RELATIVE AND ORBIFOLD GROMOV–WITTEN INVARIANTS

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Contents

1. Introduction

Gromov–Witten invariants are deformation invariant numbers associated to a smooth variety over $\mathbb{C}$ that are closely related to the numbers of curves in that variety with prescribed incidence to specified homology classes. They are defined by intersecting the homology classes in question with the virtual fundamental class on the moduli space of stable maps. There are thus two essential ingredients in Gromov–Witten theory: a proper moduli space on which to do intersection theory, and a virtual fundamental class of the expected dimension in the homology of that moduli space.

In this paper, we will be interested in counting rational curves in a smooth variety with prescribed incidence conditions, as well as prescribed tangencies along a divisor. The introduction of tangencies makes the definition of Gromov–Witten invariants more subtle. Since

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a tangency can degenerate to one of higher order, it is not obvious how to produce a proper moduli space, and since a tangency can be deformed to lower order, it is not obvious how to do the deformation theory necessary to produce a virtual fundamental class.

There are now several solutions to this problem. The first is the theory of *relative stable maps*, introduced by A. M. Li and Y. Ruan in [LR01], but also studied by Ionel–Parker [IP03, IP04], Gathmann [Gat02] and several others. In algebraic geometry it is due to J. Li ([Li01] and [Li02]). In this theory, tangencies are prevented from degenerating to higher order by allowing the target variety to expand, in close analogy to the way a Deligne–Mumford stable curve might expand to prevent a marked point from colliding with a node. The deformation theory of these curves still remains quite subtle, however.

A second solution [Cad07b] is to change the target variety by a *root construction*. The variety is replaced by a stack that is isomorphic to the original variety away from the divisor, but in which the divisor is replaced by a "stacky" version of itself with a cyclotomic stabilizer group. Provided that the stackiness of the divisor is taken to be large enough, the concept of tangency to the divisor in the original variety can be replaced with transversal contact to the stacky divisor in the root stack. In other words, the ordinary theory of twisted stable maps [AV02] applies to yield a proper moduli space and a virtual fundamental class via straightforward deformation theory. The disadvantage of this theory, as compared to relative stable maps, is that it may include extraneous information (in higher genus; see Section 1.3) and cannot be used in the degeneration formula [Li02], [AF].

Cadman and Chen applied Cadman's method to enumerate rational curves tangent to a smooth plane cubic [CC08]. Some of these numbers were also computed by Gathmann using relative Gromov–Witten invariants [Gat05]. It is no surprise that these numbers agree when they are both enumerative: they count the same thing. Remarkably, however, one is enumerative if and only if the other is, and the invariants coincide even if they are not enumerative.

Our goal in this paper is to explain this coincidence by comparing the approaches of J. Li and Cadman in genus 0. Our comparison goes by way of a third theory that combines the advantages of both while avoiding the disadvantages [AF]. This "relative–orbifold" theory furnishes a correspondence between the relative and orbifold moduli spaces.

$$\begin{tikzcd} & \overline{M}^{\mathrm{relorb}}(\sqrt[r]{X,D}) \arrow[dl, "\Psi"'] \arrow[dr, "\Phi"] & \\ \overline{M}^{\mathrm{rel}}(X,D) & & \overline{M}^{\mathrm{orb}}(\sqrt[r]{X,D}) \end{tikzcd}$$

(The notation here is temporary and will be superseded in the body of the text.) It is shown in [AF] that Ψ is an isomorphism and identifies the virtual fundamental classes, so our task is primarily to study the map Φ. This map is not an isomorphism, even in genus zero and for large r, but we will show that it nevertheless carries one virtual fundamental class to the other via push-forward, and therefore identifies the Gromov–Witten invariants.

There is a fourth theory of stable maps relative to a divisor called *logarithmic stable maps*, currently in development by several groups. We will not discuss logarithmic stable maps here, but see [Kim08] for some of the beginnings of this theory.

1.1. **Statement of the theorem.** Let X be a smooth projective variety, $D \subset X$ a smooth divisor, and fix a curve class $\beta \in H_2(X,\mathbb{Z})$. Consider a vector of nonnegative integers $\mathbf{k} = (k_1, \dots k_n)$, with $\sum k_i = \beta \cdot D$, cohomology classes $\gamma_1 \dots, \gamma_n$ where $\gamma_i \in H^*(X,\mathbb{Q})$ when $k_i = 0$ and $\gamma_i \in H^*(D,\mathbb{Q})$ when $k_i > 0$, and nonnegative integers $a_1, \dots, a_n$. For r a positive integer, denote by $\mathscr{X}_r = X_{D,r}$ the stack obtained by taking the r-th root of X along D.

Theorem 1.1. *Fix $\beta \in H_2(X,\mathbb{Z})$. If r is any sufficiently large and divisible natural number then the following relative and orbifold invariants coincide.*

$$\left\langle \prod_{i=1}^{n} \tau_{a_i}(\gamma_i, k_i) \right\rangle_{0,\beta}^{(X,D)} = \left\langle \prod_{i=1}^{n} \tau_{a_i}(\gamma_i, k_i) \right\rangle_{0,\beta}^{\mathscr{X}_r}$$

Our notation is explained in the following section. There is also a table of notation in Appendix C.

1.2. **Conventions.**

(1) Consider

$$\overline{M}^{\mathrm{rel}}(X,D) := \overline{M}^{\mathrm{rel}}_{g,(k_1,\dots,k_n)}(X,D,\beta)$$

the moduli space of relative stable maps to (X,D), where
- the source curve has genus g and n marked points,
- the i-th marked point has contact order k_i with D, and

- the homology class of the curve is β.

Let e_i^{rel} be the i-th evaluation map, where

$$\begin{aligned} e_i^{\mathrm{rel}} : \overline{M}^{\mathrm{rel}}(X,D) &\to X && \text{for } k_i = 0, \text{ and} \\ e_i^{\mathrm{rel}} : \overline{M}^{\mathrm{rel}}(X,D) &\to D && \text{for } k_i > 0. \end{aligned}$$

Let $s_i : \overline{M}^{\mathrm{rel}}(X,D) \to \overline{C}$ be the i-th section of the universal *contracted curve mapping to* X, and let $\psi_i = c_1 s_i^*(\omega_{\overline{C}/\overline{M}^{\mathrm{rel}}(X,D)})$. The stack $\overline{M}^{\mathrm{rel}}(X,D)$ admits a virtual fundamental class $[\overline{M}^{\mathrm{rel}}(X,D)]^{\mathrm{vir}}$ defined in [Li02].

With this notation we set

$$\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i,k_i) \right\rangle_{g,\beta}^{(X,D)} := \int_{[\overline{M}^{\mathrm{rel}}(X,D)]^{\mathrm{vir}}} \psi_1^{a_1} e_1^*\gamma_1 \cdots \psi_n^{a_n} e_n^*\gamma_n.$$

(2) Consider

$$\overline{M}^{\mathrm{orb}}(\mathscr{X}_r) := \overline{M}_{g,(k_1,\dots,k_n)}(\mathscr{X}_r,\beta)$$

the moduli space of stable maps to $\mathscr{X}_r$, where

- the curve has genus g and n marked points,
- the *coarse* evaluation map at the i-th marked point (defined below)

$$e_i^{\mathrm{orb}} : \overline{M}^{\mathrm{orb}}(\mathscr{X}_r) \to \underline{I}(\mathscr{X}_r)$$

  lands in the twisted sector of age k_i/r (which is isomorphic to X if $k_i = 0$ and to D if $k_i > 0$), and
- the homology class of the curve is β.

We have used the notation $\underline{I}(\mathscr{X}_r)$ for the coarse moduli space of the inertia stack of $\mathscr{X}_r$, which has r components:

$$\underline{I}(\mathscr{X}_r) \cong X \sqcup D \sqcup \cdots \sqcup D.$$

The components isomorphic to D are called twisted sectors, and are labeled by the ages $k_i/r \in (0,1) \cap (1/r)\mathbf{Z}$.

Let $s_i : \overline{M}^{\mathrm{orb}}(\mathscr{X}) \to \overline{C}$ be the i-th section of the *universal coarse curve mapping to* X, and let $\psi_i = c_1 s_i^*(\omega_{\overline{C}/\overline{M}^{\mathrm{orb}}(\mathscr{X}_r)})$. The stack $\overline{M}^{\mathrm{orb}}(\mathscr{X}_r)$ admits a virtual fundamental class $[\overline{M}^{\mathrm{orb}}(\mathscr{X}_r)]^{\mathrm{vir}}$ defined in [AGV08].

With this notation we set

$$\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i,k_i) \right\rangle_{g,\beta}^{\mathscr{X}_r} := \int_{[\overline{M}^{\mathrm{orb}}(\mathscr{X}_r)]^{\mathrm{vir}}} \psi_1^{a_1} e_1^*\gamma_1 \cdots \psi_n^{a_n} e_n^*\gamma_n.$$

1.3. **Counterexample in genus 1.** Note that Theorem 1.1 applies only to genus zero invariants. The necessity of this restriction may be seen in the following example, which was shown to us by Davesh Maulik [Mau]. Let E be an elliptic curve and let $X = E \times \mathbf{P}^1$. Let $D = X_0 \cup X_\infty$, the union of the fibers of X over 0 and $\infty \in \mathbf{P}^1$. Let $f \in H_2(X)$ be the class of a fiber of $X \to \mathbf{P}^1$. Then the relative invariant with no insertions vanishes: $\langle\rangle_{1,f}^{(X,D)} = 0$. A simple explanation for this is that the invariant remains the same when taking covering of $\mathbf{P}^1$ branched at $0, \infty$, and at the same time it is multiplied by the degree of the cover. Note that the space of genus 1 relative maps to (X, D) of class f has expected dimension 0, even though the actual dimension is 1.

Let $\mathscr{X}_{r,s}$ be the stack obtained from X via an r-th root construction on X_0 and an s-th root construction on X_∞. The space $\overline{M}_{1,0}(\mathscr{X}_{r,s}, f)$ has a 1-dimensional component and $r^2 - 1 + s^2 - 1$ components of dimension 0. The 1-dimensional component is isomorphic to the stack $\mathcal{P}_{r,s}$, obtained from $\mathbf{P}^1$ by an r-th root at 0 and an s-th root at infinity. The remaining components exist because a morphism $E \to E \times B\mu_r$ which is the identity onto the first factor is determined by a μ_r-torsor over E. There are r^2 choices for the μ_r-torsor, but the trivial torsor already appears in the 1-dimensional component.

The obstruction bundle on the 1-dimensional component is the tangent bundle, which has degree $1/r + 1/s$. The 0-dimensional components count with precisely their degree, which takes into account the automorphism group of the torsor. We may therefore calculate

$$\langle\rangle_{1,f}^{\mathscr{X}_{s,r}} = \frac{1}{r} + \frac{1}{s} + \frac{r^2-1}{r} + \frac{s^2-1}{s} = r + s.$$

We interpret this discrepancy between the relative and twisted Gromov–Witten invariants to be a result of the nontriviality of the Picard group of E. A more precise statement is postponed to a later investigation.

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2. Method of proof

2.1. **An intermediary moduli space.** In order to prove the theorem, we want to relate the moduli spaces. It is natural to relate them through a third moduli space where both relative geometry and orbifold geometry are present:

Consider

$$\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r) := \overline{M}^{\mathrm{rel}}_{g,(k_1,\dots,k_n)}(\mathscr{X}_r, \mathscr{D}_r, \beta)$$

the moduli space of relative stable maps to $(\mathscr{X}_r, \mathscr{D}_r)$, with

- genus g and n marked points,
- where the i-th marked point of the *coarse curve* has contact order k_i with D, and
- curve class β.

We denote by e_i the i-th evaluation map, where for $k_i = 0$ we have $e_i : \overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r) \to \mathscr{X}_r$ and for $k_i > 0$ we have $e_i : \overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r) \to \mathscr{D}_r$.

Let $s_i : \overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r) \to \overline{C}$ be the i-th section of the universal *coarse contracted curve mapping to* X, and let $\psi_i = c_1 s_i^*(\omega_{\overline{C}/\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r,\mathscr{D}_r)})$.

The space $\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r)$ admits a virtual fundamental class $[\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r)]^{\mathrm{vir}}$ defined in [AF].

With this notation we set

$$\left\langle \prod_{i=1}^{n} \tau_{a_i}(\gamma_i, k_i) \right\rangle_{0,\beta}^{(\mathscr{X}_r,\mathscr{D}_r)} \quad := \quad \int_{[\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r,\mathscr{D}_r)]^{\mathrm{vir}}} \psi_1^{a_1} e_1^*\gamma_1 \cdots \psi_n^{a_n} e_n^*\gamma_n.$$

2.2. **Reduction of main theorem to properties of virtual fundamental classes.** For any g, r we have a diagram of stabilization morphisms

$$\begin{tikzcd} & \overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r) \arrow[dl, "\Psi"'] \arrow[dr, "\Phi"] & \\ \overline{M}^{\mathrm{rel}}(X, D) & & \overline{M}^{\mathrm{orb}}(\mathscr{X}_r). \end{tikzcd}$$

We have defined the terms so that

- $e_i^{\mathrm{rel}} \circ \Psi = e_i^{\mathrm{relorb}} = e_i^{\mathrm{rel}} \circ \Phi$,
- $\Psi^*\overline{C}^{\mathrm{rel}} = \overline{C}^{\mathrm{relorb}} = \Phi^*\overline{C}^{\mathrm{orb}}$, therefore
- $\Psi^*\omega_{\overline{C}^{\mathrm{rel}}/\overline{M}^{\mathrm{rel}}} = \omega_{\overline{C}^{\mathrm{relorb}}/\overline{M}^{\mathrm{relorb}}} = \Phi^*\omega_{\overline{C}^{\mathrm{orb}}/\overline{M}^{\mathrm{orb}}}$, and finally
- $\Psi^* s_i^{\mathrm{rel}} = s_i^{\mathrm{relorb}} = \Phi^* s_i^{\mathrm{orb}}$.

Consequently, the projection formula gives us

$$\left\langle \prod_{i=1}^{n} \tau_{a_i}(\gamma_i, k_i) \right\rangle_{0,\beta}^{(\mathscr{X}_r, \mathscr{D}_r)} = \int_{\Psi_*([\overline{M}^{\mathrm{relorb}}]^{\mathrm{vir}})} \psi_1^{a_1} e_1^* \gamma_1 \cdots \psi_n^{a_n} e_n^* \gamma_n$$
$$= \int_{\Phi_*([\overline{M}^{\mathrm{relorb}}]^{\mathrm{vir}})} \psi_1^{a_1} e_1^* \gamma_1 \cdots \psi_n^{a_n} e_n^* \gamma_n$$

where the integrals are on $\overline{M}^{\mathrm{rel}}(X, D)$ and $\overline{M}^{\mathrm{orb}}(\mathscr{X}_r)$, respectively. The theorem is thus a consequence of the following two theorems.

Theorem 2.1. (1) *For any g, r and any twisting choice $\mathfrak{r}$ we have*

$$\Psi_*([\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r)]^{\mathrm{vir}}) = [\overline{M}^{\mathrm{rel},\mathfrak{r}}(X, D)]^{\mathrm{vir}}$$

where the obstruction theories are those defined in [AF].

(2) *The Gromov–Witten invariants defined using $[\overline{M}^{\mathrm{rel},\mathfrak{r}}(X, D)]^{\mathrm{vir}}$ coincide with those defined in 1.2(1) using $[\overline{M}^{\mathrm{rel}}(X, D)]^{\mathrm{vir}}$.*

Theorem 2.2. *If $g = 0$, then for any r sufficiently large and divisible depending on β we have*

$$\Phi_*([\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r)]^{\mathrm{vir}}) = [\overline{M}^{\mathrm{orb}}(\mathscr{X}_r)]^{\mathrm{vir}}$$

Proof of Theorem 2.1. Part (1) follows from [AF, Theorem 4.4.1]. Indeed given a twisting choice $\mathfrak{r}$, the moduli spaces

- $\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r)^{\mathfrak{r}}$ of relative twisted stable maps with twisting choice $\mathfrak{r}$ and
- $\overline{M}^{\mathrm{rel}}(X, D)^{r\cdot\mathfrak{r}}$ of relative twisted stable maps with twisting choice $r \cdot \mathfrak{r}$

are identical with identical obstruction theories (as defined in [AF, Section 4.2, Lemma C.3.3]). The equality of invariants in Part (2) is proved in [AF, Section 4.7]. □

Remark 2.2.3. We emphasize that we do not compare our virtual classes to those defined in [Li02]. See discussion in [AF, Section 4.7]. The third author has an argument for comparing these classes which goes beyond the scope of this paper.

We will prove Theorem 2.2 in Section 4 using a technique introduced by Costello.

