# Complete Moduli for Fibered Surfaces

**Dan Abramovich***
**Angelo Vistoli****

## 1 Introduction

All the schemes with which we work will be schemes over the field **Q** of rational numbers.[1]

Most of the results contained here are particular cases of our general theory of stable maps into algebraic stacks, announced in [ℵ-V1]; the details will appear in [ℵ-V2]. Here we include a rather thorough discussion of our particular case, even when it partially overlaps with the above-mentioned papers; however, the proof of the main theorem, which is very long and technical, is omitted.

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### *1.1 The problem*

Fix a base field $k$ of characteristic 0, and let $C$ be a smooth, projective, geometrically integral curve of genus $g$. By a *fibered surface over* $C$ we mean a morphism $X \to C$, with sections $\sigma_1, \dots, \sigma_\nu : C \to X$ forming a family of stable $\nu$-pointed curves of some genus $\gamma$. (This notion will be generalized below for singular $C$.)

A fibered surface naturally corresponds to a morphism $C \to \overline{\mathcal{M}}_{\gamma,\nu}$ into the moduli stack of stable $\nu$-pointed curves of genus $\gamma$. In [Vistoli2], the second author showed that $Hom(C, \overline{\mathcal{M}}_{\gamma,\nu})$ has the structure of a Deligne–Mumford stack. This stack is, in general, clearly not complete. A natural question to ask is, can one complete it in a meaningful way? Moreover, can one do this as the source curve $C$ itself moves in a family?

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[1] The theory below could be extended to the case that the characteristic is large with respect to the degrees and genera of the curves involved, but we will not pursue this here.

## *1.2 Our approach*

One construction which may spring to mind is that of *stable maps*. Recall that if $Y \subset \mathbf{P}^r$ is a projective variety, there is a complete Deligne–Mumford stack $\overline{\mathcal{M}}_g(Y,d)$ parametrizing Kontsevich stable maps of degree $d$ from curves of genus $g$ to $Y$ (see [Kont], [B-M], [F-P]). In order to reduce the possibility of later confusion, we will use the notation $\mathcal{K}_g^d(Y)$ rather than $\overline{\mathcal{M}}_g(Y,d)$ for this stack. It admits a projective coarse moduli space $\mathbf{K}_g^d(Y)$.

For instance, if $\gamma = 0$, the moduli stack $\overline{\mathcal{M}}_{0,\nu}$ is actually a projective variety. Fixing a projective embedding, we have natural, complete Deligne–Mumford stacks $\mathcal{K}_g^d(\overline{\mathcal{M}}_{0,\nu})$ parametrizing families of stable $\nu$-pointed curves of genus 0 over nodal curves of genus $g$, with a suitable stability condition. We can think of $d$, the degree of the image of $C$ via the fixed projective embedding, as an additional measure of complexity of the family of curves. One is led to ask, is there a complete Deligne–Mumford stack of stable maps $\mathcal{K}_g^d(\overline{\mathcal{M}}_{\gamma,\nu})$ in general?

We need to define our terms: given a nodal curve $C$, define a morphism $f : C \to \overline{\mathcal{M}}_{\gamma,\nu}$ to be a stable map of degree $d$ if, once projected into the coarse moduli space, we obtain a stable map $f' : C \to \overline{\mathbf{M}}_{\gamma,\nu}$ of degree $d$:

$$\begin{array}{ccc} C & \xrightarrow{f} & \overline{\mathcal{M}}_{\gamma,\nu} \\ & \searrow^{f'} & \downarrow \\ & & \overline{\mathbf{M}}_{\gamma,\nu} \end{array}$$

As we will see later, these stable maps are parametrized by a Deligne–Mumford stack. However, this stack fails to perform its first goal: it is not always complete! This can be seen via the following example:

Consider the case $\gamma = 1, \nu = 1$ of elliptic curves. It is well known (and easily follows from the formula of Grothendieck–Riemann–Roch) that given a family of stable elliptic curves $E \to C$ over a curve $C$, the degree of the corresponding moduli map $j_E : C \to \overline{\mathbf{M}}_{1,1} \simeq \mathbf{P}^1$ is divisible by 12. However, it is very easy to construct a family of moduli maps $j_t : C_t \to \mathbf{P}^1, \quad t \neq 0$, where when $t \to 0$ the curve $C_t$ breaks into two components, and the degrees of the limit map $j_0$ on these components are not divisible by 12. Thus $j_0$ is not a moduli map of a family of stable curves.

## *1.3 Compactifying the space of stable maps*

Our main goal in this paper is to correct this deficiency. In order to do so, we will enlarge the category of stable maps into $\overline{\mathcal{M}}_{\gamma,\nu}$. The source curve $\mathcal{C}$ of a new stable map $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ will acquire an orbispace structure at its nodes. Specifically, we endow it with the structure of a Deligne–Mumford stack.

