G}) = 0$ for $i \geq n_Z$.*

Proof. By [23, III.1.1] the sheaf $\mathcal{G}$ is equal to the limit $\mathcal{G} = \varinjlim \mathcal{G}_i$ over its coherent subsheaves $\mathcal{G}_i$. Furthermore by [23, II.4.17] we have

$$H^i(X, \mathcal{G}) = \varinjlim H^i(X, \mathcal{G}_i).$$

It follows that it suffices to consider the case when $\mathcal{G}$ is coherent. In this case there exists a nilpotent thickening $Z \subset Z' \subset X$ of Z in X such that the scheme-theoretic support of $\mathcal{G}$ is contained in Z'. The result therefore follows from B.3. ♠

Since U is affine and X is separated, the morphism $j : U \to X$ is an affine morphism. It follows that the sheaf $j_*j^*\mathcal{F}$ is a quasi-coherent sheaf on X and that

$$H^i(X, j_*j^*\mathcal{F}) = H^i(U, j^*\mathcal{F}).$$

Since U is affine these groups are zero for $i > 0$. Set $n_X := n_Z + 1$.

Let K (resp. I, Q) be the kernel (resp. image, cokernel) of the adjunction map $\mathcal{F} \to j_*j^*\mathcal{F}$. By B.4 we have $H^i(X, K)$ and $H^i(X, Q)$ equal to 0 for $i \geq n_Z$. Consideration of the long exact sequences associated to the short exact sequences

$$0 \to K \to \mathcal{F} \to I \to 0$$

and

$$0 \to I \to j_*j^*\mathcal{F} \to Q \to 0$$

then shows that $H^i(X, \mathcal{F}) = 0$ for $i \geq n_X$. ♠

B.5. **Base change for Ext.** Let $f : \mathcal{X} \to S$ be a finite type morphism between noetherian algebraic stacks, and assume that S is an integral scheme. Fix $L^{\cdot} \in D^{-}_{\mathrm{coh}}(\mathcal{X})$ (the derived category of bounded above complexes of $\mathcal{O}_{\mathcal{X}}$-modules with coherent cohomology sheaves) and a coherent sheaf $\mathcal{J} \in \mathrm{Coh}(\mathcal{X})$.

Lemma B.6. *For every integer n, there exists a dense open subset (which depends on n) $U \subset S$ such that for any cartesian diagram*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S, \end{array}$$

where g factors through U, the natural map

$$h^*\mathcal{E}xt^p(L^{\cdot}, \mathcal{J}) \to \mathcal{E}xt^p(\mathbb{L}h^*L^{\cdot}, \mathbb{L}h^*\mathcal{J})$$

is an isomorphism for $p \leq n$.

Proof. The assertion is local in the fppf topology on $\mathcal{X}$, S, and S'. We may therefore assume that $\mathcal{X} = \mathrm{Spec}(R)$ and $S = \mathrm{Spec}(A)$ are affine schemes, and that $L^{\cdot}$ can be represented by a bounded above complex of projective R-modules of finite type, which we again denote by $L^{\cdot}$. Let J denote the R-module corresponding to the sheaf $\mathcal{J}$, and let $F^{\cdot}$ denote the bounded below complex of finite type R-modules

$$F^{\cdot} := \mathrm{Hom}^{\cdot}(L^{\cdot}, J).$$

After shrinking on S we may assume that J is flat over A, in which case each F^j is flat over A. We need to show that after possibly replacing S by a dense affine open subset, the natural map

$$H^p(F^{\cdot}) \otimes_A A' \to H^p(F^{\cdot} \otimes_A A')$$

is an isomorphism for all ring homomorphisms $A \to A'$ and all $p \leq n$. This is a standard argument and we leave it to the reader (see for example [14, IV.9.4.3] where a similar argument is made). ♠

We would like to use this result to obtain base change properties of global Ext-groups. In order for this to be possible, however, we need certain finiteness properties of cohomology.

cohboundeddef **Definition B.7.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of noetherian algebraic stacks over a scheme S.

(i) f is *cohomologically bounded* if for every object $\mathcal{F} \in D^b_{\mathrm{qcoh}}(\mathcal{X})$ there exists an integer n such that $R^if_*\mathcal{F} = 0$ for $i \geq n$.

(ii) f is *universally cohomologically bounded* if for every morphism of noetherian algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the morphism $f' : \mathcal{X}' := \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ is cohomologically bounded.

Remark B.8. By a standard devissage, f is cohomologically bounded if and only if for every quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ there exists an integer n such that $R^i f_* \mathcal{F} = 0$ for $i \geq n$.

Remark B.9. Note that any separated representable morphism $f : \mathcal{X} \to \mathcal{Y}$ is universally cohomologically bounded by B.2.

cohboundedness

Theorem B.10. *Let $f : \mathcal{X} \to S$ be a flat finite type proper morphism of noetherian algebraic stacks with $S = \operatorname{Spec}(A)$ an integral affine scheme, and assume that f is cohomologically bounded. Let $L^{\cdot} \in D^-_{\mathrm{coh}}(\mathcal{X})$ and $\mathcal{J} \in \operatorname{Coh}(\mathcal{X})$. Then for every integer n, there exists a dense open subset (which depends on n) $U \subset S$ such that for any cartesian diagram*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' = \operatorname{Spec}(A') & \xrightarrow{g} & S, \end{array}$$

where g factors through U, the natural map

targetmorphism

$$\operatorname{Ext}^p(L^{\cdot}, \mathcal{J}) \otimes_A A' \to \operatorname{Ext}^p(\mathbb{L}h^* L^{\cdot}, \mathbb{L}h^* \mathcal{J}) \tag{B.10.1}$$

is an isomorphism for $p \leq n$.

Proof. This follows from consideration of the local-to-global spectral sequences for Ext from B.2, and standard base change properties for cohomology of coherent sheaves (see for example [30, 5.12]).