2.3. **Costello's diagram.** We restrict to genus 0 maps and construct a cartesian square

$$
\begin{tikzcd}
\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r) \arrow[r, "\Phi_X"] \arrow[d, "\sigma^{\mathrm{rel}}"'] & \overline{M}^{\mathrm{orb}}(\mathscr{X}_r) \arrow[d, "\sigma"] \\
\mathfrak{M}^{\mathrm{rel}}(\mathscr{A}, \mathcal{B}\mathbb{G}_m)' \arrow[r, "\Phi_{\mathscr{A}}"] & \mathfrak{M}(\mathscr{A})'.
\end{tikzcd}
\tag{2.3.1}
$$

Here Φ_X is the morphism denoted Φ above. We use $\mathscr{A}$ to stand for the stack $\mathscr{A} = [\mathbb{A}^1/\mathbb{G}_m]$, which includes $\mathcal{B}\mathbb{G}_m = [\{0\}/\mathbb{G}_m]$ as a closed substack (of Artin type). We show below that the diagram has the following properties:

(1) the virtual fundamental classes can be defined via perfect relative obstruction theories relative to the vertical arrows, and
(2) the hypotheses of [Cos06, Theorem 5.0.1] are satisfied for these choices.

The stacks in the bottom row of this diagram require definition, which will be given explicitly in Section 3. For the moment, we note the following.

- We define $\mathfrak{M}(\mathscr{A})$ to be the stack of triples (C, L, s), where C is a twisted curve, L a line bundle on C and s a section of L, such that the associated morphism $C \to \mathscr{A}$ is representable. The stack $\mathfrak{M}(\mathscr{A})'$ is defined in Section 3.3.2 as an open substack of $\mathfrak{M}(\mathscr{A})$.
- Let $\mathscr{T}$ be Jun Li's moduli space of expanded pairs (see Section 3.3.1 and [ACFW]). We define $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$ to be the stack of maps over $\mathscr{T}$ from twisted curves to expansions of $\mathscr{A}$ relative to the divisor $\mathcal{B}\mathbb{G}_m$. The stack $\mathfrak{M}^{\mathrm{rel}}(\mathscr{A}, \mathcal{B}\mathbb{G}_m)'$ is defined in Section 3.3.3, and it is shown there to be étale over $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$.

3. Construction of the main diagram

In this section we will construct the stacks appearing in Diagram (2.3.1). Our definitions require the notion of stable maps into the fibers of a morphism, which we summarize first. In Section 3.3.1 we describe our perspective on J. Li's moduli space of expansions of a scheme with a divisor; this section summarizes material from [ACFW]. Finally in Section 3.3.2 and 3.3.3 we construct the stacks in the bottom row of Diagram (2.3.1) and in Section 3.3.4 we show that they are algebraic.

3.1. **Stable maps into the fibers of a morphism.** We refer the reader to Appendix A for some of the details that are omitted from this section.

3.1.1. *Definitions.* Suppose that $X \to T$ is a morphism of stacks. We define $\mathfrak{M}(X/T)$ to be the stack of commutative diagrams

(3.1.1)
$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

where $C \to S$ is a family of twisted pre-stable curves with an ordered set of marked points (not illustrated in the diagram), and $f : C \to X \times_T S$ is representable. We will use the notation $\mathfrak{M}_\Gamma(X/T)$ to refer to an open and closed substack of $\mathfrak{M}(X/T)$ consisting of those diagrams as above having some fixed data; for example, $\mathfrak{M}_{g,n}(X/T)$ will denote the substack parameterizing those diagrams where C has genus g and n marked (possibly twisted) points.

Let $\overline{M}(X/T)$ be the substack of points of $\mathfrak{M}(X/T)$ where the (absolute) inertia group is unramified. Under mild conditions, $\mathfrak{M}(X/T)$ is an algebraic stack and $\overline{M}(X/T)$ is its maximal Deligne–Mumford substack, see Proposition 3.3.2 for the cases relevant in this paper.

Remark 3.1.2. The stack of maps into the fibers of a morphism is frequently defined as the space of maps into the total space whose homology class is that of a fiber. We have not taken this approach for two reasons. The first is that we need a definition in the case where the morphism is of Artin type, so we don't know how to talk about the homology class of a fiber. The second reason is more philosophical: the obstruction theory naturally associated to the space of maps with the homology class of a fiber is different from the obstruction theory on the space of maps into a fiber as defined above ([AF, Appendix C.2].

3.1.2. *The Cartesian diagram.* We will eventually need to study a commutative diagram of the following form, where the upper square is Cartesian and the lower square is not.

(3.1.3)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] \arrow[d] & Y' \arrow[d] \\ T \arrow[r] & T' \end{tikzcd}$$

In the application,

- T' will be a point,
- $Y' = \mathscr{A}$ will be the moduli space of line bundles with section,
- X' will be a smooth Deligne–Mumford stack with $X' \to Y'$ the morphism corresponding to a smooth divisor D' on X',
- T will be J. Li's moduli space of expanded pairs,
- $Y = \mathscr{A}^{\mathrm{exp}}$ will be the universal expansion of $\mathscr{A}$ along the divisor $B\mathbf{G}_m$, and
- $X = X^{\mathrm{exp}}$ will be the universal expansion of X' along the divisor D'.

This induces a Cartesian diagram

$$\begin{tikzcd} \mathfrak{M}(X/T) \arrow[r] \arrow[d] & \mathfrak{M}(X'/T') \arrow[d] \\ \mathfrak{M}(Y/T) \arrow[r] & \mathfrak{M}(Y'/T'), \end{tikzcd}$$

where all the maps are given by composition. However, the corresponding diagram with $\mathfrak{M}$ replaced by $\overline{M}$ in the upper row is not even commutative. This is because a map

$$\overline{M}(X/T) \to \overline{M}(X'/T')$$

may require the contraction of some components that become unstable after composition with the map $X \to X'$ while the map $\mathfrak{M}(Y/T) \to \mathfrak{M}(Y'/T')$ involves no contraction at all.

If we do not modify the construction, this will prevent us from applying Costello's theorem. Our solution is to replace $\mathfrak{M}(Y/T)$ by a stack that keeps track of slightly more information: in addition to a family of curves in the fibers of $Y \to T$, this stack will also parameterize a contraction of this family in the fibers of $Y' \times_{T'} T$.

The technical device used to achieve this is a stack $\mathfrak{M}(X/\overline{X}/T)$, associated to a sequence of morphisms

$$X \to \overline{X} \to T$$

with $X \to \overline{X}$ representable. This is the stack whose S-points are diagrams

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d] \\ \overline{C} \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ S \arrow[r] & T \end{tikzcd} \tag{3.1.4}$$

where the stabilization of $C \to \overline{C}$ is an isomorphism.

Given a morphism $T \to T'$ and $\overline{X} = X'_T$ for some stack X' over T', we define

$$\overline{M}(X/X'_T/T) \subset \mathfrak{M}(X/X'_T/T)$$

to be the substack of points whose inertia *relative to* $\mathfrak{M}(X'/T')$ is unramified. (Note that we are abusing notation slightly, since $\mathfrak{M}(X/X'_T/T)$ does not depend only on X'_T but also on the map $X'_T \to X'$.) Let $\overline{M}(X/T)^*$ denote the subspace of $\overline{M}(X/T)$ consisting of those diagrams (3.1.1) that can be stabilized in $\mathfrak{M}(X'/T')$ after composing with the map $X \to X'$. Then we obtain the Cartesian diagram we need:

Proposition 3.1.5. *Suppose that $X' \to T'$ is a morphism of Deligne–Mumford type. The diagram*

$$\begin{tikzcd}
\overline{M}(X/T)^* \arrow[r] \arrow[d] & \overline{M}(X'/T') \arrow[d] \\
\overline{M}(Y/Y'_T/T) \arrow[r] & \mathfrak{M}(Y'/T').
\end{tikzcd}$$

is Cartesian.

The proof (which is straightforward from the construction) is deferred to Section 3.2.

3.1.3. *Comparison of $\mathfrak{M}(X/\overline{X}/T)$ and $\mathfrak{M}(X/T)$.* We have a morphism

$$\tau : \mathfrak{M}(X/\overline{X}/T) \to \mathfrak{M}(X/T) \tag{3.1.6}$$

which forgets the middle row of (3.1.4). We will need the following lemma.

Lemma 3.1.7. *Suppose that $C \to \overline{C}$ is a morphism of n-marked twisted curves over S, the stabilization of which is an isomorphism. Then the canonical map $\mathcal{O}_{\overline{C}} \to \mathbf{R}p_*\mathcal{O}_C$ is a quasi-isomorphism.*

Proof. We claim that the fibers of $p : C \to \overline{C}$ over S are trees of genus zero curves. (Note that $C \to \overline{C}$ is representable, so the fibers are schemes.) To see this, we can suppose that S is a point. Since $\overline{C}$ is a curve, the fibers of $C \to \overline{C}$ that are not points must each have at least one special point: the point at which the fiber is attached to the rest of $\overline{C}$. Since the fibers that are not points are unstable curves, it follows that the these fibers must be connected curves of arithmetic genus zero.

Since the fibers of p have arithmetic genus zero, $H^1(p^{-1}\overline{c}, \mathcal{O}_C\big|_{p^{-1}\overline{c}})$ vanishes at each geometric point $\overline{c}$ of $\overline{C}$. Therefore, by cohomology and base change, $\mathbf{R}^1 p_*\mathcal{O}_C = 0$. Likewise, the map $\mathcal{O}_{\overline{C}} \to p_*\mathcal{O}_C$ is an isomorphism on the geometric fibers, hence is an isomorphism, again by cohomology and base change. □

Lemma 3.1.8. *The morphism* τ *of* (3.1.6) *is formally étale.*

Proof. We shall show that the natural obstruction theory vanishes. It is easy to produce a direct proof of this fact, but it is somewhat faster to rely on Illusie's relative cohomology.

For τ to be formally étale at the point corresponding to Diagram (3.1.4), it is equivalent to show that, for any square-zero extension $S \to S'$ with ideal J, any commutative diagram of solid arrows

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathfrak{M}(X/\overline{X}/T) \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & \mathfrak{M}(X/T) \end{tikzcd}$$

can be extended, uniquely up to unique isomorphism, to a commutative diagram including the dashed arrow. To show this, it is equivalent to show that the lifting problem

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"'] & C' \arrow[d, dashed] \arrow[dd, bend left] \\ \overline{C} \arrow[r, dashed] \arrow[d] & \overline{C}' \arrow[d, dashed] \\ X_S \arrow[r] & \overline{X}'_S \end{tikzcd}$$

admits a solution that is unique up to unique isomorphism. By [Ill71, Ch. III, p. 199, Prop. 2.3.2], obstructions, deformations, and infinitesimal automorphisms for such a problem are classified by

$$\mathrm{Ext}^i(\overline{C}/C; \mathbb{L}_{\overline{C}/X_S}(\log P), J), \quad i = 0, 1, 2$$

where J is the pull-back of the ideal of S in S' and P is the divisor of marked points. It therefore suffices to show that the group $\mathrm{Ext}^i(\overline{C}/C; \mathbb{L}_{\overline{C}/X_S}(\log P), J)$ vanishes for all i. There is an exact triangle [Ill71, Ch. III, p. 221, Equation (4.10.3)]

$$\begin{aligned} \mathbf{R}\mathrm{Hom}(\overline{C}/C; \mathbb{L}_{\overline{C}/X_S}(\log P), J) &\to \mathbf{R}\mathrm{Hom}(\mathbb{L}_{\overline{C}/X_S}(\log P), J) \\ &\xrightarrow{\varphi} \mathbf{R}\mathrm{Hom}(\mathbb{L}_{\overline{C}/X_S}(\log P), \mathbf{R}\pi_*\mathbf{L}\pi^*J). \end{aligned}$$

But $J \to \mathbf{R}\pi_*\mathbf{L}\pi^*J$ is a quasi-isomorphism by Lemma 3.1.7, so φ is a quasi-isomorphism, and hence $\mathrm{Ext}^i(\overline{C}/C; \mathbb{L}_{\overline{C}/X_S}(\log P), J)$ is zero for all i. □

Remark 3.1.9. In fact τ is étale: in addition to being formally étale, it is representable by algebraic spaces and locally of finite presentation. However, we will not need that fact in such generality in this paper.

3.1.4. *Obstruction theory.* If $X \to Y$ is a smooth morphism over T of Deligne–Mumford type it induces a natural obstruction theory for the morphism $\mathfrak{M}(X/T) \to \mathfrak{M}(Y/T)$. Let

$$\begin{tikzcd} & & X \arrow[d] \\ \mathfrak{C}(X/T) \arrow[r] \arrow[d, "\pi"'] \arrow[urr, bend left, "f"] & \mathfrak{C}(Y/T) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathfrak{M}(X/T) \arrow[r] & \mathfrak{M}(Y/T) \arrow[r] & T \end{tikzcd}$$

be the universal commutative diagram. There is a canonical map in the derived category of quasi-coherent sheaves on $\mathfrak{C}(X/T)$,

$$f^*\mathbb{L}_{X/Y} \to \mathbb{L}_{\mathfrak{C}(X/T)/\mathfrak{C}(Y/T)} \simeq \pi^*\mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}.$$

By adjunction, we obtain a morphism in the dervied category of quasi-coherent sheaves on $\mathfrak{M}(X/T)$,

$$\pi_! f^*\mathbb{L}_{X/Y} \to \mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}.$$

Here $\pi_!$ is the left adjoint to $\mathbf{L}\pi^*$; it exists by Grothendieck duality and the invertibility of ω_π. We denote the complex $\pi_! f^*\mathbb{L}_{X/Y}$ on the left by $\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}$.

Proposition 3.1.10. *The morphism*

$$\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)} \to \mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}$$

is a perfect relative obstruction theory of perfect amplitude in $[-1, 0]$.

Proof. To prove this, we can work locally on the base $\mathfrak{M}(Y/T)$. Since $\mathfrak{M}(X/T)$ is of Deligne–Mumford type over $\mathfrak{M}(Y/T)$, this reduces the problem to the case of a Deligne–Mumford stack over a base scheme. The rest of the proof is then very similar to [BF97, Theorem 4.5 and Proposition 6.3]. We give details in Section A.1. $\square$

Now we place ourselves in the situation of Diagram (3.1.3). The natural map

$$\tau : \mathfrak{M}(Y/Y'_T/T) \to \mathfrak{M}(Y/T)$$

is formally étale, so the obstruction theory for $\mathfrak{M}(X/T)$ relative to $\mathfrak{M}(Y/T)$ is also an obstruction theory relative to $\mathfrak{M}(Y/Y'_T/T)$. We set

$$\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/Y'_T/T)} = \mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}$$

to emphasize that we are working relative to $\mathfrak{M}(Y/Y'_T/T)$. This restricts to a perfect obstruction theory on any open substack U of $\mathfrak{M}(X/T)$, and we write $\mathbb{E}_{U/\mathfrak{M}(Y/Y'_T/T)}$ for such a restriction; below we use this for $U = \overline{M}(X/T)^*$.

Proposition 3.1.11. *Let $g : \overline{M}(X/T)^* \to \overline{M}(X'/T')$ be the canonical morphism. There is a commutative diagram*

$$\begin{tikzcd} g^*\mathbb{E}_{\overline{M}(X'/T')/\mathfrak{M}(Y'/T')} \arrow[r, "\sim"] \arrow[d] & \mathbb{E}_{\overline{M}(X/T)^*/\mathfrak{M}(Y/Y'_T/T)} \arrow[d] \\ g^*\mathbb{L}_{\overline{M}(X'/T')/\mathfrak{M}(Y'/T')} \arrow[r] & \mathbb{L}_{\overline{M}(X/T)^*/\mathfrak{M}(Y/Y'_T/T)} \end{tikzcd} \tag{3.1.12}$$

in which the upper horizontal arrow is an isomorphism and the lower horizontal arrow is the canonical morphism of cotangent complexes.

Proof. The commutativity of the diagram is not difficult to check, but it is tedious to carry out in complete detail. We refer readers with the patience for these details to Appendix A.2.

To complete the proof, we must check that the map in the upper horizontal arrow of Diagram (3.1.12) is a quasi-isomorphism. Let C be the universal curve over $\overline{M}(X/T)$, let C' be the universal curve over $\overline{M}(X'/T')$, and let $\overline{C}$ be the pullback of C' to $\overline{M}(X/T)^*$, so that the diagram

$$\begin{tikzcd} C \arrow[dr, "\epsilon"] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\pi"'] & & \\ & \overline{C} \arrow[r, "\overline{\rho}"] \arrow[d, "\overline{\pi}"] & C' \arrow[d, "\pi'"] \\ & \overline{M}_{g,n}(X/T)^* \arrow[r, "g"] & \overline{M}_{g,n}(X'/T') \end{tikzcd}$$

commutes and has a Cartesian square. Let $f' : C' \to X'$ and $f : C \to X$ be the universal maps.

Note the following.

(1) The map $q^*T_{X'/Y'} \to T_{X/Y}$ is an isomorphism since $X' \to Y'$ is smooth and $X = X' \times_{Y'} Y$.
(2) We have $\mathbf{R}\epsilon_*\mathbf{L}\epsilon^* = \mathrm{id}$ since $\mathbf{R}\epsilon_*\mathcal{O}_C = \mathcal{O}_{\overline{C}}$ by Lemma 3.1.7.
(3) Dualization commutes with pullback for perfect complexes.