To see how these come about, consider again the example of a one-parameter family of elliptic curves sketched above. Let $C \to S$ be the one-parameter family of base curves, and let $C_{\rm sm}$ be the smooth locus of this morphism. A fundamental purity lemma (see **??** below) will show that, after a suitable base change, we can extend the family $E$ of stable elliptic curves over $C_{\rm sm}$. On the other hand, if $p \in C$ is a node, then on an étale neighborhood $U$ of $p$, the curve $C$ looks like

$$uv = t^r,$$

where $t$ is the parameter on the base. By taking $r$-th roots,

$$u = u_1^r;\ v = v_1^r$$

we have a nonsingular covering $V_0 \to U$ where $u_1 v_1 = t$. The fundamental purity lemma applies to $V_0$, so the pullback of $E$ to $V_0$ extends over all of $V_0$. There is a minimal intermediate covering $V_0 \to V \to U$ such that the family extends over $V$. This $V$ gives the orbispace structure $\mathcal{C}$ over $C$.

The precise definitions will follow in section 4. We can define stable maps $f : \mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ where $\mathcal{C}$ is a Deligne–Mumford stack and $f$ is a representable morphism satisfying certain properties; such objects naturally form a 2-category. Our main theorem states that this 2-category is equivalent to a 1-category $\mathcal{F}_g^d(\gamma,\nu)$, which we call *the category of fibered surfaces*; this category forms a complete Deligne–Mumford stack (over schemes over $\mathbf{Q}$), admitting a projective coarse moduli space. Thus the original moduli problem has a solution of the same nature as that of stable curves.

We provide an explicit description of the category $\mathcal{F}_g^d(\gamma,\nu)$ of fibered surfaces in terms of charts and atlases over schemes, in analogy to Mumford's treatment of $\mathbf{Q}$-varieties in [Mum]. Thus given a stable map $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ we have a stable pointed curve $\mathcal{X} \to \mathcal{C}$, which can be described via an atlas of charts over the associated coarse moduli scheme $X \to C$.

## *1.4 Comparison with Alexeev's work*

The latter, coarse object $X \to C$, which we call a *coarse fibered surface*, has another interpretation: the associated morphism $X \to \overline{\mathbf{M}}_{\gamma,\nu}$ turns out to be a stable map in the sense of Alexeev (see [Al3]). Alexeev has shown the existence of complete moduli of (smoothable) surface stable maps, using the theory of semi-log-canonical surfaces developed in [K-SB] and [Al2]. In the last section we show that, in the case of fibered surfaces, one can use Alexeev's approach to obtain a space which coincides with the space of (smoothable) stable coarse fibered surfaces. We sketch a proof of the existence of the latter space (even as a stack) which is independent of Alexeev's boundedness result.

### *1.5 Natural generalizations in forthcoming work*

It should be evident from our work that this approach should apply to stable maps $\mathcal{C} \to \overline{\mathcal{M}}$ into any Deligne–Mumford stack which admits a projective coarse moduli scheme; moreover, the source “curve” $\mathcal{C}$ should be allowed to be “pointed” as well. See [ℵ-V1] for a discussion of the general setup and various applications that will be worked out in [ℵ-V2] and in [ℵ-C-J-V].

### *1.6 Gromov–Witten invariants*

Originally, the Kontsevich spaces of stable maps were introduced for the purpose of defining Gromov–Witten invariants. It seems likely that, using our construction, one could extend the formalism of [BF] and [B] of virtual fundamental classes to the case of stable maps into an arbitrary Deligne–Mumford stack $\overline{\mathcal{M}}$ admitting a projective coarse moduli space. This should allow one to define Gromov–Witten invariants and quantum cohomology in this generality, which may have interesting applications for specific choices of stacks $\overline{\mathcal{M}}$.

### *1.7 Acknowledgements*

We would like to thank Johan de Jong and Rahul Pandharipande for crucial discussions of ideas in this paper.

## 2 The purity lemma

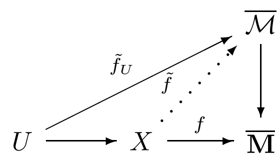
There are several results in the literature which give conditions under which a family of curves can fail to be stable only in pure codimension 1 (see [MB], [dJ-O]). For our purposes, the following case will be most useful:

**Purity Lemma 2.1.** *Let $\overline{\mathcal{M}}$ be a separated Deligne–Mumford stack, $\overline{\mathcal{M}} \to \overline{\mathbf{M}}$ the coarse moduli space. Let $X$ be a separated scheme of dimension 2 satisfying Serre's condition $S_2$. Let $P \subset X$ be a finite subset consisting of closed points, $U = X \smallsetminus P$. Assume that the local fundamental groups of $U$ around the points of $P$ are trivial.*