First after shrinking on S we may assume that $\mathcal{J}$ is flat over S and that for any morphism $g : \operatorname{Spec}(A') \to \operatorname{Spec}(A)$ the pullback map

$$h^* \mathcal{E}xt^p(L^{\cdot}, \mathcal{J}) \to \mathcal{E}xt^p(\mathbb{L}h^* L^{\cdot}, \mathbb{L}h^* \mathcal{J})$$

is an isomorphism for all $p \leq n$. By standard base change properties of cohomology of coherent sheaves as in [30, 5.12] (note that this is where the cohomological boundedness is used), it follows that after shrinking some more on S we may assume that each of the terms E_r^{pq} for $p + q \leq n$ in the spectral sequence

$$E_2^{pq} = H^p(\mathcal{X}, \mathcal{E}xt^q(L^{\cdot}, \mathcal{J})) \implies \operatorname{Ext}^{p+q}(L^{\cdot}, \mathcal{J})$$

are flat over S, and that their formation commute with arbitrary base change $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$. That B.10.1 is an isomorphism then follows from consideration of the morphism of spectral sequences

$$\begin{array}{c} E_2^{pq} = H^p(\mathcal{X}, \mathcal{E}xt^q(L^{\cdot}, \mathcal{J})) \implies \operatorname{Ext}^{p+q}(L^{\cdot}, \mathcal{J}) \\ \downarrow \\ E_2^{pq} = H^p(\mathcal{X}', \mathcal{E}xt^q(\mathbb{L}h^* L^{\cdot}, \mathbb{L}h^* \mathcal{J})) \implies \operatorname{Ext}^{p+q}(\mathbb{L}h^* L^{\cdot}, \mathbb{L}h^* \mathcal{J}). \end{array}$$

♠

Appendix C. Another boundedness theorem for Hom-stacks
By Martin Olsson

AppendixB

C.1. **Statement of Theorems.**

C.2. Let B be a scheme, and $\mathcal{X}$ and $\mathcal{Y}$ be Artin stacks of finite presentation over B with finite diagonals. Let X and Y denote the coarse moduli spaces of $\mathcal{X}$ and $\mathcal{Y}$. Assume that $\mathcal{X}$ is flat and proper over B, and that fppf locally on B there exists a finite and finitely presented flat surjection $Z \to \mathcal{X}$ with Z an algebraic space.

By [30, 1.1], we then have Artin stacks $\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$ and $\underline{\mathrm{Hom}}(\mathcal{X}, Y)$ locally of finite presentation over B with separated and quasi–compact diagonals.

athm **Theorem C.3.** *Assume that $\mathcal{Y}$ is a tame stack. Then the natural map*

$$\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y}) \to \underline{\mathrm{Hom}}(\mathcal{X}, Y)$$

is of finite type.

There is also a variant of C.3, where one does not assume that $\mathcal{X}$ admits a finite flat cover by a scheme but instead that $\mathcal{X}$ is tame (this is in fact the theorem we will apply to twisted stable maps):

athmbis **Theorem C.4.** *Let B be a scheme locally of finite type over an excellent Dedekind ring, and let $\mathcal{X}$ and $\mathcal{Y}$ be tame Artin stacks of finite presentation over B with finite diagonals. Let X (resp. Y) denote the moduli space of $\mathcal{X}$ (resp. $\mathcal{Y}$). Assume that $\mathcal{X}$ is flat and proper over B. Then $\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$ is an algebraic stack locally of finite presentation over B with quasi-compact and separated diagonal, and the map*

$$\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y}) \to \underline{\mathrm{Hom}}(\mathcal{X}, Y) = \underline{\mathrm{Hom}}(X, Y)$$

is of finite type.

Remark C.5. Note that by [4, 3.3], for any morphism $B' \to B$ the base change $\mathcal{X}_{B'} \to X_{B'}$ is the moduli space of $\mathcal{X}_{B'}$.

Remark C.6. The assumption that $\mathcal{X}/B$ is flat implies that X/B is also flat by [4, 3.3 (b)]. It follows that $\underline{\mathrm{Hom}}(X, Y)$ is an algebraic space locally of finite presentation over B.

In the context of either C.3 or C.4, one can also consider the substack

$$\underline{\mathrm{Hom}}^{\mathrm{rep}}(\mathcal{X}, \mathcal{Y}) \subset \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$$

classifying representable morphisms $\mathcal{X} \to \mathcal{Y}$. As in [30, 1.6] this substack is an open substack, and therefore the following corollary follows from C.3 and C.4:

repfinite **Corollary C.7.** *The natural map*

$$\underline{\mathrm{Hom}}^{\mathrm{rep}}(\mathcal{X}, \mathcal{Y}) \to \underline{\mathrm{Hom}}(\mathcal{X}, Y)$$

is of finite type.

The rest of this appendix is devoted to the proofs of C.3 (paragraphs C.17-C.30) and C.4 (paragraphs C.31-C.36).

A3intro **C.8.** We begin with the proof of C.3.

As in [35, 6.2], Theorem C.3 is equivalent to the following statement. For any morphism $f : \mathcal{X} \to Y$ the stack $\underline{\mathrm{Sec}}(\mathcal{X} \times_Y \mathcal{Y}/\mathcal{X})$ which to any B–scheme T associates the groupoid of sections $s : \mathcal{X} \to \mathcal{X} \times_Y \mathcal{Y}$ is of finite type. It is this latter statement that we will prove. Let $\mathcal{G}$ denote $\mathcal{X} \times_{f,Y} \mathcal{Y}$. The proof that $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is of finite type over B follows the same outline as in [35]. First (in paragraphs C.17–C.22) we reduce the proof of C.3 to the special case when $\mathcal{X} = X$ and $\mathcal{G}$ is a μ_n–gerbe for some integer n. We then prove the theorem in this special case (paragraphs C.23–C.36) using some relatively straightforward generalizations to "twisted sheaves" of theorems about the Picard functor.

The main new wrinkle is to use the base change properties of Ext explained in appendix B.

C.9. **Some results about modifications.** We use the results of appendix B to generalize three basic results about modifications in [35, §4] to tame stacks.

Passage to the maximal reduced substack.

C.10. Let S be a noetherian scheme, and let $\mathcal{G} \to \mathcal{X}$ be a finite type morphism between Artin stacks of finite type over S with finite diagonals. Assume further that $\mathcal{X}$ is flat and proper over S and that the structure morphism $\mathcal{X} \to S$ is universally cohomologically bounded in the sense of B.7. Assume further that fppf locally on S there exists a finite flat cover of $\mathcal{X}$ by an algebraic space.

finitelem **Proposition C.11.** *Let $\mathcal{X}_0 \hookrightarrow \mathcal{X}$ be a closed immersion defined by a nilpotent ideal $\mathcal{J} \subset \mathcal{O}_{\mathcal{X}}$, and assume $\mathcal{X}_0$ is flat over S. Let $\mathcal{G}_0$ denote the base change $\mathcal{G} \times_{\mathcal{X}} \mathcal{X}_0$. Then the natural map*

$$\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X}) \to \underline{\mathrm{Sec}}(\mathcal{G}_0/\mathcal{X}_0)$$

is of finite type.

Proof. This is essentially the same as in [30, 5.11].