We have

$$\begin{aligned} g^*\mathbb{E}^\vee_{\overline{M}(X'/T')/\overline{M}_{g,n}(Y'/T')} &= g^*\mathbf{R}\pi'_*f'^*T_{X'/Y'} \\ &= \mathbf{R}\pi_*\overline{\rho}^*f'^*T_{X'/Y'} \\ &= \mathbf{R}\overline{\pi}_*\mathbf{R}\epsilon_*\mathbf{L}\epsilon^*\overline{\rho}^*f'^*T_{X'/Y'} \\ &= \mathbf{R}\pi_*f^*T_{X/Y} \\ &= \mathbb{E}^\vee_{\overline{M}(X/T)^*/\overline{M}(Y/T)}. \end{aligned}$$

□

3.2. **Detailed proof of Proposition 3.1.5.** In Section 3.1.2 we defined $\overline{M}(X/X'_T/T)$ be the substack of points $s : S \to \mathfrak{M}(X/\overline{X}/T)$ such that

$$\ker\left(\underline{\mathrm{Aut}}_{\mathfrak{M}(X/X'_T/T)}(s) \to \underline{\mathrm{Aut}}_{\mathfrak{M}(X'/T')}(s)\right)$$

is unramified over S. This is the maximal substack of $\mathfrak{M}(X/X'_T/T)$ with unramified inertia relative to $\mathfrak{M}(X'/T')$.

Proposition 3.2.1. (a) *Suppose that we have a commutative diagram of algebraic stacks* (3.1.3) *with $X = X' \times_{Y'} Y$. The natural map*

$$\mathfrak{M}(X/X'_T/T) \to \mathfrak{M}(Y/Y'_T/T) \underset{\mathfrak{M}(Y'/T')}{\times} \mathfrak{M}(X'/T')$$

is an isomorphism.

(b) *Let X' and Y' be algebraic stacks over T' and write $X'_T := X' \times_{T'} T$ and $Y'_T = Y' \times_{T'} T$. The natural map*

$$\overline{M}(X/X'_T/T) \to \overline{M}(Y/Y'_T/T) \underset{\mathfrak{M}(Y'/T')}{\times} \mathfrak{M}(X'/T')$$

is an isomorphism of stacks.

Proof. (a) Set $\overline{Y} = Y'_T$ and $\overline{X} = X'_T$. By definition, an S-point of $\mathfrak{M}(Y/Y'_T/T) \times_{\mathfrak{M}(Y'/T')} \mathfrak{M}(X'/T')$ is a diagram

$$\begin{tikzcd} C \arrow[rr] \arrow[d] & & Y \arrow[d] \\ \overline{C} \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ S \arrow[r] & T \arrow[r] & T' \end{tikzcd}$$

which clearly induces a unique map $C \to X = X' \times_{Y'} Y$.

(b) If s is a point of $\mathfrak{M}(X/X'_T/T)$, then by (a) there is a cartesian diagram

$$\begin{array}{ccc} \underline{\mathrm{Aut}}_{\mathfrak{M}(X/X'_T/T)}(s) & \longrightarrow & \underline{\mathrm{Aut}}_{\mathfrak{M}(X'/T')}(s) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Aut}}_{\mathfrak{M}(Y/Y'_T/T)}(s) & \longrightarrow & \underline{\mathrm{Aut}}_{\mathfrak{M}(Y'/T')}(s). \end{array}$$

The induced map between the kernels of the two horizontal arrows is an isomorphism; one is unramified if and only if the other is.

□

For the rest of this section, we work in the setting of Proposition 3.2.1 (b), assuming in addition that X' is of Deligne–Mumford type and $X \to X'_T$ is representable.

In Section 3.1, we defined $\mathfrak{M}(X/T)^*$ be the locus of maps $C \to X$ in $\mathfrak{M}(X/T)$ such that the composed map $C \to X'$ admits a stabilization. More precisely, $\mathfrak{M}(X/T)^*$ is the locus of maps $C \to X$ where either the homology class of the image of C in X' is non-zero or $2g - 2 + n > 0$ (with g denotes the genus of C and n its number of marked points). Since the homology class of C in X' and the function $2g - 2 + n$ are both locally constant, $\mathfrak{M}(X/T)^*$ is open and closed in $\mathfrak{M}(X/T)$.

Let $\gamma : \mathfrak{M}(X/T)^* \to \overline{M}(X'/T')$ be the map which sends a map $C \to X_S$ over S to the stabilization $\overline{C}$ of the composed map $C \to X'$. Since there is a projection $C \to \overline{C}$ whose stabilization is an isomorphism, this also induces a section $\sigma : \overline{M}(X/T)^* \to \mathfrak{M}(X/X'_T/T)$ of τ.

Proposition 3.2.2. *Suppose X' is a Deligne–Mumford stack. Then the map*

$$(\sigma, \gamma) : \mathfrak{M}(X/T)^* \to \mathfrak{M}(X/X'_T/T) \underset{\mathfrak{M}(X')}{\times} \overline{M}(X'/T')$$

is an isomorphism with inverse given by τp_1 and it induces an isomorphism

$$\overline{M}(X/T)^* \to \overline{M}(X/X'_T/T) \underset{\mathfrak{M}(X'/T')}{\times} \overline{M}(X'/T') \tag{3.2.3}$$

Proof. Since σ is a section of τ, it is clear that τp_1 is left inverse to (σ, γ). In particular, (σ, γ) is an isomorphism of $\mathfrak{M}(X/T)^*$ onto a substack of $\mathfrak{M}(X/X'_T/T) \times_{\mathfrak{M}(X'/T')} \overline{M}(X'/T')$. Note that this implies that, for any $s : S \to \mathfrak{M}(X/T)^*$, we have

$$\underline{\mathrm{Aut}}_{\mathfrak{M}(X/T)}(s) \to \underline{\mathrm{Aut}}_{\mathfrak{M}(X/X'_T/T)}(s)$$

is an isomorphism of group schemes over S.

To see that τp_1 is also right inverse to (σ, γ), we must show that, for any S-point (3.1.4) of $\mathfrak{M}(X/X'_T/T)$ whose image in $\mathfrak{M}(X'/T')$ is contained in $\overline{M}(X'/T')$, the stabilization of $C \to X'$ is $\overline{C}$. Let C' denote the stabilization. Since the stabilization is initial among all factorizations of $C \to X'$ through stable maps, there is a map $C' \to \overline{C}$. On the other hand, $C' \to \overline{C}$ is also stable (its automorphism group being contained in that of C' over X') and $\overline{C}$ is the stabilization of $C \to \overline{C}$, so we obtain a section $\overline{C} \to C'$, which must be an isomorphism by the universal property. This proves the first claim.

To finish the proof, we must show that $s : S \to \mathfrak{M}(X/T)^*$ is in $\overline{M}(X/T)^*$ if and only if σs is in $\overline{M}(X/X'_T/T)$. For s to lie in $\overline{M}(X/T)^*$ means that $\underline{\mathrm{Aut}}_{\mathfrak{M}(X/T)}(s)$ is unramified over S; for σs to lie in $\overline{M}(X/X'_T/T)$ means that the group K in the exact sequence

$$0 \to K \to \underline{\mathrm{Aut}}_{\mathfrak{M}(X/X'_T/T)}(s) \to \underline{\mathrm{Aut}}_{\mathfrak{M}(X'/T')}(s)$$

is unramified over S. Since s is in $\mathfrak{M}(X/T)^*$, the group $\underline{\mathrm{Aut}}_{\mathfrak{M}(X'/T')}(s)$ is unramified. Thus, K is unramified if and only if $\underline{\mathrm{Aut}}_{\mathfrak{M}(X/X'_T/T)}(s)$ is. But we saw above that $\underline{\mathrm{Aut}}_{\mathfrak{M}(X/X'_T/T)}(s)$ is isomorphic to $\underline{\mathrm{Aut}}_{\mathfrak{M}(X/T)}(s)$. Thus s is in $\overline{M}(X/T)^*$ if and only if it is in $\overline{M}(X/X'_T/T)$. $\square$

Combining Propositions 3.2.1 and 3.2.2 now yields Proposition 3.1.5.

3.3. **Constructing the moduli stacks.**

3.3.1. *The moduli stack of expanded pairs.* The moduli space of targets $\mathscr{T}$ is defined in [Li01, Definition 4.4] (denoted there by $\mathfrak{Z}^{\mathrm{rel}}$ and called the stack of expanded relative pairs); the notation $\mathscr{T}$, as well as the name, come from [GV05]. We summarize some properties of $\mathscr{T}$ that are proved in [ACFW].

The stack $\mathscr{T}$ admits an étale cover by maps $\mathscr{A}^n \to \mathscr{T}$, $n = 0, 1, \ldots$ such that, for any monotonic injection $u : [n] \to [m]$, there is a 2-isomorphism making the diagram

$$\begin{tikzcd} \mathscr{A}^n \arrow[rr] \arrow[dr] & & \mathscr{A}^m \arrow[dl] \\ & \mathscr{T} & \end{tikzcd}$$

commute, and these 2-isomorphisms are compatible in the obvious sense. It is also proved in [ACFW] that $\mathscr{T}$ is the colimit of the $\mathscr{A}^n$, but we will not use this fact here.

There is a universal expansion $\mathscr{A}^{\mathrm{rel}}$ of $\mathscr{A}$ over $\mathscr{T}$. Setting $\mathscr{A}[n] = \mathscr{A}^{\mathrm{rel}} \times_{\mathscr{T}} \mathscr{A}^n$, it is shown in [ACFW] that $\mathscr{A}[n]$ may be identified with an open substack of $\mathscr{A}^{2n+1}$.

3.3.2. *Construction of* $\mathfrak{M}(\mathscr{A})'$. Recall that a morphism $C \to \mathscr{A}$ is equivalent to a line bundle with section on C. Therefore an object of $\mathfrak{M}_{0,n}(\mathscr{A})$ is equivalent to a triple (C, L, s), where C is an n-marked genus 0 curve over S, with a line bundle L on C, and a section s of L. Define the open substack $\mathfrak{M}_{0,n}(\mathscr{A})' \subseteq \mathfrak{M}_{0,n}(\mathscr{A})$ as follows.

An object (C, L, s) of $\mathfrak{M}_{0,n}(\mathscr{A})$ over a scheme S is an object of $\mathfrak{M}_{0,n}(\mathscr{A})'$ if for every geometric point $\xi \to S$,

- MA1 $\deg(L_\xi) = \sum_{i=1}^n \mathrm{age}_i(L_\xi)$, where age_i denotes the age at the i-th marked point, and
- MA2 for any proper subcurve $C' \subset C_\xi$, we have $-\frac{1}{2} < \deg(L\big|_{C'}) < \frac{1}{2}$.

If the geometric fiber C_ξ is smooth, with coarse moduli space $\overline{C}$, these imply that the push forward of L to $\overline{C}$ is $\mathcal{O}_{\overline{C}}$. In particular, $H^1(C_\xi, L_\xi) = 0$ in this case.

These conditions are crucial for the key technical argument in Section 4.3.

3.3.3. *Construction of* $\mathfrak{M}^{\mathrm{rel}}(\mathscr{A}, \mathcal{B}\mathbb{G}_m)'$. We construct $\mathfrak{M}^{\mathrm{rel}}(\mathscr{A}, \mathcal{B}\mathbb{G}_m)'$ through a series of definitions as follows.

Let

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathscr{A}^{\mathrm{rel}} \arrow[d] \\ S \arrow[r] & \mathscr{T} \end{tikzcd}$$

describe an S-point of $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$. The distinguished divisor of $\mathscr{A}^{\mathrm{rel}}$ pulls back to a closed substack P on C. We say that a diagram as above is in $\mathfrak{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$ if the map $C \to \mathscr{A}^{\mathrm{rel}}$ is smooth near the pre-image of the nodes and near P. The point is that when $C \to \mathscr{A}^{\mathrm{rel}}$ lifts to an expanded pair $C \to X^{\mathrm{rel}}$ with X a scheme, then the map is an object of $\mathfrak{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$ if an only if $C \to X^{\mathrm{rel}}$ is transversal to the nodes and boundary divisor of X^{rel}.

This definition makes $\mathfrak{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$ an open substack of $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$. We denote its pre-image in $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ by $\mathfrak{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$. We define $\mathfrak{M}^{\mathrm{rel}}(\mathscr{A}, \mathcal{B}\mathbb{G}_m)' = \overline{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ to be the intersection of $\mathfrak{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ and $\overline{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$.

3.3.4. *Algebraicity properties.* We prove here that the stacks considered above are algebraic. We repeatedly use

Lemma 3.3.1 (cf. [AF, Lemma B.2.1])**.** *Suppose $\mathscr{X}$ is a stack with a morphism to an algebraic stack $\mathscr{Y}$. Assume there is a smooth cover $\mathscr{U} \to \mathscr{Y}$ by an algebraic stack $\mathscr{U}$ such that $\mathscr{U} \times_{\mathscr{Y}} \mathscr{X}$ is an algebraic stack. Then $\mathscr{X}$ is algebraic.*

Proposition 3.3.2.

(1) *The stacks* $\mathfrak{M}(\mathcal{B}\mathbb{G}_m)$, $\mathfrak{M}(\mathscr{A})$, *as well as the stack* $\mathfrak{M}(\mathscr{A})'$ *defined in Section 3.3.2, are algebraic.*
(2) *The stack* $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$ *is algebraic.*
(3) *The stack* $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ *is algebraic.*
(4) *The stacks* $\mathfrak{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$, *as well as* $\mathfrak{M}^{\mathrm{rel}}(\mathscr{A}, \mathcal{B}\mathbb{G}_m)'$ *defined in Section 3.3.3, are algebraic.*
(5) *Let* $\mathscr{X}$ *be a Deligne–Mumford stack of finite type over* $\mathbb{C}$*. Then the stacks* $\mathfrak{M}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})$ *and* $\overline{M}^{\mathrm{tr}}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})$ *are algebraic. Thus* $\overline{M}^{\mathrm{tr}}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})$ *is a Deligne–Mumford stack.*

Proof. (1) To prove the algebraicity of $\mathfrak{M}(\mathcal{B}\mathbb{G}_m)$, it is sufficient to work locally in the moduli stack of Deligne–Mumford pre-stable curves $\mathfrak{M}$. It is therefore sufficient to show that for any scheme S and family of orbifold pre-stable curves C over S, the S-stack $\mathrm{Hom}_S(C, S \times \mathcal{B}\mathbb{G}_m)$ is algebraic. This follows from [Aok06, page 53] and [Bro09].

Now we may prove that $\mathfrak{M}(\mathscr{A})$ is algebraic by working locally in $\mathfrak{M}(\mathcal{B}\mathbb{G}_m)$. We must show that for any scheme S, any family of orbifold pre-stable curves C over S, and any line bundle L over C, the S-sheaf of sections of L is representable. This is just $\mathrm{Hom}_S(C, L) \times_{\mathrm{Hom}_S(C,C)} S$ where the map $S \to \mathrm{Hom}_S(C, C)$ is the one associated to the identity map on C. This is representable by [Ols06b, Theorem 1.1], since both C and L are Deligne–Mumford stacks over S.

Finally, $\mathfrak{M}(\mathscr{A})'$ is open in $\mathfrak{M}(\mathscr{A})$, which completes the proof of (1).

(2) It is sufficient to prove this locally in $\mathscr{T}$. The collection of $\mathscr{A}^n \to \mathscr{T}$ defined in Section 3.3.1 forms an étale cover, so it is sufficient to prove that $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T}) \times_{\mathscr{T}} \mathscr{A}^n$ is algebraic for each n. We have

$$\begin{aligned}\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T}) \underset{\mathscr{T}}{\times} /\mathscr{A}^n &\cong \mathfrak{M}(\mathscr{A}^{\mathrm{rel}} \underset{\mathscr{T}}{\times} \mathscr{A}^n/\mathscr{A}^n) \\ &\cong \mathfrak{M}(\mathscr{A}[n]/\mathscr{A}^n) \cong \mathfrak{M}(\mathscr{A}[n]) \underset{\mathfrak{M}(\mathscr{A}^n)}{\times} \mathscr{A}^n,\end{aligned}$$

and $\mathfrak{M}(\mathscr{A}^n) \cong \mathfrak{M}(\mathscr{A})^n$ is algebraic by (1). Since $\mathscr{A}^n$ is also algebraic, it suffices to see that $\mathfrak{M}(\mathscr{A}[n])$ is algebraic. But $\mathscr{A}[n]$ is an open substack of $\mathscr{A}^{2n+1}$ so $\mathfrak{M}(\mathscr{A}[n])$ is an open substack of the algebraic stack $\mathfrak{M}(\mathscr{A}^{2n+1})$, hence is algebrac.