*Let $f : X \to \overline{\mathbf{M}}$ be a morphism. Suppose there is a lifting $\tilde{f}_U : U \to \overline{\mathcal{M}}$:*

$$\begin{array}{ccccc} & & & & \overline{\mathcal{M}} \\ & & \nearrow^{\tilde{f}_U} & & \downarrow \\ U & \longrightarrow & X & \xrightarrow{f} & \overline{\mathbf{M}} \end{array} \tag{1}$$

*Then the lifting extends uniquely to $X$:*




(2)

**Remark 2.2.** A closely related statement has been obtained by Mochizuki in [Mo1], [Mo2].

**Proof.** By descent theory the problem is local in the étale topology, so we may replace $X$ and $\overline{\mathbf{M}}$ with the spectra of their strict henselizations at a geometric point; then we can also assume that we have a universal deformation space $V \to \overline{\mathcal{M}}$ which is *finite*. Now $U$ is the complement of the closed point, $U$ maps to $\overline{\mathcal{M}}$, and the pullback of $V$ to $U$ is finite and étale, so it has a section, because $U$ is simply connected; consider the corresponding map $U \to V$. Let $Y$ be the scheme-theoretic closure of the graph of this map in $X \times_{\overline{\mathbf{M}}} V$. Then $Y \to X$ is finite and is an isomorphism on $U$. Since $X$ satisfies $S_2$, the morphism $Y \to X$ is an isomorphism. □

**Corollary 2.3.** *Let $X$ be a smooth surface, $p \in X$ a closed point with complement $U$. Let $X \to \overline{\mathbf{M}}$ and $U \to \overline{\mathcal{M}}$ be as in the purity lemma. Then there is a unique lifting $X \to \overline{\mathcal{M}}$.*

**Corollary 2.4.** *Let $X$ be a normal crossings surface, namely a surface which is étale locally isomorphic to* Spec $k[u, v, t]/(uv)$. *Let $p \in X$ be a closed point with complement $U$. Let $X \to \overline{\mathbf{M}}$ and $U \to \overline{\mathcal{M}}$ be as in the purity lemma. Then there is a unique lifting $X \to \overline{\mathcal{M}}$.*

**Proof.** In both cases $X$ satisfies condition $S_2$ and the local fundamental group around $p$ is trivial, hence the purity lemma applies. □

# 3 Group actions on nodal curves

Fix two nonnegative integers $\gamma$ and $\nu$. An important ingredient in the theory will be formed of data as follows:

**1**. a diagram

$$\begin{array}{c} Y \\ \downarrow \circlearrowright \\ V \\ \downarrow \\ S \end{array}$$

(we think of $Y$ and $V$ as $S$-schemes);

**2**. the morphism $\rho\colon Y \to V$ comes with sections $\tau_1\colon V \to Y, \dots, \tau_\nu\colon V \to Y$, forming a family of stable $\nu$-pointed curves of genus $\gamma$ on $V$.

Given a finite group $\Gamma$, by *an action of* $\Gamma$ *on* $\rho$ we mean a pair of actions of $\Gamma$ on $V$ and $Y$ as $S$-schemes such that the morphisms $\rho$, $\tau_1, \dots, \tau_\nu$ are $\Gamma$-equivariant. Such an action induces a morphism $Y/\Gamma \to V/\Gamma$ together with sections $V/\Gamma \to Y/\Gamma$.

We adopt the convention that all families of curves are assumed to be of *finite presentation* over the base.

**Definition 3.1.** Let $\rho\colon Y \to V$ be a family of $\nu$-pointed curves of genus $\gamma$, $\Gamma$ a finite group. An action of $\Gamma$ on $\rho$ is called *essential* if no nontrivial element of $\Gamma$ leaves a geometric fiber of $\rho$ fixed.

Another way to state this condition is to require that if $\gamma$ is an element of $\Gamma$ leaving a geometric point $v_0$ of $V$ fixed, then $\gamma$ acts nontrivially on the fiber of $\rho$ on $v_0$.

The following lemma will help us replace an action of $\Gamma$ on $\rho$ by an essential action on a related family of curves.