By noetherian induction, it suffices to show that the morphism is of finite type over a dense open subset of S. We may further assume that S is reduced (since if U and V are S-schemes locally of finite type and $g : U \to V$ is a morphism, then g is of finite type if and only if the base change of g to S_{red} is of finite type), and using the same argument given in [30, paragraph following proof of 5.11] that $\mathcal{J}^2 = 0$.

It suffices to show that if $T \to \underline{\mathrm{Sec}}(\mathcal{G}_0/\mathcal{X}_0)$ is a morphism corresponding to a section $s : \mathcal{X}_0 \to \mathcal{G}_0$, then the fiber product

$$\mathcal{P} := \underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X}) \times_{\underline{\mathrm{Sec}}(\mathcal{G}_0/\mathcal{X}_0)} T$$

is of finite type over T. Furthermore, we may assume that T is an integral noetherian affine scheme. The stack $\mathcal{P}$ associates to any $w : W \to T$ the groupoid of liftings $\tilde{s} : \mathcal{X}_W \to \mathcal{G}$ over $\mathcal{X}$ of the composite

$$\mathcal{X}_{0,W} \to \mathcal{G}_0 \to \mathcal{G},$$

where the first map is the one induced by s.

To prove that $\mathcal{P}$ is quasi–compact it suffices by Noetherian induction to exhibit a dense open set $U \subset T$ such that $\mathcal{P}_U$ is quasi–compact.

Let $L_{\mathcal{G}_0/\mathcal{X}_0}$ be the cotangent complex of $\mathcal{G}_0/\mathcal{X}_0$. By B.10 (this is where the assumption of cohomological boundedness is used), after replacing T by a dense open subscheme we may assume that the groups

$$\mathrm{Ext}^i(s^*L_{\mathcal{G}_0/\mathcal{X}_0}, \mathcal{J}), \quad i = -1, 0, 1$$

are projective modules on T of finite type, and that the formation of these modules commutes with arbitrary base change on T.

By [34, 1.5] there is a canonical obstruction

$$o \in \mathrm{Ext}^1(s^*L_{\mathcal{G}_0/\mathcal{X}_0}, \mathcal{J})$$

whose vanishing is necessary and sufficient for the existence of a lifting $\tilde{s}$ as above, and the formation of this obstruction is functorial in T. After replacing T by the closed subscheme defined by the condition that o vanishes, we may assume that $o = 0$. In this case the set of isomorphism classes of liftings $\tilde{s}$ form a torsor under

$$\mathrm{Ext}^0(s^*L_{\mathcal{G}_0/\mathcal{X}_0}, \mathcal{J})$$

and the group of infinitessimal automorphisms of $\tilde{s}$ is canonically isomorphic to

$$\mathrm{Ext}^{-1}(s^*L_{\mathcal{G}_0/\mathcal{X}_0}, \mathcal{J}).$$

It follows that $\mathcal{P}$ is an $\mathrm{Ext}^{-1}(s^*L_{\mathcal{G}_0/\mathcal{X}_0}, \mathcal{J})$–gerbe over

$$\mathrm{Ext}^0(s^*L_{\mathcal{G}_0/\mathcal{X}_0}, \mathcal{J}),$$

and in particular is quasi–compact. ♠

The case of a finite morphism.

C.12. Let S be a noetherian scheme, and $\mathcal{X}/S$ a proper flat Artin stack with finite diagonal. Let $\mathcal{G} \to \mathcal{X}$ be a finite morphism. Assume that fppf locally on S, the stack $\mathcal{X}$ admits a finite flat surjection $Z \to \mathcal{X}$ with Z an algebraic space.

finitecase **Proposition C.13.** *If $\mathcal{X} \to S$ is universally cohomologically bounded, then the stack $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is of finite type and separated over S.*

Proof. To verify that $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is of finite type over S, it suffices to show that its pullback to S_{red} is of finite type. We may therefore assume that S is reduced. Furthermore, using the fact that $\mathcal{X} \to S$ is universally cohomologically bounded, we may by C.11 (applied with $\mathcal{X}_0 = \mathcal{X}_{\mathrm{red}}$) assume that $\mathcal{X}$

is also reduced.) Furthermore, by noetherian induction it suffices to exhibit a dense open subset of S over which $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is of finite type.

By Chow's lemma for Artin stacks [33, 1.1], there exists a proper surjection $h : X' \to \mathcal{X}$ with X' a projective S-scheme. After shrinking on S we may also assume that X' is flat over S (since S is reduced). Since $\mathcal{X}$ is reduced the map $\mathcal{O}_{\mathcal{X}} \to h_*\mathcal{O}_{X'}$ is injective. After shrinking on S, we may assume that the formation of $h_*\mathcal{O}_{X'}$ commutes with arbitrary base change $S' \to S$, and that for any such base change the map $\mathcal{O}_{\mathcal{X}_{S'}} \to h_*\mathcal{O}_{X'_{S'}}$ is also injective. Let $G' \to X'$ denote the pullback $\mathcal{G} \times_{\mathcal{X}} X'$.

By [30, 5.10] the stack (in fact an algebraic space) $\underline{\mathrm{Sec}}(G'/X')$ is of finite type over S. It therefore suffices to show that the pullback map

5.9.1 (C.13.1) $$\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X}) \to \underline{\mathrm{Sec}}(G'/X')$$

is of finite type.

Let $\mathcal{A}$ be the coherent sheaf of algebras defined by $\mathcal{G} = \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A})$, and let $\mathcal{B}$ denote the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras $h_*\mathcal{O}_{X'}$. For a morphism $T \to S$, let $\mathcal{A}_T$ (resp. $\mathcal{B}_T$) denote the pullback of $\mathcal{A}$ (resp. $\mathcal{B}$) to $\mathcal{X}_T := \mathcal{X} \times_S T$. Then $\underline{\mathrm{Sec}}(G'/X')$ is equal to the functor which to any scheme $T \to S$ associates the set of morphisms of $\mathcal{O}_{\mathcal{X}_T}$-algebras $\varphi : \mathcal{A}_T \to \mathcal{B}_T$ over $\mathcal{X}_T$. The stack $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is the subfunctor of morphisms φ which factor through $\mathcal{O}_{\mathcal{X}_T} \subset \mathcal{B}_T$.