(3) We have a natural morphism $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T}) \to \mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T}) \times_{\mathscr{T}} \mathfrak{M}(\mathscr{A}_{\mathscr{T}}/\mathscr{T})$. We have shown above that the stack $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$ is algebraic, and the stack $\mathfrak{M}(\mathscr{A}_{\mathscr{T}}/\mathscr{T}) = \mathfrak{M}(\mathscr{A}) \times \mathscr{T}$ is algebraic by (1). It remains to show that

$$\mathfrak{M}_S := \mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T}) \times_{\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T}) \times_{\mathscr{T}} \mathfrak{M}(\mathscr{A}_{\mathscr{T}}/\mathscr{T})} S$$

is algebraic whenever S is a scheme. Denote by $C \to S$ and $\overline{C} \to S$ the resulting families of curves as in Diagram (3.1.4). The stack $\mathfrak{M}_S$ maps to the algebraic stack $\mathrm{Hom}_S^{rep}(C, \overline{C})$ of representable maps. The locus where these maps have 0-dimensional image is closed, and in the open (and therefore algebraic) complement the stabilization $C \to C' \to \overline{C}$ of $C \to \overline{C}$ is well defined. The stack $\mathfrak{M}_S$ above factors through the the open (and therefore algebraic) locus where $C' \to \overline{C}$ is an isomorphism. The stack $\mathfrak{M}_S$ itself is therefore the algebraic substack of sections $Sect_S(D/C)$ where $D = C \times_{\mathscr{A} \times \mathscr{A}} \mathscr{A}$, where the map on the right is the diagonal and the maps on the left are $C \to \mathscr{A}^{\mathrm{rel}} \to \mathscr{A}$ and $C \to \overline{C} \to \mathscr{A}$.

(4) The stack $\mathfrak{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$ is an open substack of $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$, which is algebraic by (2) above. The stack $\mathfrak{M}^{\mathrm{rel}}(\mathscr{A}^{\mathrm{rel}}, B\mathbf{G}_m)'$ is an open substack of $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$, which is algebraic by (3) above.

(5) Since $\mathscr{X}^{\mathrm{rel}} = \mathscr{X} \times_{\mathscr{A}} \mathscr{A}^{\mathrm{rel}}$ and the map $\mathscr{X}^{\mathrm{rel}} \to \mathscr{T}$ is induced from $\mathscr{A}^{\mathrm{rel}} \to \mathscr{T}$, we have an isomorphism

$$\mathfrak{M}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T}) \xrightarrow{\sim} \mathfrak{M}(\mathscr{X}) \underset{\mathfrak{M}(\mathscr{A})}{\times} \mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T}).$$

But $\mathfrak{M}(\mathscr{X})$ is an algebraic stack by [AV02, Theorem 1.4.1 and Section 8] and both $\mathfrak{M}(\mathscr{A})$ and $\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$ are algebraic by (1) and (2), respectively.

By definition, $\overline{M}^{\mathrm{tr}}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})$ is an open substack of the maximal Deligne–Mumford substack of the algebraic stack $\mathfrak{M}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})$. □

4. Proof of the theorem

In Section 4.1 we reduce the proof of Theorem 2.2 to an application of Costello's theorem. The rest of Section 4 will be devoted to the verification of Costello's hypotheses.

4.1. **The main argument.** Let X be a smooth scheme over $\mathbb{C}$ with a smooth divisor D. Let $\mathscr{X}$ be the r-th root stack of the line bundle and section $(\mathcal{O}_X(D), 1)$ (the value of r will be specified momentarily). Let $\mathscr{D}$ be the r-th root divisor on $\mathscr{X}$.

Proposition 4.1.1. *Let X be smooth and D a Cartier divisor, and let $\sigma : X \to \mathscr{A}$ be the corresponding morphism. Then σ is smooth if and only if D is smooth.*

Proof. If σ is smooth then D is smooth as it is the inverse image of the smooth divisor $\mathcal{B}\mathbb{G}_m \subset \mathscr{A}$.

Now we assume D smooth and prove that σ is smooth. The problem is étale local on X so we may assume D is defined by a local coordinate e_1. Completing this to a regular system of parameters $(e_1, \ldots, e_n)$

with $n = \dim X$, we obtain a factorization of σ through the étale map $(e_1, \ldots, e_n) : X \to \mathbf{A}^n$. The map $\mathbf{A}^n \to \mathscr{A}$ factors through the smooth projection $\mathbf{A}^n \to \mathbf{A}^1$ on the first coordinate, and $\mathbf{A}^1 \to \mathscr{A}$ is smooth by definition. □

The proposition implies that the map $X \to \mathscr{A}$ (determined by the line bundle and section $(\mathcal{O}_X(D), 1)$) is smooth. Since $\mathscr{X} \to \mathscr{A}$ is obtained from this by base change along the r-th power map, it is smooth as well. This implies, again by the proposition, that $\mathscr{D}$ is smooth. Therefore the morphism

$$\overline{M}_{0,n}(\mathscr{X}) \to \mathfrak{M}_{0,n}(\mathscr{A})$$

has a perfect relative obstruction theory of perfect amplitude in $[-1, 0]$. We denote the relative virtual class by $\left[\overline{M}_{0,n}(\mathscr{X})/\mathfrak{M}_{0,n}(\mathscr{A})\right]^{\mathrm{vir}}$.

Now we select the integer r determining the root stack $\mathscr{X}$. Note that, as in the proof of Theorem 2.1, this choice in turn determines a twisting choice $\mathfrak{r}$ in the sense of [AF, Definition 3.4.1] for relative stable maps to X of class β, namely $\mathfrak{r}(c_1, \ldots, c_k) = r$ for every multiset $\{c_1, \ldots, c_k\}$ of contact orders occuring in the moduli space.

Definition 4.1.2. *Let β be an effective class in $H_2(X, \mathbb{Z})$ and let $d = D.\beta$. Set $\kappa = \min_{0 \le \gamma \le \beta} D.\gamma$, the minimum taken over all classes γ such that both γ and $\beta - \gamma$ are effective. Let r be an integer such that*

1. *$r > 2d$,*
2. *$r > -2\kappa$, and*
3. *if j is an integer such that $1 \le j \le d - \kappa$ then j divides r.*

Remark 4.1.3. The integer r is chosen carefully for Proposition 4.3.4 to apply. Since the conditions on r only bound it from below and impose divisibility, it is clear that an integer r satisfying the conditions exists.

Let Γ stand for the numerical data $(g = 0, n, \beta)$. The assumptions on r made above guarantee that the universal curve over $\overline{M}_\Gamma(\mathscr{X})$ satisfies Properties MA1 and MA2 of Section 3.3.2, hence that the map $\overline{M}_\Gamma(\mathscr{X}) \to \mathfrak{M}_{0,n}(\mathscr{A})$ factors through $\mathfrak{M}_{0,n}(\mathscr{A})'$. We therefore have a commutative diagram

$$\begin{tikzcd}
\overline{M}^{\mathrm{tr}}_\Gamma(\mathscr{X}^{\mathrm{rel}}/\mathscr{T}) \arrow[r, "\Phi_{\mathscr{X}}"] \arrow[d] & \overline{M}_\Gamma(\mathscr{X}) \arrow[d] \\
\mathfrak{M}^{\mathrm{rel}}_{0,n}(\mathscr{A}, \mathcal{B}\mathbb{G}_m)' \arrow[r, "\Phi_{\mathscr{A}}"] & \mathfrak{M}_{0,n}(\mathscr{A})',
\end{tikzcd}$$

which is cartesian by Proposition 3.1.5. This is diagram 2.3.1 with a more specific notation that includes the relevant discrete data. We will apply [Cos06, Theorem 5.0.1] to this diagram. The hypotheses of the theorem are the following (with numbers corresponding to the bullets in Costello's statement):

(1a) $\overline{M}^{\mathrm{tr}}_{\Gamma}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})$ is a Deligne–Mumford stack by Proposition 3.3.2.
(1b) $\overline{M}_{\Gamma}(\mathscr{X})$ is a Deligne–Mumford stack by [AV02, Theorem 1.4.1].
(2,3) $\overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ and $\mathfrak{M}_{0,n}(\mathscr{A})$ are Artin stacks of the same pure dimension. The morphism $\mathscr{A}^{\mathrm{rel}} \to \mathscr{A}_{\mathscr{T}}$ is representable, so by [AV02, Section 8.3] the morphism $\overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T}) \to \mathfrak{M}_{0,n}(\mathscr{A})'$ is of relative Deligne–Mumford type. It is of pure degree 1 by Proposition 4.1.4).
(4) $\Phi_{\mathscr{X}}$ is proper by [AF, Corollary 3.4.8].
(5) The obstruction theory for $\overline{M}^{\mathrm{tr}}_{\Gamma}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T}) \to \overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ on the left is equivalent to the one obtained by pulling back that of the morphism $\overline{M}_{\Gamma}(\mathscr{X}) \to \mathfrak{M}_{0,n}(\mathscr{A})'$ on the right (Proposition 3.1.11).

Hypothesis (2,3) is implied by

Proposition 4.1.4. *There are dense, unobstructed, open substacks*

$$\overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})'' \subset \overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T}) \qquad \textit{(Section 4.2)}$$
$$\mathfrak{M}_{0,n}(\mathscr{A})'' \subset \mathfrak{M}_{0,n}(\mathscr{A})' \qquad \textit{(Section 4.3)}$$

on which $\Phi'_{\mathscr{A}}$ induces an isomorphism.

Proof. For $\overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})''$ we take the fiber of $\overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ above the open point of $\mathscr{T}$. Since $\mathscr{A}^{\mathrm{rel}}$ and $\mathscr{A}_{\mathscr{T}}$ are isomorphic over the open point of $\mathscr{T}$, Diagram (3.1.4) simplifies to

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathscr{A} \arrow[d] \\ \overline{C} \arrow[r] \arrow[d] & \mathscr{A} \\ S. & \end{tikzcd}$$

Since this point lies in $\overline{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$, it can have no continuous automorphisms fixing the map $\overline{C} \to \mathscr{A}$. This means that $C \to \overline{C}$ is stable, hence an isomorphism, since $\overline{C}$ is the stabilization of the map $C \to \overline{C}$. Thus the top line in the diagram above is determined from the rest of the diagram. This implies that $\Phi_{\mathscr{A}}$, which forgets the top line above, is an embedding on $\overline{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})''$.

By the definition of $\overline{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$, a map $C \to \mathscr{A}$ lies in the image of $\Phi_{\mathscr{A}}$ if and only if it is transverse to the divisor $\mathcal{B}\mathbb{G}_m \subset \mathscr{A}$, i.e., if and only if the map $C \to \mathscr{A}$ is smooth near the pre-image of $\mathcal{B}\mathbb{G}_m$. This is an open condition on $\mathfrak{M}(\mathscr{A})'$, hence defines an open substack $\mathfrak{M}(\mathscr{A})''$ onto which $\Phi_{\mathscr{A}}$ defines an isomorphism.

To complete the proof, we check in Section 4.2 that $\overline{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})''$ is dense in $\overline{M}^{\mathrm{tr}}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ and in Section 4.3 that $\mathfrak{M}(\mathscr{A})''$ is dense in $\mathfrak{M}(\mathscr{A})'$. $\square$

In order to check Hypothesis (4), we relate our notation to that of [AF]. Our stack $\overline{M}^{\mathrm{tr}}_{\Gamma}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})$ is isomorphic to a union of open and closed substacks $\mathcal{K}^{\mathfrak{r}}_{\Gamma'}(X, D)$ of [AF] where $\mathfrak{r}$ is equal to our twisting choice (Definition 4.1.2) and the union is taken as Γ' ranges among all "data for a pair" ([AF, Convention 3.1.2]) compatible with Γ (effectively, over all partitions of $\{1, \ldots, n\}$ into two sets, denoted M and N in loc.cit.). By [AF, Corollary 3.4.8], each $\mathcal{K}^{\mathfrak{r}}_{\Gamma}(X, D)$ is proper; since $\overline{M}_{\Gamma}(\mathscr{X})$ is separated ([AV02, Theorem 1.4.1]) this implies $\Phi_{\mathscr{X}}$ is proper, which is Hypothesis (4).

We deduce from Costello's theorem that

$$\Phi_*\big[\overline{M}^{\mathrm{tr}}_{\Gamma}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})/\mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})'\big]^{\mathrm{vir}} = [\overline{M}_{\Gamma}(\mathscr{X})/\mathfrak{M}_{0,n}(\mathscr{A})]^{\mathrm{vir}}.$$

where here and later we omit the subscript $\mathscr{X}$ of $\Phi_{\mathscr{X}}$ when no confusion is likely.

In Section 4.2, we show that $\mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})'$ is smooth, which implies by a standard compatibility result reviewed in Proposition A.4.1 that

$$\big[\overline{M}^{\mathrm{tr}}_{\Gamma}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})/\mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})'\big]^{\mathrm{vir}} = \big[\overline{M}^{\mathrm{tr}}_{\Gamma}(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})\big]^{\mathrm{vir}}.$$

Noting that $\mathscr{A} \to \mathcal{B}\mathbb{G}_m$ is smooth, we apply a similar compatibility result reviewed in Proposition A.4.2 to the sequence $\overline{M}_{\Gamma}(\mathscr{X}) \to \mathfrak{M}_{0,n}(\mathscr{A}) \to \mathfrak{M}_{0,n}(\mathcal{B}\mathbb{G}_m)$ to conclude

$$\big[\overline{M}_{\Gamma}(\mathscr{X})/\mathfrak{M}_{0,n}(\mathscr{A})\big]^{\mathrm{vir}} = \big[\overline{M}_{\Gamma}(\mathscr{X})/\mathfrak{M}_{0,n}(\mathcal{B}\mathbb{G}_m)\big]^{\mathrm{vir}}.$$

Lemma 4.1.5. *The stack $\mathfrak{M}(B\mathbf{G}_m)$ is smooth and unobstructed.*

Proof. The obstructions to deforming a line bundle on C lie in $H^2(C, \mathcal{O}_C)$, which vanishes if C is a curve. Hence $\mathfrak{M}(B\mathbf{G}_m)$ is smooth over $\mathfrak{M}$, which is itself smooth. $\square$

Lemma 4.1.5 permits us again to apply Proposition A.4.1 to the sequence $\overline{M}_{0,n}(\mathscr{X}) \to \mathfrak{M}_{0,n}(\mathcal{B}\mathbb{G}_m) \to$ (point) and find

$$\big[\overline{M}_{\Gamma}(\mathscr{X})/\mathfrak{M}_{0,n}(\mathcal{B}\mathbb{G}_m)\big]^{\mathrm{vir}} = \big[\overline{M}_{\Gamma}(\mathscr{X})\big]^{\mathrm{vir}}.$$

Taken together, the equalities above imply

$$\Phi_* \big[\overline{M}^{\mathrm{tr}}_\Gamma(\mathscr{X}^{\mathrm{rel}}/\mathscr{T})\big]^{\mathrm{vir}} = \big[\overline{M}_\Gamma(\mathscr{X})\big]^{\mathrm{vir}},$$

which is the conclusion of Theorem 2.2.

Our remaining task is to complete the proof of Proposition 4.1.4.

4.2. **A dense open substack of $\mathfrak{M}^{\mathrm{tr}}_{g,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$.** We abbreviate $\mathfrak{M}^{\mathrm{tr}}_{g,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ to $\mathfrak{M}$. Let $f : \mathfrak{C} \to \mathscr{A}^{\mathrm{rel}}$ be the universal morphism and write

$$\mathbb{L} := \mathrm{Cone}\big(f^*\mathbb{L}_{\mathscr{A}^{\mathrm{rel}}/\mathscr{T}}(\log D) \to \mathbb{L}_{\mathfrak{C}/\mathfrak{M}}(\log P)\big)$$

where D is the universal divisor on $\mathscr{A}^{\mathrm{rel}}$ and P is the divisor of marked points on $\mathfrak{C}$. If $\pi : \mathfrak{C} \to \mathfrak{M}$ is the projection, then there is a natural map

$$\pi_!\mathbb{L}[-1] \to \mathbb{L}_{\mathfrak{M}/\mathscr{T}}.$$

Lemma 4.2.1. *This is a relative obstruction theory for $\mathfrak{M} \to \mathscr{T}$.*

This is a standard generalization of [BF97, Proposition 6.3] in combination with [BL00, Proposition A.1]. See Appendix A.3 for more details.

Proposition 4.2.2. *The morphism $\mathfrak{M}^{\mathrm{tr}}_{g,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T}) \to \mathscr{T}$ is smooth and unobstructed.*

Proof. Let $\mathbb{E} = \pi_!\mathbb{L}[-1]$. We must show that $\mathbb{E}$ has perfect amplitude in $[0,1]$. Since $\mathbb{E}$ is certainly perfect, it suffices to see that $\mathbb{E}^\vee[-1] = R\pi_*\big(\mathbb{L}^\vee\big)$ has perfect amplitude in $[0,1]$. By semicontinuity, it is enough to see that, when S is a geometric point of $\mathfrak{M}$ and $f : C \to \mathscr{A}^{\mathrm{rel}} \times_{\mathscr{T}} S$ is the corresponding morphism, we have

$$\mathrm{Ext}^p(\mathbb{L}\big|_C, \mathcal{O}_C) = 0$$

for $p \geq 2$ (where $\mathbb{L}\big|_C$ denotes the pullback of $\mathbb{L}$ to C via the canonical map $C \to \mathfrak{C}$).