**Lemma 3.2.** *Let $V$ be the spectrum of a local ring $R$, $Y \to V$ a flat family of nodal curves, $\Gamma_0$ a group acting compatibly on $Y$ and $V$. Suppose that the action of $\Gamma_0$ on the residue field $k$ of $R$ and on the fiber $Y_0$ of $Y$ over* Spec $k$ *is trivial. Then $Y/\Gamma_0 \to V/\Gamma_0$ is again a flat family of nodal curves.*

**Remark 3.3.** It is easy to give an example showing that this fails in positive characteristic.

**Proof.** We need to show that the map $\rho\colon Y/\Gamma_0 \to V/\Gamma_0$ is flat, and the natural map $Y_0 \to \rho^{-1}(\operatorname{Spec} k)$ is an isomorphism. Replacing $R$ with the completion of its strict henselization we may assume that $R$ is complete and $k$ is algebraically closed. Choose a rational point $p \in X(k)$, and let $M$ be the completion of the local ring of $Y$ at $p$. Write $M \otimes_R k$ as a quotient of a power series algebra $k[[t_1, \dots, t_r]] = k[[\mathbf{t}]]$, and lift the images of the $t_i$ in $M \otimes_R k$ to invariant elements in $M$. We get a surjective homomorphism $R[[\mathbf{t}]] \to M$ which is equivariant, if we let $\Gamma_0$ act on $R[[\mathbf{t}]]$ leaving the $t_i$ fixed. Then the result follows from the lemma below.

**Lemma 3.4.** *Let $M$ be a finitely generated $R[[\mathbf{t}]]$-module which is flat over $R$, such that $\Gamma_0$ acts trivially on $M \otimes_R k$. Then $M^{\Gamma_0}$ is flat over $R^{\Gamma_0}$, and the natural homomorphism $M^{\Gamma_0} \otimes_{R^{\Gamma_0}} R \to M$ is an isomorphism.*

**Proof.** We have $R[[\mathbf{t}]]^{\Gamma_0} = R^{\Gamma_0}[[\mathbf{t}]]$, and from this we see that the statement holds when $M$ is free over $R[[\mathbf{t}]]$. In general, take a finite set of generators of the finite $k[[\mathbf{t}]]$-module $M \otimes_R k$ and lift them to a set of invariant generators of $M$. We obtain an equivariant surjective homomorphism $F \to M$, where $F$ is a free $R[[t]]$-module. Let $K$ be the kernel:

$$0 \to K \to F \to M \to 0.$$

Note that by the finite presentation assumption on $Y \to V$, $K$ is finitely generated. Since $M$ is flat we get $Tor_i(K, \cdot) = Tor_{i+1}(M, \cdot) = 0$, thus $K$ is also flat. Moreover, from the exact sequence

$$0 \to K \otimes_R k \to F \otimes_R k \to M \otimes_R k \to 0$$

we get that $\Gamma_0$ acts trivially on $K \otimes_R k$. We obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & K^{\Gamma_0} \otimes_{R^{\Gamma_0}} R & \longrightarrow & F^{\Gamma_0} \otimes_{R^{\Gamma_0}} R & \longrightarrow & M^{\Gamma_0} \otimes_{R^{\Gamma_0}} R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the middle column is an isomorphism. It follows that the right column is surjective; since $K$ is also flat and $\Gamma_0$ acts trivially on $K \otimes_R k$ we can apply the same argument to $K$, and it follows that the left column is also surjective. Therefore the right column is also injective, as desired.

Applying the same argument to the module $K$, we have that the left column is also an isomorphism, and the first arrow in the top row is injective; the flatness of $M^{\Gamma_0}$ over $R^{\Gamma_0}$ then follows from Grothendieck's local criterion of flatness. □

The following lemma will help us identify a situation where an essential action is free:

**Lemma 3.5.** *Let $G$ be a group of automorphisms of a stable $\nu$-pointed curve $X$ of genus $\gamma$ over an algebraically closed field, and assume that the quotient $X/G$ is also a stable $\nu$-pointed curve of genus $\gamma$. Then $G$ is trivial.*

**Proof.** If $C$ is a complete curve we denote by $\mathrm{CH}^i(C)$ the Chow group of classes of cycles of codimension $i$ on $C$; so $\mathrm{CH}^0(C)$ is a direct sum of $n$ copies of $\mathbf{Z}$, where $n$ is the number of irreducible components of $C$, while $\mathrm{CH}^1(C)$ is the divisor class group of $C$. If $\mathcal{F}$ is a coherent sheaf on $C$ we denote by $\tau^C(\mathcal{F}) \in \mathrm{CH}(C)$ its Riemann–Roch class, and by $\tau_i^C(\mathcal{F}) \in \mathrm{CH}^i(C)$ its component of codimension $i$; $\tau_0^C(\mathcal{F})$ is the cycle associated with $\mathcal{F}$.

Set $Y = X/G$, and call $\pi\colon X \to Y$ the projection. Call $\Delta_X$ and $\Delta_Y$ the divisors corresponding to the distinguished points in $X$ and $Y$ respectively, $\omega_X$ and $\omega_Y$ the dualizing sheaves.