Fix a morphism $\varphi : \mathcal{A}_T \to \mathcal{B}_T$ over a noetherian scheme T defining a T-valued point of $\underline{\mathrm{Sec}}(G'/X')$, and set

$$P := \underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X}) \times_{\underline{\mathrm{Sec}}(G'/X'),\varphi} T.$$

To prove that P is of finite type, we may again assume that T is reduced, and by noetherian induction it suffices to show that $P \to T$ is of finite type over a dense open subset of T. Let M denote the cokernel of the map $\mathcal{O}_{\mathcal{X}_T} \hookrightarrow \mathcal{B}_T$, and let $\bar{\varphi} : \mathcal{A}_T \to M$ be the map induced by φ. Let Q be the cokernel of $\bar{\varphi}$, and let K be the kernel of the map $M \to Q$ so that there is an exact sequence

Qseq (C.13.2) $$0 \to K \to M \to Q \to 0.$$

After shrinking on T we may assume that Q is flat over T in which case C.13.2 remains exact after arbitrary base change $T' \to T$. In this case, let $\mathcal{Z} \subset \mathcal{X}_T$ be the support of the coherent sheaf K, and let $W \subset T$ be the image of $\mathcal{Z}$ (which is closed since $\mathcal{X}_T \to T$ is proper). Then P is represented by the complement of W in T.

To verify that $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is separated, note that we already know that the diagonal is quasi-compact and separated. Therefore it suffices to verify the valuative criterion for properness. This amounts to the following. Assume that S is the spectrum of a valuation ring and let $\eta \in S$ be the generic point. Assume given two sections $s_1, s_2 : \mathcal{X} \to \mathcal{G}$ whose restrictions to η are equal. Then we need to show that $s_1 = s_2$. For this note that since $\mathcal{G}$ is finite over $\mathcal{X}$, we have $\mathcal{G} = \underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{A})$ for some coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras $\mathcal{A}$ on $\mathcal{X}$, and the sections s_1 and s_2 are specified by two morphisms of $\mathcal{O}_{\mathcal{X}}$-algebras

$\rho_1, \rho_2 : \mathcal{A} \to \mathcal{O}_{\mathcal{X}}$. Let $j : \mathcal{X}_\eta \hookrightarrow \mathcal{X}$ be the inclusion of the generic fiber. Since $\mathcal{X}$ is flat over S, the natural map $\mathcal{O}_{\mathcal{X}} \to j_*\mathcal{O}_{\mathcal{X}_\eta}$ is an inclusion. Therefore it suffices to show that the composite maps

$$\mathcal{A} \xrightarrow{\rho_i} \mathcal{O}_{\mathcal{X}} \longrightarrow j_*\mathcal{O}_{\mathcal{X}_\eta}$$

are equal. Equivalently that the maps on the generic fiber

$$\rho_{i,\eta} : \mathcal{A}_\eta \to \mathcal{O}_{\mathcal{X}_\eta}$$

are equal, which holds by assumption. ♠

Behavior with respect to proper modifications of $\mathcal{X}$.

P:C.14 **C.14.** Let S be a noetherian scheme, and let $\mathcal{X}$ and $\mathcal{Y}$ be Artin stacks of finite type over S with finite diagonals. Assume that the following conditions hold:

(i) The formation of the coarse spaces $\pi_{\mathcal{X}} : \mathcal{X} \to X$ and $\pi_{\mathcal{Y}} : \mathcal{Y} \to Y$ commutes with arbitrary base change $S' \to S$.
(ii) $\mathcal{X}$ is proper and flat over S, and fppf locally on S there exists a finite flat surjection $Z \to \mathcal{X}$ with Z an algebraic space.
(iii) The coarse space X of $\mathcal{X}$ is flat over S (note that X is automatically proper over S since $\mathcal{X}$ is proper over S).
(iv) The morphism $\mathcal{X} \to S$ is universally cohomologically bounded.

modprop **Proposition C.15.** *Let $m : \mathcal{X}' \to \mathcal{X}$ be a proper representable surjection with $\mathcal{X}'$ a proper and flat algebraic stack over S with finite diagonal. Let $f : \mathcal{X} \to Y$ be a morphism, and let $\mathcal{G}$ (resp. $\mathcal{G}'$) denote the pullback along f (resp. $f \circ m$) of $\mathcal{Y}$. Assume that $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ and $\underline{\mathrm{Sec}}(\mathcal{G}'/\mathcal{X}')$ are algebraic stacks locally of finite type over S with quasi-compact diagonals. Then the pullback map*

pullmap (C.15.1) $$\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X}) \to \underline{\mathrm{Sec}}(\mathcal{G}'/\mathcal{X}')$$

is of finite type.

Remark C.16. Note that since m is representable, the stack $\mathcal{X}'$ also admits a finite flat surjection from an algebraic space.

Proof. This follows from the same argument proving [35, 4.13]. The cohomological boundedness assumption is necessary in order to apply C.11. ♠

proofstart

C.17. **Dévissage.** Now we begin the proof of the statement that $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ in C.8 is of finite type over B. In the rest of the proof that $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is of finite type over B, we work under the assumptions of C.14.

By a standard limit argument as discussed for example in [30, §2] we can without loss of generality assume that B is of finite type over $\mathbb{Z}$.

C.18. The assertion that $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is of finite type is fppf local on B, and therefore we may assume that there exists a finite flat surjection $Z \to \mathcal{X}$ with Z an algebraic space. Let $Z^{(i)}$ denote the i–fold fiber product of Z with itself over $\mathcal{X}$. Then by the description of $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ in terms of the

$\underline{\mathrm{Sec}}(\mathcal{G} \times_{\mathcal{X}} Z^{(i)}/Z^{(i)})$ given in [30, 3.3] it suffices to show that for $i = 1, 2, 3$ the stacks $\underline{\mathrm{Sec}}(\mathcal{G} \times_{\mathcal{X}} Z^{(i)}/Z^{(i)})$ are of finite type. We may therefore assume that $\mathcal{X} = X$.

Furthermore, we can without loss of generality assume that B is integral. By noetherian induction, it suffices to exhibit a dominant morphism $B' \to B$ such that the restriction of $\underline{\mathrm{Sec}}(\mathcal{G}/X)$ to B' is of finite type.