We use the local-to-global spectral sequence, which gives

$$H^p(C, \underline{\mathrm{Ext}}^q(\mathbb{L}\big|_C, \mathcal{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}(\mathbb{L}\big|_C, \mathcal{O}_C).$$

The proposition will follow once we show

1. $\underline{\mathrm{Ext}}^1(\mathbb{L}\big|_C, \mathcal{O}_C)$ is supported in dimension 0 (which implies that $H^p(C, \underline{\mathrm{Ext}}^1(\mathbb{L}\big|_C, \mathcal{O}_C)) = 0$ for $p > 0$), and
2. $\underline{\mathrm{Ext}}^q(\mathbb{L}\big|_C, \mathcal{O}_C) = 0$ for $q > 1$.

Indeed, these properties show that the E_2 term of the spectral sequence is concentrated in positions $(p,q) \in \{(0,0),(0,1),(1,0)\}$ which implies both that it degenerates at the E_2 term and that $\mathrm{Ext}^p(\mathbb{L}\big|_C, \mathcal{O}_C) = 0$ for $p \geq 2$.

Let $U \subset \mathscr{A}^{\mathrm{rel}}$ be the complement of D in the smooth locus of the map $\mathscr{A}^{\mathrm{rel}} \to \mathscr{T}$. Since U is étale over $\mathscr{T}$ we have $\mathbb{L}_{U/\mathscr{T}} = 0$ and hence

$$\mathbb{L}\big|_{f^{-1}U} \simeq \Omega_C.$$

It is well-known tht $\underline{\mathrm{Ext}}^q(\Omega_C, \mathcal{O}_C) = 0$ for $q > 1$ and is supported on the nodes for $q = 1$.

The map $C \to \mathscr{A}^{\mathrm{rel}}$ is transverse to the singularities and to the distinguished divisor D. Therefore, if x is either a node or the distinguished divisor of $\mathscr{A}^{\mathrm{rel}}$ then there is an open neighborhood V_x of x such that $f^{-1}V_x \to \mathscr{A}^{\mathrm{rel}} \times_{\mathscr{T}} S$ is smooth. Therefore $\mathbb{L}\big|_{f^{-1}V_x}$ is a vector bundle, and $\underline{\mathrm{Ext}}^q(\mathbb{L}\big|_{f^{-1}V_x}, \mathcal{O}_C\big|_{f^{-1}V_x}) = 0$ for $q > 0$.

Since $f^{-1}U$ and the $f^{-1}V_x$ cover C, this completes the proof. □

Definition 4.2.3. *Let $\mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})'' \subset \mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ be the pre-image of the open point in $\mathscr{T}$ and let $\overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})''$ be its intersection with $\overline{M}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$.*

Corollary 4.2.4. *The open inclusions*

$$\mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})'' \subset \mathfrak{M}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$$
$$\overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})'' \subset \overline{M}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$$

are dense.

Proof. The claim for $\overline{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ follows immediately from that for $\mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$. Let t be a point of $\mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ that is not in $\mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})''$. Choose a scheme T, the spectrum of a complete Noetherian local ring, and a map $T \to \mathscr{T}$ whose restriction to the closed point is the image of t and whose open point maps to the open point of $\mathscr{T}$. By the proposition, there is a formal section of $\mathfrak{M}^{\mathrm{tr}}_{0,n}(\mathscr{A}^{\mathrm{rel}}/\mathscr{A}_{\mathscr{T}}/\mathscr{T})$ over T, coinciding with t on the closed point. It suffices to show this section can be algebraized.

Since $\mathscr{A}[n]$ is open in $\mathscr{A}^{2n+1}$, this is a question of algebraizing vector bundles and their sections. It therefore has an answer by Grothendieck's existence theorem. □

4.3. A dense open substack of $\mathfrak{M}(\mathscr{A})'$.

Lemma 4.3.1. *Let (C, L, s) be a geometric point of $\mathfrak{M}_{0,n}(\mathscr{A})'$. Assume C is smooth. Let C' be a square-zero thickening of C with ideal $\mathcal{O}_C$.*

Let L' be an extension of L to C'. Then s can be extended to a nonzero section of L' on C'.

Proof. The obstruction for this deformation problem lies in $H^1(C, L)$; when the obstruction vanishes, solutions form a torsor under $H^0(C, L)$. To show that a deformation exists it suffices to show that $H^1(C, L) = 0$. To show that the section can be deformed to be non-zero, we must show in addition $H^0(C, L) \neq 0$. Our assumptions in Section 3.3.2 on $\mathfrak{M}_{0,n}(\mathscr{A})'$ imply that

$$L \cong \mathcal{O}_C\Big(\sum_{y\in C} \operatorname{age}_y(L)y\Big).$$

Writing $\pi : C \to \overline{C}$ for the projection on the coarse moduli space, this implies that $\pi_* L = \mathcal{O}_C$. But $H^i(C, L) = H^i(\overline{C}, \pi_* L)$ and C has genus zero, so the lemma follows. $\square$

Lemma 4.3.2. *If (C, L, s) is a point of $\mathfrak{M}_{0,n}(\mathscr{A})'$ then*

(1) *if x is a node whose stabilizer group is non-trivial, then s vanishes on at least one of the components containing x, and*
(2) *if x is a node whose stabilizer group is trivial and $s(x) = 0$, then s vanishes identically on both components containing x.*

Proof. Let (C, L, s) be a point of $\mathfrak{M}_{0,n}(\mathscr{A})'$. Suppose that C_1 and C_2 are two irreducible components of C meeting at an orbifold node x. Since $\operatorname{age}_x L\big|_{C_1} + \operatorname{age}_x L\big|_{C_2} = 1$, one of these, say the first, must be at least $\frac{1}{2}$. But then if s does not vanish on C_1, we have $\deg L\big|_{C_1} \geq \operatorname{age}_x L\big|_{C_1} \geq \frac{1}{2}$, contradicting Condition MA2 (Section 3.3.2).

Now let the notation be as above, but assume that x is a node with trivial stabilizer. If s vanishes at x but does not vanish identically on C_i then $\deg L\big|_{C_i} \geq 1$, once again contradicting Condition MA2. Thus if s is an ordinary node, s cannot vanish at x unless it vanishes identically on both components containing x. $\square$

Definition 4.3.3. *Let $\mathfrak{M}_{0,n}(\mathscr{A})'' \subseteq \mathfrak{M}_{0,n}(\mathscr{A})'$ be the open substack consisting of triples (C, L, s) such that C is smooth and s is not identically* 0.

Proposition 4.3.4. *The substack $\mathfrak{M}_{0,n}(\mathscr{A})'' \subset \mathfrak{M}_{0,n}(\mathscr{A})'$ is dense.*

Proof. Step 1: smoothing the vanishing locus of s. We will write $\mathfrak{M}_{0,n}(\mathscr{A})^{\text{sm.zeros}}$ for the open substack consisting of those (C, L, s) in $\mathfrak{M}_{0,n}(\mathscr{A})'$ for which the vanishing locus of s inside C is smooth (i.e., is a disjoint union of reduced stacky points and smooth curves). Equivalently, $\mathfrak{M}_{0,n}(\mathscr{A})^{\text{sm.zeros}}$ is the locus where s does not vanish on two adjacent irreducible components of C.

By Lemma 4.1.5, a node joining two components of C on which s vanishes can be smoothed along with the line bundle. The section extends as the zero section on the resulting smoothed component. It follows that $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{sm.zeros}}$ is dense in $\mathfrak{M}_{0,n}(\mathscr{A})'$. Thus, to prove Proposition 4.3.4 it remains only to prove that $\mathfrak{M}_{0,n}(\mathscr{A})''$ is dense in $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{sm.zeros}}$.

Step 2: smoothing the orbifold nodes: setup. Define the open substack $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{ord.nodes}} \subset \mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{sm.zeros}}$ to be the locus where C has trivial stack structure at all of its nodes. In Steps 2–7, we will show that $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{ord.nodes}}$ is dense in $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{sm.zeros}}$.

On an object (C, L, s) of $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{sm.zeros}}$ having an orbifold node, the section s vanishes identically on a component C_0 of C through the node. Let $C' \subset C$ be the locus where s does not vanish and let $C_1, \ldots, C_k$ be the connected components of the closure of C' that meet C_0. Let p_i be the intersection of C_0 with C_i. We claim that the nodes p_i can be smoothed simultaneously.

Step 3: Isolating the relevant curve. To obtain this smoothing, it suffices to assume that C is the union of $C_0, \ldots, C_k$: any node connecting C_i to another component of $\{s = 0\}$ must be twisted, so if we can deform the section on this subcurve, the deformed section will automatically vanishes on such nodes. It follows that the deformation can be glued to a trivial deformation of the rest of the curve.

Step 4: Removing markings way from C_0. We further simplify the object: if there is a twisted marking $x \in C \setminus C_0$, then let $\pi : C \to C'$ be the morphism which forgets this twisted marking. We can recover L from $\pi_* L$ by the formula

$$L = \pi^* \pi_* L \otimes \mathcal{O}_C(age_x(L)x).$$

This can be done in families, so it is equivalent to deform either (C, L, s) or $(C', \pi_* L, \pi_* s)$ (where the marked point remains on C', but is untwisted). So we may assume that all twisted markings of C lie on C_0, though we may no longer assume that $-\frac{1}{2} < \deg L\big|_{C_i} < \frac{1}{2}$ for each $i \neq 0$.

Step 5: Reduction to positive degree on C_0. Choose a general point $x \in C_0$, and replace C_0 with the square root of C_0 at x, replacing L with $L \otimes \mathcal{O}(x/2)$, and s with its image in $L \otimes \mathcal{O}(x/2)$. By the same argument as before, it suffices to deform this new triple. Therefore, we can assume that the degree of L restricted to C_0 is positive, and that the total degree of L is less than 1.

Let m be the number of twisted marked points on C_0 and define $a_1, \ldots, a_m$ to be the ages of L at these points. For $1 \leq i \leq k$, let b_i be the age of $L\big|_{C_i}$ at the node. Since $\deg(L\big|_{C_i}) < 1$ and $L\big|_{C_i}$ has a nontrivial section, it follows that $\deg(L\big|_{C_i}) = b_i$. Since $\deg(L) = \sum a_i$, it follows that $\deg(L\big|_{C_0}) = \sum a_i - \sum b_i$.

STEP 6: LIFTING TO A WEIGHTED PROJECTIVE SPACE. Choose a positive integer $\tilde{r}$ such that $a_i \cdot \tilde{r} \in \mathbf{Z}$ and $b_i \cdot \tilde{r} \in \mathbf{Z}$ for all i. Then $L^{\otimes \tilde{r}}$ is pulled back from the coarse moduli space of C and has non-negative degree on each component. The complete linear system of $L^{\otimes \tilde{r}}$ together with the section s define a representable morphism $C \to \mathbf{P}^N_{H,\tilde{r}}$ of degree $\tilde{r}(\sum a_i)$, where $H \subseteq \mathbf{P}^N$ is a hyperplane, and $\mathbf{P}^N_{H,\tilde{r}} = \sqrt[\tilde{r}]{\mathbf{P}^N, H}$ is the $\tilde{r}$-th root stack.

The moduli space of twisted stable maps to $\mathbf{P}^N_{H,\tilde{r}}$ of genus 0, degree $\tilde{r}(\sum a_i)$, having n marked points, m of which are twisted with contact types $\tilde{r} \cdot a_1, \ldots, \tilde{r} \cdot a_m$, has expected dimension $\tilde{r} \sum a_i N + n + N - 3$; see [Cad07a, 3.5.1].

STEP 7: DIMENSION ESTIMATE. Now suppose the map $C \to \mathbf{P}^N_{H,\tilde{r}}$ could not be deformed to a map where C has fewer nodes. We calculate the dimension of the component of the space of stable maps in which it lies as follows:

(1) for each $i > 0$, deformations of $C_i \to \mathbf{P}^N_{H,\tilde{r}}$ marked by p_i are unobstructed and therefore have dimension precisely $\tilde{r} b_i N + 1 + N - 3$.
(2) Deformations of $C_0 \to \mathbf{P}^N_{H,\tilde{r}}$ inside H marked by $p_1, \ldots, p_k$ and the m twisted marked points are identical to deformations of $\underline{C}_0 \to \mathbf{P}^{N-1}$ with as maky marked points, and have precisely dimension $\tilde{r}(\sum a_i - \sum b_i)N + m + k + N - 4$.
(3) We have $n-m$ additional untwisted marked points, contributing as many dimensions to moduli.
(4) The conditions for these maps to glue are independent by Kleiman's Bertini argument [Kle74, Theorem 2]. We have $N-1$ independent conditions for each p_i.

We compute:

$$\begin{aligned}
&\tilde{r}(\sum a_i - \sum b_i)N + m + k + N - 4 \\
&\qquad\qquad + \sum_{i=1}^{k} (\tilde{r} b_i N + 1 + N - 3 - (N-1)) + n - m \\
&\qquad = \tilde{r} \sum a_i N + n + k + N - 4 + \sum_{i=1}^{k} (-1) \\
&\qquad = \tilde{r} \sum a_i N + n + N - 4.
\end{aligned}$$

Since the dimension is greater than or equal to the expected dimension, it follows that the map $C \to \mathbf{P}^N_{H,r}$ can be smoothed. We then get the required deformation of (C, L, s) by pulling back $\mathcal{O}(H^{1/r})$ together with its tautological section.

It now remains to prove that $\mathfrak{M}_{0,n}(\mathscr{A})''$ is dense in $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{ord.nodes}}$.

Step 8: smoothing non-orbifold nodes. Now, we show that $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{sm}}$, the open substack of $\mathfrak{M}_{0,n}(\mathscr{A})'$ consisting of (C, L, s) with C smooth, is dense in $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{ord.nodes}}$. By Lemma 4.1.5, there is no obstruction to deforming L as C is smoothed, and the obstruction to deforming s as C and L are deformed lies in $H^1(C, L)$. But since (C, L, s) is in $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{ord.nodes}}$ the vanishing locus of s must be discrete by Lemma 4.3.2. Therefore L must have non-negative degree on every component of C, whence $H^1(C, L) = 0$. The obstruction to carrying s along with the deformation of (C, L) therefore vanishes and (C, L, s) can be deformed into $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{sm}}$.

Step 9: deforming to a nonzero section. Lemma 4.3.1 implies that $\mathfrak{M}_{0,n}(\mathscr{A})''$ is dense in $\mathfrak{M}_{0,n}(\mathscr{A})^{\mathrm{sm}}$, which completes the proof. $\square$

Appendix A. Proofs of standard obstruction results

A.1. Proposition 3.1.10.

Lemma A.1.1. *Suppose $\mathbb{E} \to \mathbb{L} \to \mathbb{F}$ is an exact triangle in $D_{\leq s}(A)$ for some abelian category A. The following are equivalent.*

(i) *the map $H^p(\mathbb{E}) \to H^p(\mathbb{L})$ is an isomorphism for $p > t$ and surjective for $p = t$,*
(ii) *$\mathbb{F}$ is concentrated in degrees $< t$,*
(iii) *for every $J \in A$, the map*
$$\mathrm{Ext}^{-p}(\mathbb{L}, J) \to \mathrm{Ext}^{-p}(\mathbb{E}, J)$$
is an isomorphism for $p > t$ and injective for $p = t$,
(iv) *$\mathrm{Ext}^{-p}(\mathbb{F}, J) = 0$ for all $p \geq t$ and all $J \in A$.*

Proof. (i) $\Leftrightarrow$(ii) follows immediately by considering the long exact sequence in cohomology
$$\cdots \to H^{s-1}(\mathbb{F}) \to H^s(\mathbb{E}) \to H^s(\mathbb{L}) \to H^s(\mathbb{F}) \to 0.$$

Likewise, (iii) $\Leftrightarrow$(iv) follows from the long exact sequence
$$0 \to \mathrm{Ext}^{-s}(\mathbb{F}, J) \to \mathrm{Ext}^{-s}(\mathbb{L}, J) \to \mathrm{Ext}^{-s}(\mathbb{E}, J) \to \mathrm{Ext}^{-s+1}(\mathbb{F}, J) \to \cdots .$$

It is obvious that (ii) implies (iii). For the converse, it is sufficient, by descending induction, to assume that $\mathbb{F} \in D_{\leq t}(A)$. In that case, it follows by taking $J = H^t(\mathbb{F})$. $\square$

Proposition A.1.2. *Suppose that $X \to Y$ is representable morphism of algebraic stacks. Then the morphism $\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)} \to \mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}$ constructed in Section 3.1.4 is an obstruction theory. If $X \to Y$ is smooth then it is of perfect amplitude in $[-1, 0]$.*

For brevity, we enote $\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}$ by $\mathbb{E}$ and $\mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}$ by $\mathbb{L}$. By the Lemma above, it's enough to prove that

$$\mathrm{Ext}^i(\mathbb{L}, J) \to \mathrm{Ext}^i(\mathbb{E}, J)$$

is injective for $i = 1$ and bijective for $i = 0$ and that $\mathbb{E}$ is perfect in degrees $[-1, 0]$ if $X \to Y$ is smooth.