A local calculation reveals that $\chi\big(\omega_X(\Delta_X)\big) \geq \chi\big(\pi^*\omega_Y(\Delta_Y)\big)$. Indeed, for this calculation we may replace $X$ and $Y$ by their normalization, where we add markings on the normalized curves at the points above the nodes. Then the natural pullback $\pi^*\omega_{Y^{nor}} \to \omega_{X^{nor}}$ extends to a homomorphism of invertible sheaves $\pi^*\omega_{Y^{nor}}(\Delta_{Y^{nor}}) \to \omega_{X^{nor}}(\Delta_{X^{nor}})$ which is injective, and whose cokernel is supported on the points of ramification of $\pi$ not in $\Delta_{X^{nor}}$.

Thus we get an inequality of Euler characteristics of sheaves:

$$\begin{aligned}\chi\big(\omega_X(\Delta_X)\big) &\geq \chi\big(\pi^*\omega_Y(\Delta_Y)\big)\\ &= \chi\big(\pi_*\pi^*\omega_Y(\Delta_Y)\big)\\ &= \chi\big(\omega_Y(\Delta_Y)\otimes\pi_*\mathcal{O}_X\big)\\ &= \chi\big(\omega_Y(\Delta_Y)\big)+\chi\big(\omega_Y(\Delta_Y)\otimes(\pi_*\mathcal{O}_X/\mathcal{O}_Y)\big).\end{aligned}$$

By hypothesis

$$\chi\big(\omega_X(\Delta_X)\big)=\gamma+\nu-1=\chi\big(\omega_Y(\Delta_Y)\big).$$

From this, together with the theorem of Grothendieck–Riemann–Roch, we obtain

$$\begin{aligned}0 &\geq \chi\big(\omega_Y(\Delta_Y)\otimes(\pi_*\mathcal{O}_X/\mathcal{O}_Y)\big)\\ &= \int_Y \mathrm{ch}\big(\omega_Y(\Delta_Y)\big)\tau^Y(\pi_*\mathcal{O}_X/\mathcal{O}_Y)\\ &= \int_Y \mathrm{c}_1\big(\omega_Y(\Delta_Y)\big)\cap\tau_0^Y(\pi_*\mathcal{O}_X/\mathcal{O}_Y)+\int_Y\tau_1^Y(\pi_*\mathcal{O}_X/\mathcal{O}_Y)\\ &= \int_Y \mathrm{c}_1\big(\omega_Y(\Delta_Y)\big)\cap(\pi_*[X]-[Y])+\int_X\tau_1^X(\mathcal{O}_X)-\int_Y\tau^Y(\mathcal{O}_Y)\\ &= \int_Y \mathrm{c}_1\big(\omega_Y(\Delta_Y)\big)\cap(\pi_*[X]-[Y])+\chi(\mathcal{O}_X)-\chi(\mathcal{O}_Y)\\ &= \int_Y \mathrm{c}_1\big(\omega_Y(\Delta_Y)\big)\cap(\pi_*[X]-[Y]).\end{aligned}$$

But $\omega_Y(\Delta_Y)$ is an ample line bundle on $Y$ and $\pi_*[X]-[Y]$ is an effective cycle; the only possibility is that $\pi_*[X]-[Y]=0$. This implies that $\pi$ is birational, so $G$ is trivial. □

# 4 Fibered surfaces: definitions

If $C\to S$ is a flat family of nodal curves, we denote by $C_{\mathrm{sm}}$ the open subscheme of $C$ consisting of points where the morphism $C\to S$ is smooth.

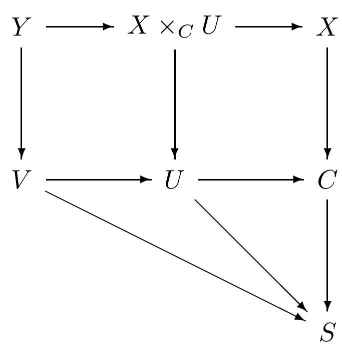
**Definition 4.1.** Let $C\to S$ be a flat (not necessarily proper) family of nodal curves, $X\to C$ a proper morphism with one-dimensional fibers, and $\sigma_1,\dots,\sigma_\nu\colon C\to X$ sections of $\rho$. We will say that

$$\begin{array}{c}X\\ \downarrow\\ C\\ \downarrow\\ S\end{array}$$

is a *family of generically fibered surfaces* if $X$ is flat over $S$, and the restriction of $\rho$ to $C_{\mathrm{sm}}$ is a flat family of stable pointed curves. If $S$ is the spectrum of a field we will refer to $X \to C$ as a generically fibered surface.

**Remark 4.2.** Notice that we do not require the morphism $X \to C$ to be flat.

**Definition 4.3.** A *chart* $(U, Y \to V, \Gamma)$ for a family of generically fibered surfaces $X \to C \to S$ consists of a diagram




together with a group action $\Gamma \subset \mathrm{Aut}_S(Y \to V)$ satisfying:

1. The morphism $U \to C$ is étale;
2. $V \to S$ is a flat (but not necessarily proper) family of nodal curves;
3. $\rho\colon Y \to V$ is a flat family of stable $\nu$-pointed curves of genus $\gamma$,
4. the action of $\Gamma$ on $\rho$ is essential;
5. we have isomorphisms of $S$-schemes $V/\Gamma \simeq U$ and $Y/\Gamma \simeq U \times_C X$ compatible with the projections $Y/\Gamma \to V/\Gamma$ and $U \times_C X \to U$, such that the sections $U \to U \times_C X$ induced by the $\sigma_i$ correspond to the sections $V/\Gamma \to Y/\Gamma$.