C.19. By the same argument used in [35, 6.4] we then reduce the proof of C.3 to the case when X is integral and $\mathcal{G} \to X$ has a section $s : X \to \mathcal{G}$. Let $\underline{\mathrm{Aut}}(s)$ denote the group scheme of automorphisms of s (a scheme over X). Since X is reduced there exists a dense open subset $U \subset X$ such that the restriction of $\underline{\mathrm{Aut}}(s)$ to U is flat over U. Let $H \subset \underline{\mathrm{Aut}}(s)$ denote the scheme–theoretic closure of $\underline{\mathrm{Aut}}(s)|_U$. By [17, Premiére partie 5.2.2] there exists a blow-up $X' \to X$ with center a proper closed subspace of X such that the strict transform of H in $\underline{\mathrm{Aut}}(s)$ is flat over X'. After further shrinking on S we may assume that X' is flat over S. Since the map

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \to \underline{\mathrm{Sec}}(\mathcal{G} \times_X X'/X')$$

is of finite type by C.15, we may therefore replace X by X' and hence can assume that H is a finite flat subgroup scheme of $\underline{\mathrm{Aut}}(s)$. Since all the geometric fibers of H are linearly reductive being closed subgroup schemes of linearly reductive groups (see [4, 2.7]), the X-group scheme H is in fact linearly reductive. If $\widetilde{X}$ is the normalization of X the map

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \to \underline{\mathrm{Sec}}(\mathcal{G} \times_X \widetilde{X}/\widetilde{X})$$

is also of finite type by C.15. This enables us to reduce to the following situation: X is normal, $s : X \to \mathcal{G}$ is a section, and $H \subset \underline{\mathrm{Aut}}(s)$ is a finite closed [20] subgroup scheme which is flat and linearly reductive over X and over a dense open subset of X the group scheme H is equal to $\underline{\mathrm{Aut}}(s)$. ←20

Lemma C.20. *The morphism $\pi : \mathcal{B}H \to \mathcal{G}$ induced by s identifies $\mathcal{B}H$ with the normalization of $\mathcal{G}_{\mathrm{red}}$.*

Proof. Etale locally on X we can write $\mathcal{G} = [W/GL_n]$ for some n, where W is an X–scheme. Indeed by [4, 3.2] we can étale locally on X write $\mathcal{G} = [Z/G]$ with G a linearly reductive group scheme. Choosing any embedding $G \hookrightarrow GL_n$ we take W to be the quotient of $Z \times GL_n$ by the diagonal action of G.

Let $P \to X$ be the pullback $X \times_{s,\mathcal{G}} W$. Then P is a GL_n–torsor over X with a GL_n–equivariant morphism $f : P \to W$. Since P is smooth over X the space P is in particular normal, and since s is proper and quasi–finite the morphism f is finite. After replacing X by an étale cover we may assume given a section $p \in P$. Then $\underline{\mathrm{Aut}}(s)$ is the closed subgroup scheme of $GL_{n,X}$

[20] (Martin) removed normal as it is not used

given by the fiber product of the diagram

$$\begin{array}{ccc} & & W \times GL_n \\ & & \Big\downarrow{\scriptstyle\rho} \\ & & W \times W \xleftarrow{\Delta\circ f\circ p} X, \end{array}$$

where $\Delta : W \to W \times W$ is the diagonal and ρ is the map sending (v, g) to $(v, g(v))$. In particular, we obtain an embedding $H \subset GL_{n,X}$. Let $\overline{P}$ denote the quotient P/H. The space $\overline{P}$ is normal. By the construction the map f induces a morphism $\bar{f} : \overline{P} \to W^*$, where W^* denotes the normalization of W_{red}. This map $\bar{f}$ is finite, surjective, and birational and hence by Zariski's Main Theorem an isomorphism. On the other hand, we have $\overline{P} \simeq W \times_{\mathcal{G}} \mathcal{B}H$, and hence if $\mathcal{G}^*$ denotes the normalization of $\mathcal{G}_{\text{red}}$ we find that the base change of $\mathcal{B}H \to \mathcal{G}^*$ to $W \times_{\mathcal{G}} \mathcal{G}^*$ is an isomorphism. It follows that $\mathcal{B}H \to \mathcal{G}^*$ is an isomorphism. ♠

C.21. Let $\mathcal{G}^*$ denote the normalization of $\mathcal{G}$, so that $\mathcal{G}^*$ is a gerbe over X. As in [35, 3.14], after shrinking on B we may assume that for every field valued point $b \in B(k)$ the fiber $\mathcal{G}^*_b \to \mathcal{G}_b$ is the normalization of $\mathcal{G}_{b,\text{red}}$ and that X_b is normal. It then follows that the map

$$\underline{\mathrm{Sec}}(\mathcal{G}^*/X) \to \underline{\mathrm{Sec}}(\mathcal{G}/X)$$

is surjective on field valued points and hence surjective. We may therefore replace $\mathcal{G}$ by $\mathcal{G}^*$ and therefore may assume that $\mathcal{G}$ is a gerbe over X.

Next we consider the case of a gerbe. If the generic point of B has characteristic 0, then after shrinking on B we can assume $\mathcal{G}$ is Deligne–Mumford in which case the result follows from [35, 1.1]. We may therefore assume that the generic point of B has characteristic $p > 0$, and hence after shrinking on B may assume that B is an $\mathbb{F}_p$–scheme. In this case, for any B–scheme T and $t \in \mathcal{G}(T)$, the automorphism group scheme G_t of t is canonically an extension

$$1 \to \Delta_t \to G_t \to H_t \to 1,$$

where Δ_t is locally diagonalizable and H_t is étale over T. Indeed, the group scheme G_t is tame so we can take Δ_t to be the subfunctor of G_t classifying elements of G_t killed by some power of p. This normal subgroup Δ_t is functorial in the pair (T, t).

Define $\mathcal{H}$ to be the stack (with respect to the fppf topology) associated to the prestack whose objects are the same as the objects of $\mathcal{G}$ but for which a morphism between two $t, t' \in \mathcal{G}(T)$ is defined to be the set

$$\Delta_t \backslash \underline{\mathrm{Hom}}(t, t') = \Delta_t \backslash \underline{\mathrm{Hom}}(t, t')/\Delta_{t'} = \underline{\mathrm{Hom}}(t, t')/\Delta_{t'}.$$

Then $\mathcal{H}$ is a Deligne–Mumford stack and there is a natural map $\mathcal{G} \to \mathcal{H}$ over X. By [35, 1.1] the stack $\underline{\mathrm{Sec}}(\mathcal{H}/X)$ is of finite type. On the other hand, for any section $s : X \to \mathcal{H}$, the fiber product

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \times_{\underline{\mathrm{Sec}}(\mathcal{H}/X), s} B$$

is isomorphic to $\underline{\mathrm{Sec}}(\mathcal{G} \times_{\mathcal{H},s} X/B)$. Now the stack $\mathcal{G} \times_{\mathcal{H},s} X$ is a gerbe over X whose stabilizer groups are all diagonalizable. This further reduces the proof to the case when the H_t's in the above discussion are all trivial.

B.33 **C.22.** In this case, there exists a canonical locally diagonalizable group scheme Δ on X such that $\mathcal{G}$ is bound by Δ. Indeed fppf locally on X there exists a section $s : X \to \mathcal{G}$ giving rise to a group scheme Δ_s. If $s' : X \to \mathcal{G}$ is a second section then locally s and s' are isomorphic so we obtain an isomorphism $\Delta_s \simeq \Delta_{s'}$. This isomorphism is independent of the choice of the isomorphism between s and s' since Δ_s and $\Delta_{s'}$ are abelian. It follows that the Δ_s's descend to a group scheme Δ on X.