The last property is the easiest: assuming X is smooth over Y, we obtain $\mathbb{E}$ as the dual of the derived pushforward of the vector bundle $f^*T_{X/Y}$, where $f : C \to X$ is the universal map. Since C has one-dimensional fibers over $\mathfrak{M}(X/T)$, we have $H^i(\pi^{-1}s, f^*T_{X/Y}) = 0$ for $i > 1$ and every geometric point s of $\mathfrak{M}(X/T)$. The difference of the ranks of $H^i(\pi^{-1}s, f^*T_{X/Y})$ for $i = 0, 1$ is the Euler characteristic of $f^*T_{X/Y}$, which is locally constant. Therefore $R\pi_*f^*T_{X/Y}$ is perfect of perfect amplitude in $[0, 1]$, hence its dual $\mathbb{E}$ is perfect of perfect amplitude in $[-1, 0]$.

We will reduce the rest of the proof to

Lemma A.1.3. *The maps*

$$\underline{\mathrm{Ext}}^i(\mathbb{L}, J) \to \underline{\mathrm{Ext}}^i(\mathbb{E}, J) \tag{A.1.4}$$

are injective for $i = 1$ and an isomorphism for $i = 0$.

Using the lemma we can prove that $\mathbb{E}$ is an obstruction theory. The hypothesis clearly implies the bijectivity of $\mathrm{Ext}^0(\mathbb{L}, J) \to \mathrm{Ext}^0(\mathbb{E}, J)$. On the other hand, we obtain the following commutative diagram with exact rows from the local-to-global spectral sequence.

$$\begin{tikzcd}
0 \arrow[r] & H^1(\underline{\mathrm{Ext}}^0(\mathbb{L}, J)) \arrow[r] \arrow[d, "A"'] & \mathrm{Ext}^1(\mathbb{L}, J) \arrow[r] \arrow[d, "B"'] & H^0(\underline{\mathrm{Ext}}^1(\mathbb{L}, J)) \arrow[d, "C"'] \\
0 \arrow[r] & H^1(\underline{\mathrm{Ext}}^0(\mathbb{E}, J)) \arrow[r] & \mathrm{Ext}^1(\mathbb{E}, J) \arrow[r] & H^0(\underline{\mathrm{Ext}}^1(\mathbb{E}, J))
\end{tikzcd}$$

Since A is a bijection, the injectivity of C implies the injectivity of B. □

The rest of this section will be devoted to a proof of the lemma. We will interpret the map $\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)} \to \mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}$ in terms of infinitesimal obstructions and deformations. We begin by proving the injectivity of (A.1.4) in the the case $i = 1$.

Let $S \to S'$ be a morphism over $\mathfrak{M}(X/T) \to \mathfrak{M}(Y/T)$, so that S and S' have families of curves C and C' over them, with $C = C' \times_{S'} S$, and there are morphisms $f : C \to X$ and $C' \to Y$. Let $S \to S''$ be an infinitesimal extension over S' with ideal J, inducing an infinitesimal extension $C \to C'' = C' \times_{S'} S''$ with ideal $\pi_S^* J$, where $\pi_S : C \to S$ is the projection. Assume that S' is a scheme and that $S \to S'$ is representable. (This assumption is necessary to apply the results of [Ols06a].)

We study the relationship between the following two lifting problems. Consider first the commutative diagram of solid lines

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d] \\ C'' \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

By [Ols06a, Theorem 1.5] there is an obstruction in $\beta \in \mathrm{Ext}^1(f^* \mathbb{L}_{X/Y}, \pi^* J)$ whose annihilation is equivalent to the existence of a dashed arrow making the diagram commute. If a solution exists then the set of solutions forms a torsor under $\mathrm{Ext}^0(f^* \mathbb{L}_{X/Y}, \pi^* J)$.

Second, we have the diagram

$$\begin{tikzcd} S \arrow[r, "\mathrm{id}"] \arrow[d] & S \arrow[d] \\ S'' \arrow[r] \arrow[ur, dashed] & S'. \end{tikzcd}$$

This lifting problem is obstructed by $\mathrm{Ext}^1(\mathbb{L}_{S/S'}, J)$ and solutions, if any exist, form a torsor under $\mathrm{Ext}^0(\mathbb{L}_{S/S'}, J)$ (again by loc.cit.).

The map $\mathbb{E}_{S/S'} \to \mathbb{L}_{S/S'}$ is induced from the diagram

$$\pi_S^* \mathbb{L}_{S/S'} \xrightarrow{\sim} \mathbb{L}_{C/C'} \leftarrow \mathbb{L}_{C/Y} \leftarrow f^* \mathbb{L}_{X/Y}$$

using the adjunction $(\pi_{S!}, \pi_S^*)$. These induce maps

$$\begin{aligned} \mathrm{Ext}^i(\mathbb{L}_{S/S'}, J) \to \mathrm{Ext}^i(\pi_S^* \mathbb{L}_{S/S'}, \pi_S^* J) &\xleftarrow{\sim} \mathrm{Ext}^i(\mathbb{L}_{C/C'}, \pi_S^* J) \\ &\to \mathrm{Ext}^i(\mathbb{L}_{C/Y}, \pi_S^* J) \to \mathrm{Ext}^i(f^* \mathbb{L}_{X/Y}, \pi_S^* J) \end{aligned}$$

for each i. Let α be an element of $\mathrm{Ext}^1(\mathbb{L}_{S/S'}, J)$.

1. Corresponding to α there is a square-zero thickening $S \to S''$ with ideal J. We can view α as the obstruction to the existence of a lift in the diagram below.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S \arrow[d] \\ S'' \arrow[r] \arrow[ur, dashed] & S' \end{tikzcd}$$

2. By [Ols06a, Lemma 4.12], the image of α in $\mathrm{Ext}^1(\pi_S^* \mathbb{L}_{S/S'}, \pi_S^* J)$ obstructs the lifting problem
$$\begin{tikzcd} C \arrow[r] \arrow[d] & S \arrow[d] \\ C'' \arrow[r] \arrow[ur, dashed] & S'. \end{tikzcd}$$
3. This obstruction is the image of the standard obstruction to the lifting problem
$$\begin{tikzcd} C \arrow[r] \arrow[d] & C \arrow[d] \\ C'' \arrow[r] \arrow[ur, dashed] & C' \end{tikzcd}$$
under the isomorphism $\mathrm{Ext}^1(\mathbb{L}_{C/C'}, \pi_S^* J) \xrightarrow{\sim} \mathrm{Ext}^1(\pi_S^* \mathbb{L}_{S/S'}, \pi_S^* J)$. If C and C' were schemes instead of Deligne–Mumford stacks, this would follow directly from the construction of the obstruction [Ill71, III.2.2.1.2]; the case of Deligne–Mumford stacks can be proved either by following Illusie's construction but working in the étale site instead of the Zariski site, or by following Olsson and reducing to the case of schemes by taking simplicial resolutions. We omit the details.
4. The image of our class along the map $\mathrm{Ext}^1(\pi_S^* \mathbb{L}_{C/C'}, \pi_S^* J) \to \mathrm{Ext}^1(\mathbb{L}_{C/Y}, \pi_S^* J)$ is the obstruction to lifting
$$\begin{tikzcd} C \arrow[r] \arrow[d] & C \arrow[d] \\ C'' \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$
If everything were a scheme, this would be immediate from the construction of the obstruction. It can be proved in our context by working with simplicial resolutions by disjoint unions of affine schemes.
5. And the image of this in $\mathrm{Ext}^1(f^* \mathbb{L}_{X/Y}, \pi_S^* J)$ obstructs the existence of a lift for
$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d] \\ C'' \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$
Again, this is immediate from the definition in the case of schemes and proved in general by a simplicial resolution.

Now, suppose that $S' = \mathfrak{M}(Y/T)$ and that this final obstruction vanishes. Then there is a map $C'' \to X$ lifting the diagram above, so there is a lift for

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathfrak{M}(X/T) \arrow[d] \\ S'' \arrow[r] \arrow[ur, dashed] & \mathfrak{M}(Y/T). \end{tikzcd}$$

Thus the original obstruction α must have been zero. In other words, the map $\mathrm{Ext}^1(\mathbb{L}_{S/\mathfrak{M}(Y/T)}, J) \to \mathrm{Ext}^1(\mathbb{E}_{S/\mathfrak{M}(Y/T)}, J)$ is injective. Applying this with $u : S \to \mathfrak{M}(X/T)$ varying among smooth maps from schemes to $\mathfrak{M}(X/T)$, we find that

$$\underline{\mathrm{Ext}}^1(\mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}, J) \to \underline{\mathrm{Ext}}^1(\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}, J)$$

is injective (since $\underline{\mathrm{Ext}}^1(\mathbb{L}_{S/\mathfrak{M}(Y/T)}, J) \to \underline{\mathrm{Ext}}^1(u^*\mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}, J)$ is an isomorphism, cf. [Ols06a, Lemma 4.7]).

This implies that $H^0(\underline{\mathrm{Ext}}^1(\mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}, J)) \to H^0(\underline{\mathrm{Ext}}^1(\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}, J))$ is injective.

Now we turn to the bijectivity of (A.1.4) in the case $i = 0$. Consider any element $\beta \in \mathrm{Ext}^0(\mathbb{L}_{S/S'}, J)$. We may represent this by a choice of dashed arrow making the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S \arrow[d] \\ S[J] \arrow[r] \arrow[ur, dashed] & S' \end{tikzcd} \tag{A.1.5}$$

commute. Following the chain of diagrams above with S'' replaced by $S[J]$ and C'' replaced by $C[\pi_S^* J]$ we find the image of β in $\mathrm{Ext}^1(f^*\mathbb{L}_{X/Y}, \pi_S^* J)$ to be the induced choice of dashed arrow making

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d] \\ C[\pi_S^* J] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{A.1.6}$$

commute. Now putting $S' = \mathfrak{M}(Y/T)$, any such arrow comes from a unique dashed arrow

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathfrak{M}(X/T) \arrow[d] \\ S[J] \arrow[r] \arrow[ur, dashed] & \mathfrak{M}(Y/T) \end{tikzcd}$$

(since $\mathfrak{M}(X/T)$ is a moduli space!). If S is smooth over $\mathfrak{M}(X/T)$ and affine then this lift is induced from a lift of Diagram (A.1.5). Letting S vary among all affine schemes smooth over $\mathfrak{M}(X/T)$, we deduce that

$$\underline{\mathrm{Ext}}^0(\mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}, J) \to \underline{\mathrm{Ext}}^0(\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}, J) \tag{A.1.7}$$

is surjective.

Suppose finally that the lift of Diagram (A.1.6) factors through the retraction onto C over the retraction of $S[J]$ onto S (i.e., the image of β in $\mathrm{Ext}^0(f^*\mathbb{L}_{X/Y}, \pi_S^* J)$ is zero). Then the map $S[J] \to \mathfrak{M}(X/T)$ factors through S, which is to say that β lies in the image of $\mathrm{Ext}^0(\mathbb{L}_{S/\mathfrak{M}(X/T)}, J)$ inside $\mathrm{Ext}^0(\mathbb{L}_{S/\mathfrak{M}(Y/T)}, J)$. But the exactness of the sequence

$$0 \to \mathrm{Ext}^0(\mathbb{L}_{S/\mathfrak{M}(X/T)}, J) \to \mathrm{Ext}^0(\mathbb{L}_{S/\mathfrak{M}(Y/T)}, J) \to \mathrm{Ext}^0(h^*\mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}, J)$$

where h denotes the map $S \to \mathfrak{M}(X/T)$, implies that the image of β in $\mathrm{Ext}^0(h^*\mathbb{L}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}, J)$ vanishes. This implies that the morphism (A.1.7) is injective and completes the proof that $\mathbb{E}_{\mathfrak{M}(X/T)/\mathfrak{M}(Y/T)}$ is an obstruction theory. □

A.2. **Proposition 3.1.11.** This can be proved directly from the definitions by chasing adjunctions. This is unpleasant, though, so we prefer to interpret Proposition 3.1.11 in deformation theoretic terms.

To keep the notation readable, we will abbreviate

$$\begin{aligned}
\mathbb{E}_{\overline{X}/\overline{Y}} &= \mathbb{E}_{\overline{M}_{g,n}(\overline{X}/T)/\mathfrak{M}_{g,n}(\overline{Y}/T)} \\
\mathbb{E}_{X/Y} &= \mathbb{E}_{\overline{M}_{g,n}(X/T)^*/\mathfrak{M}_{g,n}(Y/\overline{Y}/T)} = \mathbb{E}_{\overline{M}_{g,n}(X/T)^*/\mathfrak{M}_{g,n}(Y/T)} \\
\mathbb{F}_{\overline{X}/\overline{Y}} &= \mathbb{L}_{\overline{M}_{g,n}(\overline{X}/T)/\mathfrak{M}_{g,n}(\overline{Y}/T)} \\
\mathbb{F}_{X/Y} &= \mathbb{F}_{\overline{M}_{g,n}(X/T)^*/\mathfrak{M}_{g,n}(Y/\overline{Y}/T)} = \mathbb{F}_{\overline{M}_{g,n}(X/T)^*/\mathfrak{M}_{g,n}(Y/T)}.
\end{aligned}$$

We'll also omit the $/T$ notation for moduli spaces of maps, since everything is over T (i.e., $\overline{M}_{g,n}(\overline{X})$ will mean $\overline{M}_{g,n}(\overline{X}/T)$).

We wish to show that the diagram

$$\begin{tikzcd}
h^*\mathbb{E}_{\overline{X}/\overline{Y}} \arrow[r, "\sim"] \arrow[d] & \mathbb{E}_{X/Y} \arrow[d] \\
h^*\mathbb{F}_{\overline{X}/\overline{Y}} \arrow[r] & \mathbb{F}_{X/Y}
\end{tikzcd}$$

in $D(\overline{M}_{g,n}(\overline{X}))$ commutes and that the upper horizontal map is an isomorphism. It is equivalent to show that the maps induced on the functors that these objects co-represent are the same. By the naturality of the local-to-global spectral sequence for Ext, it is therefore enough to show that the diagram

$$\begin{tikzcd} \underline{\mathrm{Ext}}^p(\mathbb{F}_{X/Y}, J) \arrow[r] \arrow[d] & \underline{\mathrm{Ext}}^p(h^*\mathbb{F}_{\overline{X}/\overline{Y}}, J) \arrow[d] \\ \underline{\mathrm{Ext}}^p(\mathbb{E}_{X/Y}, J) \arrow[r] & \underline{\mathrm{Ext}}^p(h^*\mathbb{E}_{\overline{X}/\overline{Y}}, J) \end{tikzcd}$$

commutes for all p. Since $\mathbb{E}_{\overline{X}/\overline{Y}}$ has perfect amplitude within $[-1, 0]$, it is sufficient to prove this for $p = 0, 1$.

Since the definition of the obstruction theory is compatible with flat base change, it is enough to prove this locally with respect to the moduli spaces in question. We therefore take S and S' to be affine and smooth over $\overline{M}_{g,n}(X/T)^*$ and $\overline{M}_{g,n}(\overline{X}/T)$, respectively, fitting into a commutative diagram

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "u"'] & S' \arrow[d] \\ \overline{M}_{g,n}(X)^* \arrow[r] \arrow[d] & \overline{M}_{g,n}(\overline{X}) \arrow[d] \\ \mathfrak{M}_{g,n}(Y) \arrow[r] & \mathfrak{M}_{g,n}(\overline{Y}). \end{tikzcd}$$

corresponding to a diagram

$$\begin{tikzcd} & & X \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ C \arrow[r] \arrow[d] \arrow[urr, bend left] & C' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \overline{Y} \arrow[dl] \\ S \arrow[r] & S' \arrow[r] & T & \end{tikzcd}$$

where C and C' are twisted curves over S and S', respectively.

We must now show that the lower square commutes in the diagram

$$\begin{tikzcd} \mathrm{Ext}^p(\mathbb{L}_{S/\mathfrak{M}_{g,n}(Y)}, J) \arrow[r] \arrow[d] & \mathrm{Ext}^p(g^*\mathbb{L}_{S'/\mathfrak{M}_{g,n}(\overline{Y})}, J) \arrow[d] \\ \mathrm{Ext}^p(u^*\mathbb{F}_{X/Y}, J) \arrow[r] \arrow[d] & \mathrm{Ext}^p(u^*h^*\mathbb{F}_{\overline{X}/\overline{Y}}, J) \arrow[d] \\ \mathrm{Ext}^p(u^*\mathbb{E}_{X/Y}, J) \arrow[r] & \mathrm{Ext}^p(u^*h^*\mathbb{E}_{\overline{X}/\overline{Y}}, J). \end{tikzcd}$$

We note, however

1. the vertical maps in the upper square are isomorphisms for $p = 1$ since S and S' are smooth over $\overline{M}_{g,n}(X)^*$ and $\overline{M}_{g,n}(\overline{X})$, respectively, and
2. by the naturality of the construction of $\mathbb{E}_{X/Y}(S)$ (resp. $\mathbb{E}_{\overline{X}/\overline{Y}}(S')$), the vertical maps in the outer rectangle are the ones induced by the maps $\mathbb{E}_{X/Y}(S) \to \mathbb{L}_{S/\mathfrak{M}_{g,n}(Y)}$ (resp. $\mathbb{E}_{\overline{X}/\overline{Y}}(S') \to \mathbb{L}_{S'/\mathfrak{M}_{g,n}(\overline{Y})}$) in Section 3.1.4.