**Lemma 4.4.** *Let $(U, Y \to V, \Gamma)$ be a chart for a family of generically fibered surfaces $X \to C \to S$. If $S' \to S$ is an arbitrary morphism, then $(S' \times_S U, S' \times_S Y \to S' \times_S V, \Gamma)$ (with the obvious definitions of the various maps and of the action of $\Gamma$ on $S' \times_S Y$ and $S' \times_S V$) is a chart for the family of generically fibered surfaces $S' \times_S X \to S' \times_S C \to S'$.*

**Proof.** Conditions (1) through (4) in the definition are immediately verified for the pullback diagram. The only point that requires a little care is to check that $(S' \times_S V)/\Gamma \simeq S' \times_S U$ and $(S' \times_S Y)/\Gamma \simeq (S' \times_S U) \times_C X$, which requires the hypothesis that $S$ be a scheme over $\mathbf{Q}$. □

**Proposition 4.5.** **1**. *Let $(U, Y \to V, \Gamma)$ be a chart for a family of generically fibered surfaces $X \to C \to S$. Let $V' \subset V$ be the inverse image of $C_{\mathrm{sm}}$. Then*

*(a) the action of $\Gamma$ on $V'$ is free; and*

*(b) the natural morphism $Y|_{V'} \to V' \times_S X$ is an isomorphism.*

**2**. *Furthermore, if $t_0$ is a geometric point of $S$ and $v_0$ a nodal point of the fiber $V_{t_0}$ of $V$ over $t_0$, then*

*(a) the stabilizer $\Gamma'$ of $v_0$ is a cyclic group which sends each of the branches of $V_{t_0}$ to itself.*

*(b) If $n$ is the order of $\Gamma'$, then a generator of $\Gamma'$ acts on the tangent space of each branch by multiplication with a primitive $n$-th root of 1.*

*In particular, $V' = V_{\mathrm{sm}}$.*

**Proof.** The claim (1b), that the natural morphism $Y|_{V'} \to V' \times_S X$ is an isomorphism, is a consequence of the statement (1a).

Let $v_0$ be a geometric point of the inverse image of $U_{\mathrm{sm}}$ in $V$, $\Gamma'$ its stabilizer. By definition $\Gamma'$ acts faithfully on the fiber $Y_0$ of $Y$ on $v_0$. The fiber $X_0$ of $X$ over the image of $v_0$ in $C$ is the quotient of $Y_0$ by $\Gamma'$; from Lemma 3.5 above we get that $\Gamma'$ is trivial, as claimed.

For part (2) of the proposition we observe that if the stabilizer $\Gamma'$ of $v_0$ did not preserve the branches of $V_{t_0}$ then the quotient $V_{t_0}/\Gamma'$, which is étale at the point $v_0$ over the fiber $U_{t_0}$, would be smooth over $S$ at $v_0$, so $v_0$ would be in the inverse image of $U_{\mathrm{sm}}$. From part (1) of the proposition it would follow that $\Gamma'$ is trivial, a contradiction.

So $\Gamma'$ acts on each of the two branches individually. The action on each branch must be faithful because it is free on the complement of the set of nodes; this means that the representation of $\Gamma'$ in each of the tangent spaces to the branches is faithful, and this implies the final statement. □

**Definition 4.6.** A chart is called *balanced* if for any nodal point of any geometric fiber of $V$, the two roots of 1 describing the action of a generator of the stabilizer on the tangent spaces to each branch of $V$ are inverse to each other.

It is easy to see that a chart is balanced if and only if it admits a deformation to a smooth curve.

## *4.7 The transition scheme of two charts*

Let $X \to C \to S$ be a family of generically fibered surfaces, $(U_1, Y_1 \to V_1, \Gamma_1)$, $(U_2, Y_2 \to V_2, \Gamma_2)$ two charts; call $\mathrm{pr}_i \colon V_1 \times_C V_2 \to V_i$ the $i^{\mathrm{th}}$

projection. Consider the scheme

$$I = \operatorname*{Isom}_{V_1 \times_C V_2}(\mathrm{pr}_1^* Y_1, \mathrm{pr}_2^* Y_2)$$