Since the Cartier dual of Δ is a locally constant sheaf of finite abelian groups on X_{et}, there exists a finite étale covering $X' \to X$ such that the pullback of Δ to X' is a diagonalizable group scheme. Since

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \to \underline{\mathrm{Sec}}(\mathcal{G} \times_X X'/X')$$

is of finite type by C.15, this reduces the proof to the case when $\mathcal{G}$ is a gerbe over X bound by a diagonalizable group scheme Δ. In other words, when $\mathcal{G}$ corresponds to a cohomology class in $H^2(X, \Delta)$. Write $\Delta = \mu_{n_1} \times \cdots \times \mu_{n_r}$ so that

$$H^2(X, \Delta) = \prod_i H^2(X, \mu_{n_i}).$$

Using the resulting decomposition of the cohomology class of $\mathcal{G}$, we see that $\mathcal{G}$ is isomorphic to a product $\mathcal{G}_1 \times_X \cdots \times_X \mathcal{G}_r$, where $\mathcal{G}_i$ is a μ_{n_i}–gerbe over X. Then

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \simeq \prod_i \underline{\mathrm{Sec}}(\mathcal{G}_i/X).$$

This therefore finally reduces the proof to the case of a μ_n–gerbe over X which is the case treated in the following subsection.

Gtwisted

C.23. **$\mathcal{G}$–twisted line bundles.**

case1 **C.24.** Fix an integer n. Assume that $\mathcal{X} = X$ and that $\mathcal{G}$ is a μ_n–gerbe over X. Assume furthermore that if $f : X \to B$ denotes the structural morphism then the map $\mathcal{O}_B \to f_*\mathcal{O}_X$ is an isomorphism, and the same remains true after arbitrary base change $B' \to B$. This implies that for any scheme T and object $s \in \mathcal{G}(T)$ we have an isomorphism $\mu_n \simeq \underline{\mathrm{Aut}}(s)$ and these isomorphisms are functorial in the pair (T, s). If $\mathcal{L}$ is a line bundle on $\mathcal{G}$ then for any such pair (T, s) we obtain a line bundle $s^*\mathcal{L}$ on T which also comes equipped with an action of $\mu_n = \underline{\mathrm{Aut}}(s)$. The line bundle $\mathcal{L}$ is called *a $\mathcal{G}$–twisted invertible sheaf on X* if this action of μ_n coincides with the standard action induced by the embedding $\mu_n \hookrightarrow \mathbb{G}_m$.

Note that if χ denotes the standard character $\mu_n \hookrightarrow \mathbb{G}_m$ then for any line bundle $\mathcal{L}$ on $\mathcal{G}$ the action of $\mu_n = \underline{\mathrm{Aut}}(s)$ on $s^*\mathcal{L}$ is given by χ^i for some integer i (locally constant on T). It follows that to check that a line bundle $\mathcal{L}$ is $\mathcal{G}$–twisted it suffices to verify that the two actions coincide for pairs (T, s) with T the spectrum of an algebraically closed field.

If $\mathcal{L}$ is a $\mathcal{G}$–twisted sheaf on X, then $\mathcal{L}^{\otimes n}$ descends canonically to an invertible sheaf on X since the stabilizer actions of μ_n are trivial. We usually write just $\mathcal{L}^{\otimes n}$ for the sheaf on X obtained from $\mathcal{L}^{\otimes n}$.

B.18 **Proposition C.25.** *There is a natural equivalence between* $\underline{\mathrm{Sec}}(\mathcal{G}/X)$ *and the stack which to any B–scheme T associates the groupoid of pairs* $(\mathcal{L}, \iota)$*, where $\mathcal{L}$ is a $\mathcal{G}$–twisted sheaf on X and* $\iota : \mathcal{L}^{\otimes n} \simeq \mathcal{O}_X$ *is an isomorphism of invertible sheaves on X.*

Proof. If $s : X \to \mathcal{G}$ is a section, then there is a canonical action of $\mu_n = \underline{\mathrm{Aut}}(s)$ on $\mathcal{F} := s_*\mathcal{O}_X$ and hence $\mathcal{F}$ decomposes canonically as $\oplus_{\chi \in \mathbb{Z}/(n)} \mathcal{F}_\chi$.

Lemma C.26. *Each $\mathcal{F}_\chi$ is locally free of rank 1 on $\mathcal{G}$ and for any two $\chi, \epsilon \in \mathbb{Z}/(n)$ the natural map $\mathcal{F}_\chi \otimes \mathcal{F}_\epsilon \to \mathcal{F}_{\chi+\epsilon}$ is an isomorphism. If $\chi_1 : \mu_n \to \mathbb{G}_m$ denotes the standard inclusion, then $\mathcal{F}_{\chi_1}$ is a $\mathcal{G}$–twisted sheaf on X.*

Proof. The choice of the section s identifies $\mathcal{G}$ with $B\underline{\mathrm{Aut}}(s) \simeq \mathcal{B}\mu_n \times X$. The fiber product $X \times_{s, \mathcal{B}\mu_n, s} X$ is equal to $X \times \mathcal{B}\mu_n$ from which we see that $s^*\mathcal{F}$ is equal to $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathbb{Z}[X]/(X^n - 1)$ with the natural action of μ_n. From this the lemma follows. ♠

In particular, by sending a section $s : X \to \mathcal{G}$ to $\mathcal{F}_{\chi_1}$ with the isomorphism $\mathcal{F}_{\chi_1}^{\otimes n} \simeq \mathcal{O}_{\mathcal{G}}$ we obtain a functor

sectwist (C.26.1) $$\underline{\mathrm{Sec}}(\mathcal{G}/X) \to (\text{stack of pairs } (\mathcal{L}, \iota) \text{ as in C.25}).$$