For $p = 1$, everything therefore comes down to showing that the diagram

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathbb{L}_{S/\mathfrak{M}_{g,n}(Y)}, J) \arrow[r] \arrow[d] & \mathrm{Ext}^1(g^*\mathbb{L}_{S'/\mathfrak{M}_{g,n}(\overline{Y})}, J) \arrow[d] \\
\mathrm{Ext}^1(\mathbb{E}_{X/Y}(S), J) \arrow[r] & \mathrm{Ext}^1(\mathbb{E}_{\overline{X}/\overline{Y}}(S), J)
\end{tikzcd}$$

commutes. A class $\alpha \in \mathrm{Ext}^1(\mathbb{L}_{S/S'}, J)$ can be represented by a square-zero extension S'' of S by J over S'. The upper horizontal arrow carries this extension to the (cotangent) obstruction to lifting the diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & S' \arrow[d] \\
S'' \arrow[r] \arrow[ur, dashed] & \mathfrak{M}_{g,n}(\overline{Y})
\end{tikzcd}$$

and the right vertical arrow carries this to the (cotangent) obstruction to lifting

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & C' \arrow[r] & \overline{X} \arrow[d] \\
C'' \arrow[rr] \arrow[urr, dashed] & & \overline{Y}.
\end{tikzcd} \tag{A.2.1}$$

On the other hand, the left vertical arrow takes α to the (cotangent) obstruction to lifting

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & X \arrow[d] \\
C'' \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd} \tag{A.2.2}$$

The (cotangent) obstruction to lifting (A.2.1) is the one induced from the cotangent obstruction to lifting (A.2.2), so this proves what we need for $p = 1$.

The $p = 0$ case is easier. We take $S = \overline{M}(X)$. Then a class $\beta \in \mathrm{Ext}^0(\mathbb{F}_{X/Y}, J)$ is a lift in the diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \overline{M}_{g,n}(Y) \arrow[d] \\
S[J] \arrow[r] \arrow[ur, dashed] & \mathfrak{M}_{g,n}(Y)
\end{tikzcd}$$

corresponding to a diagram

(A.2.3)
$$\begin{tikzcd}
 & & X \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\
C \arrow[r] \arrow[d] \arrow[urr, bend left] & C[\pi^* J] \arrow[r] \arrow[d] \arrow[ur, dashed] & Y \arrow[r] \arrow[d] & \overline{Y} \arrow[dl] \\
S \arrow[r] & S[J] \arrow[r] & T.
\end{tikzcd}$$

The dashed arrow is the image of β via the map to $\mathrm{Ext}^0(\mathbb{E}_{X/Y}, J)$. We obtain the image in $\mathrm{Ext}^0(\mathbb{E}_{\overline{X}/\overline{Y}}, J)$ by forgetting the $X \to Y$ part of the diagram.

On the other hand, the map $\mathrm{Ext}^0(\mathbb{F}_{\overline{X}/\overline{Y}}, J) \to \mathrm{Ext}^0(g^*\mathbb{F}_{\overline{X}/\overline{Y}}, J)$ sends β to the induced lift of

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \overline{M}(\overline{X}) \arrow[d] \\
S[J] \arrow[r] \arrow[ur, dashed] & \mathfrak{M}_{g,n}(\overline{Y}),
\end{tikzcd}$$

which clearly induces the same lift of the diagram

$$\begin{tikzcd}
 & & \overline{X} \arrow[d] \\
C \arrow[r] \arrow[d] \arrow[urr, bend left] & C[\pi^* J] \arrow[r] \arrow[d] \arrow[ur, dashed] & \overline{Y} \arrow[d] \\
S \arrow[r] & S[J] \arrow[r] & T
\end{tikzcd}$$

as Diagram (A.2.3) did. This completes the proof of the commutativity of Diagram (3.1.12).

As for the fact that $h^* E_{\overline{X}/\overline{Y}} \to E_{X/Y}$ is an isomorphism, note that

$$h^* E^\vee_{\overline{X}/\overline{Y}} = R\pi_{S'!}\overline{f}^* T_{\overline{X}/\overline{Y}}$$
$$E^\vee_{X/Y} = R\pi_{S!} f^* T_{X/Y}$$

where $f : C \to X$ and $\overline{f} : C \to \overline{X}$ are the structural maps, and the induced map

$$R\pi_{S!} f^* T_{X/Y} \to R\pi_{S'!}\overline{f}^* T_{\overline{X}/\overline{Y}}$$

is the natural morphism associated to the isomorphism $f^* T_{X/Y} \xrightarrow{\sim} \overline{f}^* T_{\overline{X}/\overline{Y}}$; this is an isomorphism by cohomology and base change.

A.3. **Lemma 4.2.1.** Here we construct an obstruction theory for $\mathfrak{M}(X/T)$ relative to T. First we assume there are no marked points (this is essentially a harmless thing to do since the deformation of marked points is unobstructed). Let $\mathfrak{M}$ denote $\mathfrak{M}(X/T)$ to keep the notation shorter.

Suppose S is affine and $u : S \to \mathfrak{M}$ is smooth. Let $f : C \to X$ and $p : C \to S$ be the corresponding map and curve. We have a commutative diagram in $D(C)$ with distinguished rows and columns:

$$\begin{tikzcd}
f^* p^* \mathbb{L}_T \arrow[r] \arrow[d] & f^* \mathbb{L}_X \arrow[r] \arrow[d] & f^* \mathbb{L}_{X/T} \arrow[d] \\
p^* \mathbb{L}_S \arrow[r] \arrow[d] & \mathbb{L}_C \arrow[r] \arrow[d] & \mathbb{L}_{C/S} \arrow[d] \\
p^* \mathbb{L}_{S/T} \arrow[r] & \mathbb{L}_{C/X} \arrow[r] & \mathbb{L}_{C/X_S}.
\end{tikzcd}$$

Shifting the exact triangle in the bottom row gives a map

$$\mathbb{L}_{C/X_S}[-1] \to p^* \mathbb{L}_{S/T}$$

and so by adjunction we have the map in the derived category

$$\mathbb{E}(S) := p_! \mathbb{L}_{C/X_S}[-1] \to \mathbb{L}_{S/T}.$$

Applying this with $S = \mathfrak{M}$, we get $\mathbb{E} := \mathbb{E}(\mathfrak{M})$. Since in general $\mathbb{L}_{C/X_S}$ is the pullback of $\mathbb{L}_{\mathfrak{C}/X_{\mathfrak{M}}}$, we have $\mathbb{E}(S) = u^* \mathbb{E}(\mathfrak{M})$. We show that $\mathbb{E}(\mathfrak{M}) \to \mathbb{L}_{\mathfrak{M}/T}$ is an obstruction theory using the chart $S \to \mathfrak{M}$.

As in Section A.1, it is sufficient to show that

$$\begin{aligned}
&\operatorname{Ext}^{-1}(u^* \mathbb{L}_{\mathfrak{M}/T}, J) \to \operatorname{Ext}^{-1}(u^* \mathbb{E}, J) && \text{is bijective,} \\
&\operatorname{Ext}^{0}(u^* \mathbb{L}_{\mathfrak{M}/T}, J) \to \operatorname{Ext}^{0}(u^* \mathbb{E}, J) && \text{is bijective, and} \\
&\operatorname{Ext}^{1}(\mathbb{L}_{S/T}, J) \to \operatorname{Ext}^{1}(\mathbb{E}(S), J) && \text{is injective.}
\end{aligned}$$

For the second of these, we represent a class $\alpha \in \mathrm{Ext}^1(\mathbb{L}_{S/T}, J)$ by a square-zero extension S' of S over T with ideal J. Suppose that the class induced by α in $\mathrm{Ext}^1(\mathbb{E}(S), J) = \mathrm{Ext}^2(\mathbb{L}_{C/X_S}, p^*J)$ is zero.

The map

$$\mathrm{Ext}^1(\mathbb{L}_{S/T}, J) \to \mathrm{Ext}^1(p^*\mathbb{L}_{S/T}, p^*J)$$

sends α to the class $p^*\alpha$ representing the extension of ringed topoi (in the étale site of C)

$$(C, p^{-1}\mathcal{O}_S) \to (C, p^{-1}\mathcal{O}_{S'}) \tag{A.3.1}$$

over T that is induced by pullback. Our assumption on α implies that the image of $p^*\alpha$ in $\mathrm{Ext}^1(\mathbb{L}[-1], J)$ is zero. Thus $p^*\alpha$ is the image of some $\alpha' \in \mathrm{Ext}^1(\mathbb{L}_{C/X}, J)$. But $\mathrm{Ext}^1(\mathbb{L}_{C/X}, J)$ is the group of deformations

$$\begin{tikzcd} C \arrow[r, dashed] \arrow[d] & C' \arrow[dl, dashed] \\ X & \end{tikzcd}$$

The image of α' in $\mathrm{Ext}^1(p^*\mathbb{L}_{S/T}, p^*J)$ is the extension (A.3.1) obtained composition with the map $X \to T$, followed by pushout via the map of ringed topoi $(C, \mathcal{O}_C) \to (C, p^{-1}\mathcal{O}_S)$. Thus to say that $p^*\alpha$ is the image of some α' means precisely that there is a diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] & S' \arrow[r] & T. \end{tikzcd}$$

Thus the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathfrak{M} \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$

has a lift. Since α obstructs the existence of such a lift, we conclude that α is zero. This proves that $\mathrm{Ext}^1(\mathbb{L}_{S/\mathscr{T}}, J) \to \mathrm{Ext}^1(\mathbb{E}(S), J)$ is injective.

For the surjectivity in Ext^0, suppose that

$$\beta \in \mathrm{Ext}^0(u^*\mathbb{E}, J) = \mathrm{Ext}^1(f^*\mathbb{L}_{\mathfrak{C}/X_{\mathfrak{M}}}, p^*J) = \mathrm{Ext}^1(\mathbb{L}_{C/X_S}, p^*J).$$

This induces a class in $\mathrm{Ext}^1(\mathbb{L}_{C/X}, p^*J)$ whose image in $\mathrm{Ext}^1(p^*\mathbb{L}_{S/T}, p^*J)$ is trivial. This corresponds to a diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \\ C' \arrow[ur] & \end{tikzcd}$$

where C' is a square-zero extension of C with ideal p^*J. As above, the map $\mathrm{Ext}^0(\mathbb{L}_{C/X}, p^*J) \to \mathrm{Ext}^1(p^*\mathbb{L}_{S/T}, p^*J)$ factors through $\mathrm{Ext}^0(\mathbb{L}_{C/T}, p^*J)$, sending the diagram above to the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & T \\ C' \arrow[ur] & \end{tikzcd}$$

This is in turn carried to the extension of ringed topoi

$$\begin{tikzcd} (C, p^{-1}\mathcal{O}_S) \arrow[r] \arrow[d] & T \\ (C, \mathscr{B}) \arrow[ur] & \end{tikzcd}$$

obtained by pushout. By assumption, this class is trivial, so the oblique arrow factors through a section of the vertical arrow. In particular, $\mathscr{B} \cong p^{-1}\mathcal{O}_S + p^*J$ and we have a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] & S[J] \arrow[r] & T. \end{tikzcd}$$

inducing a lift of the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathfrak{M} \arrow[d] \\ S[J] \arrow[r] \arrow[ur, dashed] & T. \end{tikzcd}$$

This corresponds to a class in $\mathrm{Ext}^0(u^*\mathbb{L}_{\mathfrak{M}/T}, J)$ that induces β. Thus the map $\mathrm{Ext}^0(u^*\mathbb{L}_{\mathfrak{M}/T}, J) \to \mathrm{Ext}^0(u^*\mathbb{E}, J)$ is surjective.

For the injectivity, note that $\gamma \in \operatorname{Ext}^0(u^*\mathbb{L}_{\mathfrak{M}/\mathscr{T}}, J)$ corresponds to a lift of the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathfrak{M} \arrow[d] \\ S[J] \arrow[r, "0"] \arrow[ur, dashed] & T. \end{tikzcd}$$

The image $p^*\gamma$ in $\operatorname{Ext}^0(p^*u^*\mathbb{L}_{\mathfrak{M}/T}, p^*J)$ is the class of the lift of the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathfrak{M} \arrow[d] \\ C[p^*J] \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$

induced by composition. The hypothesis that the image of $p^*\gamma$ in $\operatorname{Ext}^0(\mathbb{L}_{C/X_S}[-1], J)$ be zero implies that it is induced from some $\gamma' \in \operatorname{Ext}^0(\mathbb{L}_{\mathfrak{C}/X}, J)$, which means that there exists a dashed arrow completing the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C[p^*J] \arrow[r, dashed] \arrow[d] \arrow[rr, bend left, "0"] & \mathfrak{C} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] & S[J] \arrow[r] \arrow[rr, bend right, "0"'] & \mathfrak{M} \arrow[r] & T. \end{tikzcd}$$

Thus the map $S[J] \to \mathfrak{M}$ is isomorphic to the zero tangent vector at $u : S \to \mathfrak{M}$. This shows that γ is zero, so that $\operatorname{Ext}^0(u^*\mathbb{L}_{\mathfrak{M}/T}, J) \to \operatorname{Ext}^0(u^*\mathbb{E}_{\mathfrak{M}/T}, J)$ is injective.

Since X may be of Artin type over T, we still need to show that $\operatorname{Ext}^{-1}(u^*\mathbb{L}_{\mathfrak{M}/T}, J) \to \operatorname{Ext}^{-1}(u^*\mathbb{E}_{\mathfrak{M}/T}, J)$ is bijective. Take the surjectivity first. A class

$$\delta \in \operatorname{Ext}^{-1}(u^*\mathbb{E}_{\mathfrak{M}/T}, J) = \operatorname{Ext}^{-1}(\mathbb{L}_{\mathfrak{C}/X_{\mathfrak{M}}}[-1], p^*J) = \operatorname{Ext}^0(\mathbb{L}_{\mathfrak{C}/X_{\mathfrak{M}}}, p^*J)$$

induces a class in $\operatorname{Ext}^0(f^*\mathbb{L}_{\mathfrak{C}/X}, J)$. This class can be viewed as a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C[p^*J] \arrow[d, "0"] \\ C[p^*J] \arrow[r, "0"'] \arrow[ur, "\mathrm{id}"] & X \end{tikzcd}$$

(in other words, a 2-automorphism of the zero map $C[p^*J] \to X$). Since the image of this class in $\operatorname{Ext}^0(f^*\pi^*\mathbb{L}_{\mathfrak{M}/T}, p^*J)$ is zero, we deduce that there is a commutative diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & C[p^*J] \arrow[r, "\mathrm{id}"'] \arrow[d] \arrow[rr, bend left] & C[p^*J] \arrow[r, "0"'] \arrow[d] & X \arrow[d] \\
S \arrow[r] & S[J] \arrow[r, "\mathrm{id}"] \arrow[rr, bend right, "0"'] & S[J] \arrow[r, "0"] & T
\end{tikzcd}$$

where the 2-morphism making the triangle on the bottom commute is the identity. This means precisely that we have a 2-automorphism of the zero map $S[J] \to \mathfrak{M}$ such that the induced 2-automorphism of the zero map $S[J] \to T$ is the identity. This gives us a class in $\mathrm{Ext}^{-1}(u^*\mathbb{L}_{\mathfrak{M}/T}, J)$.

Finally, we check the injectivity of $\mathrm{Ext}^{-1}(u^*\mathbb{L}_{\mathfrak{M}/T}, J) \to \mathrm{Ext}^{-1}(u^*\mathbb{E}, J)$. Given $\epsilon \in \mathrm{Ext}^{-1}(u^*\mathbb{L}_{\mathfrak{M}/T}, J)$ we have, as above, a 2-automorphism of the zero map $S[J] \to \mathfrak{M}$ inducing the identity on the zero map $S[J] \to T$. This induces a 2-automorphism of the zero map $C[p^*J] \to \mathfrak{M}$ inducing a the identity 2-automorphism over T. This corresponds to the image of ϵ in $\mathrm{Ext}^{-1}(f^*\pi^*\mathbb{L}_{\mathfrak{M}/T}, p^*J) = \mathrm{Ext}^{-1}(p^*u^*\mathbb{L}_{\mathfrak{M}/T}, p^*J)$. If $p^*\epsilon$ induces the zero class in $\mathrm{Ext}^{-1}(u^*\mathbb{E}, J) = \mathrm{Ext}^{0}(f^*\mathbb{L}_{\mathfrak{C}/X_{\mathfrak{M}}}, p^*J)$ then it is the image of some $\epsilon' \in \mathrm{Ext}^{-1}(\mathbb{L}_{\mathfrak{C}/X}, p^*J)$. This corresponds to a 2-automorphism of the zero map $C[p^*J] \to \mathfrak{C}$ inducing the trivial 2-automorphism of the zero map $C[p^*J] \to X$. Since this class induces $p^*\epsilon$, the diagram

$$\begin{tikzcd}
C[p^*J] \arrow[r, "\mathrm{id}"'] \arrow[d] \arrow[rr, bend left] & C[p^*J] \arrow[r, "0"'] \arrow[d] & X \arrow[d] \\
S[J] \arrow[r, "\mathrm{id}"] \arrow[rr, bend right] & S[J] \arrow[r, "0"] & T
\end{tikzcd}$$

is (2-)commutative (the 2-morphisms making the upper and lower triangles commute are the respective identity arrows). Thus the 2-automorphism ϵ of the zero $S[J] \to \mathfrak{M}$ that induces the above diagram must be the identity, and we are done.