over $V_1 \times_C V_2$ representing the functor of isomorphisms of the two families $\mathrm{pr}_1^* Y_1$ and $\mathrm{pr}_2^* Y_2$. There is a section of $I$ over the inverse image $\widetilde{V}$ of $C_{\mathrm{sm}}$ in $V_1 \times_C V_2$ which corresponds to the isomorphism $\mathrm{pr}_1^* Y_1|_{\widetilde{V}} \simeq \mathrm{pr}_2^* Y_2|_{\widetilde{V}}$ coming from the fact that both $\mathrm{pr}_1^* Y_1$ and $\mathrm{pr}_2^* Y_2$ are pullbacks to $\widetilde{V}$ of the restriction of $X$ to $C_{\mathrm{sm}}$. We will call the scheme-theoretic closure $R$ of this section in $I$ the *transition scheme* from $(U_1, Y_1 \to V_1, \Gamma_1)$ to $(U_2, Y_2 \to V_2, \Gamma_2)$; it comes equipped with two projections $R \to V_1$ and $R \to V_2$. There is also an action of $\Gamma_1 \times \Gamma_2$ on $I$, defined as follows. Let $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$, and $\phi\colon \mathrm{pr}_1^* Y_1 \simeq \mathrm{pr}_2^* Y_2$ an isomorphism over $V_1 \times_C V_2$; then define $(\gamma_1, \gamma_2) \cdot \phi = \gamma_2 \circ \phi \circ \gamma_1^{-1}$. This action of $\Gamma_1 \times \Gamma_2$ on $I$ is compatible with the action of $\Gamma_1 \times \Gamma_2$ on $V_1 \times_C V_2$, and leaves $R$ invariant. It follows from the definition of an essential action that the action of $\Gamma_1 = \Gamma_1 \times \{1\}$ and $\Gamma_2 = \{1\} \times \Gamma_2$ on $I$ is free.

**Definition 4.8.** Two charts $(U_1, Y_1 \to V_1, \Gamma_1)$ and $(U_2, Y_2 \to V_2, \Gamma_2)$ are compatible if their transition scheme $R$ is étale over $V_1$ and $V_2$.

Let us analyze this definition. Start from two charts $(U_1, Y_1 \to V_1, \Gamma_1)$ and $(U_2, Y_2 \to V_2, \Gamma_2)$. Fix two geometric points $v_1\colon \operatorname{Spec} \Omega \to V_1$ and $v_2\colon \operatorname{Spec} \Omega \to V_2$ mapping to the same geometric point $v_0\colon \operatorname{Spec} \Omega \to C$, and call $\Gamma_i' \subseteq \Gamma_i$ the stabilizer of $v_i$. Also call $V_1^{\mathrm{sh}}$, $V_2^{\mathrm{sh}}$ and $C^{\mathrm{sh}}$ the spectra of the strict henselizations of $V_1$, $V_2$ and $C$ at the points $v_1, v_2$ and $v_0$ respectively. The action of $\Gamma_i$ on $V_i$ induces an action of $\Gamma_i'$ on $V_i^{\mathrm{sh}}$. Also, call $Y_i^{\mathrm{sh}}$ the pullback of $Y_i$ to $V_i^{\mathrm{sh}}$; there is an action of $\Gamma_i'$ on $Y_i^{\mathrm{sh}}$ compatible with the action of $\Gamma_i'$ on $V_i$.

**Proposition 4.9.** *The two charts are compatible if and only if for any pair of geometric points $v_1$ and $v_2$ as above there exist an isomorphism of groups $\eta\colon \Gamma_1' \simeq \Gamma_2'$ and two compatible $\eta$-equivariant isomorphisms $\phi\colon V_1^{\mathrm{sh}} \simeq V_2^{\mathrm{sh}}$ and $\psi\colon Y_1^{\mathrm{sh}} \to Y_2^{\mathrm{sh}}$ of schemes over $C^{\mathrm{sh}}$.*

**Proof.** Consider the spectrum $(V_1 \times_C V_2)^{\mathrm{sh}}$ of the strict henselization of $V_1 \times_C V_2$ at the point $(v_1, v_2)\colon \operatorname{Spec} \Omega \to V_1 \times_C V_2$, and call $R^{\mathrm{sh}}$ the pullback of $R$ to $(V_1 \times_C V_2)^{\mathrm{sh}}$. Assume that the two charts are compatible. The action of $\Gamma_1 \times \Gamma_2$ on $I$ described above induces an action of $\Gamma_1' \times \Gamma_2'$ on $R^{\mathrm{sh}}$, compatible with the action of $\Gamma_1' \times \Gamma_2'$ on $(V_1 \times_C V_2)^{\mathrm{sh}}$. The action of $\Gamma_1' = \Gamma_1' \times \{1\}$ on the inverse image of $C_{\mathrm{sm}}$ in $R^{\mathrm{sh}}$ is free, and its quotient is the inverse image of $C_{\mathrm{sm}}$ in $V_2^{\mathrm{sh}}$; but $R^{\mathrm{sh}}$ is finite and étale over $V_2^{\mathrm{sh}}$, so the action of $\Gamma_1'$ on all of $R^{\mathrm{sh}}$ is free, and $R^{\mathrm{sh}}/\Gamma_1' = V_2$. Analogously the action of $\Gamma_2'$ on $R^{\mathrm{sh}}$ is free, and $R^{\mathrm{sh}}/\Gamma_2' = V_1$.