Conversely, given a $\mathcal{G}$–twisted sheaf $\mathcal{L}$ with an isomorphism $\iota : \mathcal{L}^{\otimes n} \simeq \mathcal{O}_X$ we can form the cyclic algebra $\mathcal{A} = \oplus_{i=0}^{n-1} \mathcal{L}^{\otimes i}$ with multiplication induced by the natural maps $\mathcal{L}^{\otimes i} \otimes \mathcal{L}^{\otimes j} \to \mathcal{L}^{\otimes i+j}$ and the map ι. We can then consider

$$\mathrm{Spec}_{\mathcal{G}}(\mathcal{A}) \to \mathcal{G}.$$

If $\mathcal{G} = \mathcal{B}\mu_n \times X$ and $X \to \mathcal{G}$ is the map defined by the trivial torsor, then the restriction of $(\mathcal{L}, \iota)$ to X is an invertible sheaf L on X with an isomorphism $L^{\otimes n} \simeq \mathcal{O}_X$ and the restriction of $\mathcal{A}$ to X is just the cyclic algebra $\oplus_{i=0}^{n-1} L^{\otimes i}$ with μ_n acting $L^{\otimes i}$ through the character $u \mapsto u^i$. It follows that the projection $\mathrm{Spec}_{\mathcal{G}}(\mathcal{A}) \to X$ is an isomorphism and hence defines a point of $\underline{\mathrm{Sec}}(\mathcal{G}/X)$. We leave to the reader that this defines an inverse to C.26.1. ♠

Let $\mathcal{P}ic^{\mathcal{G}}_{X/B}$ denote the stack over B which to any B–scheme T associates the groupoid of $\mathcal{G}_T$–twisted invertible sheaves on X_T (where $\mathcal{G}_T$ denotes $\mathcal{G} \times_B T$ etc.).

Proposition C.27. (i) *The stack $\mathcal{P}ic^{\mathcal{G}}_{X/B}$ is an algebraic stack locally of finite presentation over B.*

(ii) *The sheaf (with respect to the fppf topology) associated to the presheaf*

$$T \mapsto \{\textit{isomorphism classes in } \mathcal{P}ic^{\mathcal{G}}_{X/B}\}$$

is representable by a separated algebraic space locally of finite presentation over B.

Proof. This follows for example by the same argument used in [5, Appendix]. ♠

C.28. There is a natural morphism

Picmap (C.28.1)
$$\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B} \to \underline{\mathrm{Pic}}_{X/B}, \quad \mathcal{L} \mapsto \mathcal{L}^{\otimes n}.$$

Denote by $\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B}[n]$ the inverse image of the identity $e = [\mathcal{O}_X] \in \underline{\mathrm{Pic}}_{X/B}$. Another description of this space is as follows. The map C.28.1 lifts naturally to a morphism
$$\pi^* : \mathcal{P}ic^{\mathcal{G}}_{X/B} \to \mathcal{P}ic_{X/B}, \quad \mathcal{L} \mapsto \mathcal{L}^{\otimes n}.$$
Let $e : B \to \mathcal{P}ic_{X/B}$ be the morphism corresponding to $\mathcal{O}_X$, and set
$$\mathcal{P}ic^{\mathcal{G}}_{X/B}[n] := \mathcal{P}ic^{\mathcal{G}}_{X/B} \times_{\mathcal{P}ic_{X/B}, e} B.$$
The stack $\mathcal{P}ic^{\mathcal{G}}_{X/B}[n]$ classifies pairs $(\mathcal{L}, \iota)$, where $\mathcal{L}$ is a $\mathcal{G}$–twisted sheaf on X and $\iota : \mathcal{L}^{\otimes n} \simeq \mathcal{O}_X$ is an isomorphism. The space $\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B}[n]$ is the coarse moduli space of $\mathcal{P}ic^{\mathcal{G}}_{X/B}[n]$ via the map sending $(\mathcal{L}, \iota)$ to the class of $\mathcal{L}$. In fact, $\mathcal{P}ic^{\mathcal{G}}_{X/B}[n]$ is a μ_n–gerbe over $\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/S}[n]$. Indeed any point P of $\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B}[n](B)$ can fppf–locally on B be represented by a $\mathcal{G}$–twisted sheaf $\mathcal{L}$ with $\mathcal{L}^{\otimes n} \simeq \mathcal{O}_X$. Conversely for any two pairs $(\mathcal{L}, \iota)$ and $(\mathcal{L}', \iota')$ defining the same point of $\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B}[n]$, the two $\mathcal{G}$–twisted sheaves $\mathcal{L}$ and $\mathcal{L}'$ become isomorphic after making an fppf base change on B. Then ι and ι' differ by a section of $f_*\mathcal{O}^*_{\mathcal{G}} \simeq \mathbb{G}_m$. After making another fppf base change on B so that this unit becomes an n–th power, the two pairs $(\mathcal{L}, \iota)$ and $(\mathcal{L}', \iota')$ become isomorphic. Moreover, this isomorphism is unique up to multiplication by an element of $\mu_n(B)$.

Proposition C.29. *The algebraic space $\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B}[n]$ is of finite type over B.*

Proof. To prove this statement it suffices by a standard limit argument to consider the case when B is a noetherian scheme. In this case by noetherian induction it suffices to find a dominant morphism $B' \to B$ such that the restriction of $\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B}$ to B' is of finite type. We may therefore also assume that B is a noetherian affine integral scheme. By Chow's lemma there exists a proper surjection $P \to \mathcal{G}$ with P an algebraic space. After shrinking on B we may assume that P is also flat over B. Then $\mathcal{G} \times_X P$ is trivial and by C.15 the pullback map
$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \simeq \mathcal{P}ic^{\mathcal{G}}_{X/B}[n] \to \mathcal{P}ic^{\mathcal{G}}_{\mathcal{G}\times_X P/P}[n] \simeq \underline{\mathrm{Sec}}(\mathcal{G} \times_X P/P)$$
is of finite type. Consequently the map
$$\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B}[n] \to \underline{\mathrm{Pic}}^{\mathcal{G}}_{P/B}[n]$$
is also of finite type. It therefore suffices to show $\underline{\mathrm{Pic}}^{\mathcal{G}}_{P/B}[n]$ is of finite type.

Now in the case when $\mathcal{G} = \mathcal{B}\mu_n \times X$ there exists a globally defined $\mathcal{G}$-twisted sheaf $\mathcal{L}$ with $\mathcal{L}^{\otimes n}$ trivial. Choose one such sheaf $\mathcal{L}$. If $p : \mathcal{G} \to X$

denotes the projection, then for any invertible sheaf $\mathcal{M}$ on X with $\mathcal{M}^{\otimes n} \simeq \mathcal{O}_X$, the sheaf $\mathcal{L} \otimes p^*\mathcal{M}$ is also a $\mathcal{G}$–twisted sheaf. Conversely, if $\mathcal{N}$ is a $\mathcal{G}$–twisted sheaf with $\mathcal{N}^{\otimes n}$ trivial, then $\mathcal{L} \otimes \mathcal{N}^{-1}$ descends uniquely to an n–torsion sheaf on X. Using this one sees that $\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B}[n]$ is a trivial torsor under $\underline{\mathrm{Pic}}_{X/B}[n]$. The proposition therefore follows from [22, 6.27 and 6.28]. ♠

proofend **C.30.** Since $\underline{\mathrm{Sec}}(\mathcal{G}/X) \simeq \mathcal{P}ic^{\mathcal{G}}_{X/B}[n]$ is a μ_n–gerbe over $\underline{\mathrm{Pic}}^{\mathcal{G}}_{X/B}[n]$ this completes the proof of C.3 in the special case of C.24 and hence also the proof in general. ♠