A.4. **Compatibility of obstruction theories in triangles.** The purpose of this subsection is to prove Proposition A.4.1, comparing absolute and relative virtual fundamental classes. This would be a well-known fact if $\mathfrak{M}(\mathscr{A})$ were smooth. It is not smooth, but the image of $\overline{M}_{0,n}(X)$ in $\mathfrak{M}(\mathscr{A})$ is LCI, so we are able to show that the two classes agree.

The following is in Proposition A.1 in [BL00] (see also [KKP03, Theorem 1], or [Man08]).

Proposition A.4.1. *Suppose $X \xrightarrow{p} Y \xrightarrow{q} Z$ is a sequence of morphisms of Artin stacks, p and qp are of DM type, q is smooth. Suppose that the morphisms have compatible obstruction theories and the obstruction theory of q coincides with the cotangent complex. Then $p^! q^! = (qp)^!$.*

Proof. We have a commutative diagram on X:

$$\begin{array}{ccc} \mathfrak{C}_{X/Y} & \longrightarrow & \mathfrak{C}_{X/Z} \\ \downarrow & & \downarrow \\ \mathfrak{E}_{X/Y} & \longrightarrow & \mathfrak{E}_{X/Z}. \end{array}$$

It suffices to show that this diagram is Cartesian, for $[X/Y]^{\mathrm{vir}} = 0^!_{\mathfrak{E}_{X/Y}}[\mathfrak{C}_{X/Y}]$ and $[X/Z]^{\mathrm{vir}} = 0^!_{\mathfrak{E}_{X/Z}}[\mathfrak{C}_{X/Z}]$. By the compatibility of the obstruction theories with the canonical obstruction theory, the fiber of $\mathfrak{E}_{X/Y} \to \mathfrak{E}_{X/Z}$ is $p^* T_{Y/Z}$; a straightforward calculation will show that $\mathfrak{C}_{X/Y} \to \mathfrak{C}_{X/Z}$ is surjective with fiber $p^* T_{Y/Z}$, which proves that $\mathfrak{C}_{X/Y} \to \mathfrak{C}_{X/Z} \times_{\mathfrak{E}_{X/Z}} \mathfrak{E}_{Y/Z}$ is an isomorphism. ☐

In general, if $Y \to Z$ is any morphism, the relative obstruction theory for $\mathfrak{M}(Y) \to \mathfrak{M}(Z)$ is given by

$$\mathrm{Ext}(\mathfrak{C}(Y)/\mathfrak{M}(Y); f^* L_{Y/Z}, \mathcal{O}_{\mathfrak{C}(Y)})^\vee.$$

If $Y \to Z$ is smooth, then this is perfect in $[-1, 0]$.

Proposition A.4.2. *Let $X \xrightarrow{p} Y \xrightarrow{q} Z$ be a sequence of smooth morphisms (could be lci or maybe even general?). Then the relative obstruction theories in the sequence $\mathfrak{M}(X) \to \mathfrak{M}(Y) \to \mathfrak{M}(Z)$ are compatible.*

Proof. Consider the diagram,

$$\begin{array}{ccccc} X & \xrightarrow{p} & Y & \xrightarrow{q} & Z \\ {\scriptstyle f}\uparrow & & {\scriptstyle g}\uparrow & & \uparrow \\ \mathfrak{C}(X) & \xrightarrow{\mathfrak{C}(f)} & \mathfrak{C}(Y) & \xrightarrow{\mathfrak{C}(g)} & \mathfrak{C}(Z) \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow \\ \mathfrak{M}(X) & \xrightarrow{\mathfrak{M}(f)} & \mathfrak{M}(Y) & \xrightarrow{\mathfrak{M}(g)} & \mathfrak{M}(Z). \end{array}$$

If ϕ is a morphism with a relative dualizing complex ω, denote by $\phi_!$ the functor $A \mapsto R\phi_*(A \otimes \omega)$. This functor is left adjoint to $L\phi^*$.

Adjunction gives a canonical map $\alpha_! C(p)^* \mathbb{F} \to \mathfrak{M}(p)^* \beta_! \mathbb{F}$ for any complex of coherent sheaves, $\mathbb{F}$, on $\mathfrak{C}(Y)$. If $\mathbb{F}$ is a perfect complex then so is $\omega \otimes \mathbb{F}$, so this map is an isomorphism by cohomology and base change (Mumford, Chapter II, §5, Theorem).

On $\mathfrak{C}(X)$, we have a commutative diagram with exact rows,

$$\begin{tikzcd}
f^*p^*L_{Y/Z} \arrow[r] \arrow[d, equal] & f^*L_{X/Z} \arrow[r] \arrow[dd] & f^*L_{X/Y} \arrow[dd] \\
\mathfrak{C}(p)^*g^*L_{Y/Z} \arrow[d] & & \\
\mathfrak{C}(p)^*L_{\mathfrak{C}(Y)/\mathfrak{C}(Z)} \arrow[r] & L_{\mathfrak{C}(X)/\mathfrak{C}(Z)} \arrow[r] & L_{\mathfrak{C}(X)/\mathfrak{C}(Y)} \\
\mathfrak{C}(p)^*\beta^*L_{\mathfrak{M}(Y)/\mathfrak{M}(Z)} \arrow[u, "\sim"] \arrow[d, equal] & & \\
\alpha^*\mathfrak{M}(p)^*L_{\mathfrak{M}(Y)/\mathfrak{M}(Z)} \arrow[r] & \alpha^*L_{\mathfrak{M}(X)/\mathfrak{M}(Z)} \arrow[r] \arrow[uu] & \alpha^*L_{\mathfrak{M}(X)/\mathfrak{M}(Z)} \arrow[uu]
\end{tikzcd}$$

The vertical arrows in the bottom half of the diagram are equivalences. Applying the adjunction $(\pi_!, \pi^*)$, we get the commutative diagram with exact rows,

$$\begin{tikzcd}
\alpha_!\mathfrak{C}(p)^*g^*L_{Y/Z} \arrow[r] \arrow[d] & \alpha_!f^*L_{X/Z} \arrow[r] \arrow[d] & \alpha_!f^*L_{X/Y} \arrow[d] \\
L_{\mathfrak{M}(Y)/\mathfrak{M}(Z)} \arrow[r] & L_{\mathfrak{M}(X)/\mathfrak{M}(Z)} \arrow[r] & L_{\mathfrak{M}(X)/\mathfrak{M}(Y)}.
\end{tikzcd}$$

As already noted, cohomology and base change an equivalence $\alpha_!\mathfrak{C}(p)^*g^*L_{Y/Z} \simeq \mathfrak{M}(p)^*\beta_!g^*L_{Y/Z}$. Moreover, the diagram,

$$\begin{tikzcd}
\alpha_!\mathfrak{C}(p)^*g^*L_{Y/Z} \arrow[rr] \arrow[dr] & & \mathfrak{M}(p)^*\beta_!g^*L_{Y/Z} \arrow[dl] \\
& \mathfrak{M}(p)^*L_{\mathfrak{M}(Y)/\mathfrak{M}(Z)} &
\end{tikzcd}$$

commutes. To prove that this diagram commutes, it is sufficient to show that the diagram,

$$\begin{tikzcd}
\alpha_!\alpha^*\mathfrak{M}(p)^*F \arrow[rr] \arrow[dr] & & \mathfrak{M}(p)^*\beta_!\beta^*F \arrow[dl] \\
& \mathfrak{M}(p)^*F &
\end{tikzcd}$$

commutes. But $\alpha_!\alpha^*\mathfrak{M}(p)^*F$ can be identified with $\alpha_*\omega_{\mathfrak{C}(X)/\mathfrak{M}(X)} \otimes \mathfrak{M}(p)^*F$ and $\mathfrak{M}(p)^*\beta_!\beta^*F$ can be identified with $\mathfrak{M}(p)^*(F \otimes \omega_{\mathfrak{C}(Y)/\mathfrak{M}(Y)})$. Under these identifications, the above map is simply the canonical isomorphism

$$\alpha_* \omega_{\mathfrak{C}(X)/\mathfrak{M}(X)} \otimes \mathfrak{M}(p)^* F \xrightarrow{\sim} \mathfrak{M}(p)^* (\beta_* \omega_{\mathfrak{C}(Y)/\mathfrak{M}(Y)} \otimes F)$$

(using $\mathfrak{C}(p)^* \omega_{\mathfrak{C}(Y)/\mathfrak{M}(Y)} = \omega_{\mathfrak{C}(X)/\mathfrak{M}(X)}$ and the isomorphism $\mathfrak{M}(p)^* \beta_* \omega_{\mathfrak{C}(Y)/\mathfrak{M}(Y)} \simeq \alpha_* \mathfrak{C}(p)^* \omega_{\mathfrak{C}(Y)/\mathfrak{M}(Y)}$ by cohomology and base change). □

Appendix B. Obstruction theories and local complete intersections

Proposition B.1. *The map $\mathfrak{M}(\mathscr{A})' \to \mathfrak{M}(B\mathbf{G}_m)$ is a local complete intersection morphism.*

This proposition is not used in the proof of Theorem 2.2, but the lemma we use to prove it seems to be of independent interest.

Lemma B.2. *Let $\mathcal{M} \to \mathcal{N}$ be a representable, finite type morphism of locally Noetherian algebraic stacks and let $\mathbb{E} \to \mathbb{L}_{\mathcal{M}/\mathcal{N}}$ be a perfect relative obstruction theory. Suppose that $\mathcal{N}$ is smooth and that generically, $h^{-1}(\mathbb{E}) = 0$. Then $\mathbb{E} \to \mathbb{L}_{\mathcal{M}/\mathcal{N}}$ is an isomorphism, and in particular, $\mathcal{M} \to \mathcal{N}$ is a local complete intersection morphism.*

Proof. We begin by reducing to the case where $\mathcal{M} \to \mathcal{N}$ is an embedding of affine schemes. It suffices to prove the lemma after a base change by a smooth presentation $V \to \mathcal{N}$. Under such a base change, $\mathbb{E} \to \mathbb{L}_{\mathcal{M}/\mathcal{N}}$ pulls back to a perfect relative obstruction theory on $\mathcal{M} \times_{\mathcal{N}} V \to V$. So we may assume that $\mathcal{N} = \operatorname{Spec} S$ is an affine noetherian scheme. Now it suffices to prove the lemma after an étale base change $U \to \mathcal{M}$, where U is an affine scheme of finite type over S.

Let $\iota : U \to W$ be an embedding into an affine scheme $W = \operatorname{Spec} A$ which is smooth over $\mathcal{N}$. Let I be the ideal of U in W. Since $\iota^* \mathbb{L}_{W/\mathcal{N}}$ is a vector bundle in degree 0 and $h^0(E) \to h^0(\mathbb{L}_{U/\mathcal{N}})$ is an isomorphism, $\iota^* \mathbb{L}_{W/\mathcal{N}} \to \mathbb{L}_{U/\mathcal{N}}$ lifts uniquely to $\iota^* \mathbb{L}_{W/\mathcal{N}} \to \mathbb{E}$. Let $\mathbb{F} = Cone(\iota^* \mathbb{L}_{W/\mathcal{N}} \to \mathbb{E})$. Then we have a morphism of distinguished triangles:

$$\begin{tikzcd}
\iota^* L_{W/\mathcal{N}} \arrow[r] \arrow[d, "="] & \mathbb{E} \arrow[r] \arrow[d] & \mathbb{F} \arrow[r] \arrow[d] & \iota^* \mathbb{L}_{W/\mathcal{N}}[1] \arrow[d, "="] \\
\iota^* \mathbb{L}_{W/\mathcal{N}} \arrow[r] & \mathbb{L}_{U/\mathcal{N}} \arrow[r] & \mathbb{L}_{U/W} \arrow[r] & \iota^* \mathbb{L}_{W/\mathcal{N}}[1].
\end{tikzcd}$$

By taking long exact sequences, we see that $\mathbb{F}$ is represented by a vector bundle in degree -1 which surjects onto $h^{-1}(\mathbb{L}_{U/W}) = I/I^2$.

The assumption that $h^{-1}(\mathbb{E})$ is generically 0 implies that there is a dense open subset of U over which $\mathbb{F}^{-1} \to I/I^2$ is an isomorphism.

By restricting to a smaller open set, we may assume that $\mathbb{F}^{-1}$ is free of rank d. Then a basis of $\mathbb{F}$ determines elements $x_1, \dots, x_d \in I$ which generate I modulo I^2. In other words $I/(x_1, \dots, x_d)$ is generated by the image of I^2. Thus $I \cdot I/(x_1, \dots, x_d) = I/(x_1, \dots, x_d)$ and Nakayama's lemma implies that there is an element $a \in A$ such that $a \equiv 1$ modulo I and $aI \subseteq (x_1, \dots, x_d)$ [Mat89, 2.2]. Since a does not vanish on U, we may invert a and assume that $I = (x_1, \dots, x_d)$. To show that $x_1, \dots, x_d$ is a regular sequence, it suffices to show that $\operatorname{depth}(I, A) = d$ [Mat89, p.131].

By assumption, U has a dense open set which is a local complete intersection. It follows that d is the codimension of U in W. But any proper ideal I of a Cohen-Macaulay ring A has $\operatorname{depth}(I, A) = \operatorname{ht}(I)$ [Mat89, 17.4], so U is a local complete intersection and I/I^2 is free with basis $x_1, \dots, x_d$. This shows that $F^{-1} \to I/I^2$ is an isomorphism, which implies that $\mathbb{E} \to \mathbb{L}_{U/\mathcal{N}}$ is an isomorphism. □

Proof of Proposition B.1. By the lemma and the definition of $\mathbb{E} \to \mathbb{L}_P$, it suffices to show that $H^1(C_{\overline{s}}, L_{\overline{s}}) = 0$ for a general geometric point $\overline{s} \to \mathfrak{M}(\mathscr{A})'$. If $C_{\overline{s}}$ is smooth, with coarse moduli space $\underline{C}$, then it follows from the definition of $\mathfrak{M}(\mathscr{A})'$ that the pushforward of $L_{\overline{s}}$ to $\underline{C}$ is $\mathscr{O}_{\underline{C}}$. In particular, $H^1(C_{\overline{s}}, L_{\overline{s}}) = 0$ in this case. So it suffices to show that $C_{\overline{s}}$ is smooth for general $\overline{s}$, which is Proposition 4.3.4. □

Appendix C. Notation index

$\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i, k_i) \right\rangle_{0,\beta}^{(X,D)}$	relative GW invariant	1.1.0, p.3
$\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i, k_i) \right\rangle_{0,\beta}^{\mathscr{X}_r}$	orbifold GW invariant	1.1.0, p.3
$\overline{M}^{\mathrm{rel}}(X, D)$	moduli of relative stable maps	1.2.0, p.3
$\overline{M}^{\mathrm{orb}}(\mathscr{X}_r)$	moduli of orbifold stable maps	1.2.0, p.4
$\underline{I}(\mathscr{X}_r)$	coarse mod. sp. of inertia stack	1.2.0, p.4
$\overline{M}^{\mathrm{relorb}}(\mathscr{X}_r, \mathscr{D}_r)$	relative orbifold moduli space	2.1.0, p.6
$\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i, k_i) \right\rangle_{0,\beta}^{(\mathscr{X}_r,\mathscr{D}_r)}$	relative orbifold GW invariant	2.1.0, p.6
$\mathfrak{M}^{\mathrm{rel}}(\mathscr{A}, \mathcal{B}\mathbb{G}_m)'$	special open subset of $\mathfrak{M}(\mathscr{A}, \mathcal{B}\mathbb{G}_m)$	2.3.0, p.8
$\mathfrak{M}(\mathscr{A})'$	special open subset of $\mathfrak{M}(\mathscr{A})$	2.3.0, p.8
$\mathscr{A}$	moduli of line bundle with section	2.3.0, p.8

$\mathfrak{M}(\mathscr{A})$	curves with line bundle and section	2.3.0, p.8
$\mathfrak{M}(\mathscr{A}^{\mathrm{rel}}/\mathscr{T})$	curves with line bundle, section, and expansion	2.3.0, p.8
$\overline{M}(X/X'_T/T)$	substack of $\mathfrak{M}(X/X'_T/T)$ with unramified inertia relative to $\mathfrak{M}(X'/T')$	3.2.0, p.15
$\mathfrak{M}(X/T)^*$	substack of curves $\mathfrak{M}(X/T)$ admitting a contraction	3.2.0, p.16
$\mathscr{T}$	J. Li's moduli space of targets	3.3.1, p.17
$\mathfrak{M}_{0,n}(\mathscr{A})'$	special open substack of $\mathfrak{M}_{0,n}(\mathscr{A})'$	3.3.2, p.18

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