Proof of C.4

[21]

21→

C.30 **C.31.** *Algebraicity of* $\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$. By the case of algebraic spaces, the stack $\underline{\mathrm{Hom}}(X, Y)$ is an algebraic space locally of finite presentation over B. To prove C.4 it therefore suffices, as in the proof of C.3, to show that for any morphism $f : X \to Y$ the stack $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is algebraic locally of finite presentation over B, where $\mathcal{G}$ denotes the pullback of $\mathcal{Y}$ along the morphism

$$\mathcal{X} \longrightarrow X \xrightarrow{f} Y.$$

This is shown exactly as in [30, 5.3-5.8] by verifying Artin's conditions. The key point is Grothendieck's existence theorem for Artin stacks which is shown in [30, A.1].

Note that this implies that if $\mathcal{X}/B$ is a proper flat tame stack and $\mathcal{G} \to \mathcal{X}$ is a morphism with $\mathcal{G}$ a tame stack of finite presentation over B, then $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is an algebraic stack locally of finite presentation over B, as it is equal to the fiber product of the diagram

$$\begin{tikzcd} & \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{G}) \arrow[d] \\ S \arrow[r, "\mathrm{id}_{\mathcal{X}}"] & \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{X}). \end{tikzcd}$$

C.32. *The diagonal of* $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ *is quasi-compact and separated.*

L:C.33 **Lemma C.33.** *If $\mathcal{X}/B$ is a proper and flat tame stack, and $I \to \mathcal{X}$ is a finite morphism, then $\underline{\mathrm{Sec}}(I/\mathcal{X})$ is separated and of finite type over B.*

Proof. For the quasi-compactness of the diagonal, it suffices to show that given two sections $s_1, s_2 : \mathcal{X} \to I$, the fiber product M of the diagram

$$\begin{tikzcd} & S \arrow[d, "s_1 \times s_2"] \\ \underline{\mathrm{Sec}}(I/\mathcal{X}) \arrow[r, "\Delta"] & \underline{\mathrm{Sec}}(I/\mathcal{X}) \times_S \underline{\mathrm{Sec}}(I/\mathcal{X}) \end{tikzcd}$$

[21] (Martin) I reorganized the proof of C.4

is quasi-compact. For this we may assume that S is reduced, and furthermore by noetherian induction it suffices to show that the map $M \to S$ is quasi-compact over some dense open in S. Let $\mathcal{X}_0$ denote the fiber product of the diagram

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \Big\downarrow{\scriptstyle s_1 \times s_2} \\ I & \xrightarrow{\Delta} & I \times_{\mathcal{X}} I. \end{array}$$

Since I is finite over $\mathcal{X}$, the diagonal of $I/\mathcal{X}$ is a closed immersion which implies that the projection $j : \mathcal{X}_0 \to \mathcal{X}$ is also a closed immersion. The S-space M represents the functor which to any scheme T/S associates the unital set if the base change

$$\mathcal{X}_{0,T} \to \mathcal{X}_T$$

is an isomorphism, and the empty set otherwise. Now to prove that M is of finite type over S, we may by shrinking if necessary also assume that $\mathcal{X}_0$ is flat over S. In this case, if $\mathcal{Z} \subset \mathcal{X}$ denotes the support of the ideal sheaf of $\mathcal{X}_0$, then M is represented by the complement of the closed (since $\mathcal{X}/S$ is proper) image of $\mathcal{Z}$ in S. This proves the quasi-compactness of the diagonal of $\underline{\mathrm{Sec}}(I/\mathcal{X})$.

To verify that the diagonal of $\underline{\mathrm{Sec}}(I/\mathcal{X})$ is proper, one verifies the valuative criterion using the same argument as in the proof of C.13.

The proof that $\underline{\mathrm{Sec}}(I/\mathcal{X})$ is of finite type over B now proceeds exactly as in the proof of C.13. ♠

Now let $\mathcal{X}$ and $\mathcal{Y}$ be finite type tame B-stacks with $\mathcal{X}$ proper and flat over B, and let X and Y be the coarse moduli space of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Fix a morphism $f : X \to Y$, and let $\mathcal{G} \to \mathcal{X}$ be the pullback of $\mathcal{Y}$ along the composite

$$\mathcal{X} \longrightarrow X \xrightarrow{f} Y.$$

Lemma C.34. *The diagonal of* $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ *is quasi-compact and separated.*

Proof. Let $s_1, s_2 : \mathcal{X} \to \mathcal{G}$ be two sections, and let I denote the fiber product of the diagram (which is a finite stack over $\mathcal{X}$ since $\mathcal{G}$ has finite diagonal)

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \Big\downarrow{\scriptstyle s_1 \times s_2} \\ \mathcal{G} & \xrightarrow{\Delta} & \mathcal{G} \times_{\mathcal{X}} \mathcal{G}. \end{array}$$

Then the fiber product of the diagram

$$\begin{array}{ccc} & & B \\ & & \Big\downarrow{\scriptstyle s_1 \times s_2} \\ \underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X}) & \xrightarrow{\Delta} & \underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X}) \times_B \underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X}) \end{array}$$

is isomorphic to $\underline{\mathrm{Sec}}(I/\mathcal{X})$ which is quasi-compact and separated by C.33. ♠

C.35. *Proposition C.15 still holds.* Now an examination of the proof of C.15, following [35, 4.13], shows that once we know that in the case when $\mathcal{X}$ is tame the stack $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is locally of finite type with separated and quasi-compact diagonal, then the proof of C.15 carries over also in the case when $\mathcal{X}$ does not necessarily admit a finite flat surjection from an algebraic space but instead is tame.

B.45 **C.36.** *Completion of proof.* To prove that $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is quasi-compact, it suffices by noetherian induction to exhibit a dense open subset of B where this is so. Furthermore, we may without loss of generality assume that B is integral. By Chow's lemma for Artin stacks [33, 1.1] there exists a proper surjection $X' \to \mathcal{X}$ with X' an algebraic space. After shrinking on B we may assume that X' is flat over B. Applying C.15 to the stack $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ and using the case when $\mathcal{X}$ is an algebraic space we get that $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is quasi-compact. This completes the proof of C.4.

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