Now, each of the connected components of $R^{\mathrm{sh}}$ maps isomorphically onto both $V_1$ and $V_2$, because $V_i$ is the spectrum of a strictly henselian ring and the projection $R^{\mathrm{sh}} \to V_i$ is étale; this implies in particular that the order of $\Gamma_1$ is the same as the number $n$ of connected components, and likewise for $\Gamma_2$. Fix one of these components, call it $R_0^{\mathrm{sh}}$; then we get isomorphisms $R_0^{\mathrm{sh}} \simeq V_i$, which yield an isomorphism $\phi\colon V_1 \simeq V_2$.

Call $\Gamma'$ the stabilizer of the component $R_0^{\mathrm{sh}}$ inside $\Gamma_1' \times \Gamma_2'$; the order of $\Gamma'$ is at least $|\Gamma_1' \times \Gamma_2'|/n = n^2/n = n$. But the action of $\Gamma_2'$ on $R^{\mathrm{sh}}$ is free, and so $\Gamma' \cap \Gamma_2 = \{1\}$; this implies that the order of $\Gamma'$ is $n$, and the projection $\Gamma' \to \Gamma_1$ is an isomorphism. Likewise the projection $\Gamma' \to \Gamma_2$ is an isomorphism, so from these we get an isomorphism $\eta\colon \Gamma_1 \to \Gamma_2$, and it is easy to check that the isomorphism of schemes $\phi\colon V_1 \simeq V_2$ is $\eta$-equivariant.

There is also an isomorphism of the pullbacks of $Y_1^{\mathrm{sh}}$ and $Y_2^{\mathrm{sh}}$ to $R_0^{\mathrm{sh}}$, coming from the natural morphism $R_0^{\mathrm{sh}} \to I$, which induces an isomorphism $\psi\colon Y_1^{\mathrm{sh}} \to Y_2^{\mathrm{sh}}$. This isomorphism is compatible with $\phi$, and is it also $\eta$-equivariant.

Let us prove the converse. Suppose that there exist $\eta$, $\phi$ and $\psi$ as above. Then there is a morphism $\sigma\colon V_1^{\mathrm{sh}} \times \Gamma_1' \to I$ which sends a point $(v_1, \gamma_1)$ of $V_1^{\mathrm{sh}} \times \Gamma_1'$ into the point of $I$ lying over the point $(v_1, \phi\gamma_1 v_1) = (v_1, \eta(\gamma_1)\phi v_1)$ corresponding to the isomorphism $\gamma_1\psi$ of the fiber of $Y_1$ over $v_1$ with the fiber of $Y_2$ over $\phi\gamma_1 v_1$. The morphism $\sigma$ is an isomorphism of $V_1^{\mathrm{sh}} \times \Gamma_1'$ with $R^{\mathrm{sh}}$ in the inverse image of $C_{\mathrm{sm}}$; it also follows from the fact that the action of $\Gamma'$ on $Y_1 \to V_1$ is essential that $\sigma$ is injective. Since the inverse image of $C_{\mathrm{sm}}$ is scheme-theoretically dense in $R^{\mathrm{sh}}$ and $V_1^{\mathrm{sh}} \times \Gamma_1$ is unramified over $V_1$ we see that $\sigma$ is an isomorphism; it follows that $R^{\mathrm{sh}}$ is étale over $V_1^{\mathrm{sh}}$; analogously it is étale over $V_2^{\mathrm{sh}}$. So $R$ is étale over $V_1$ and $V_2$ at the points $v_1$ and $v_2$; since this holds for all $v_1$ and $v_2$ mapping to the same point of $C$, the conclusion follows. □

In 6.14 and 6.15 below, we give two examples of incompatible charts on the same coarse fibered surface.

## *4.10 The product chart*

Given two compatible charts $(U_1, Y_1 \to V_1, \Gamma_1)$ and $(U_2, Y_2 \to V_2, \Gamma_2)$, the graph $Y \subseteq (Y_1 \times_{V_1} R) \times_R (R \times_{V_2} Y_2)$ over $R$ of the canonical isomorphism of the two families $Y_1 \times_{V_1} R$ and $R \times_{V_2} Y_2$ is invariant under the action of $\Gamma_1 \times \Gamma_2$, and it is a family of pointed curves on $R$. Then

$$(U_1 \times_C U_2, Y \to R, \Gamma_1 \times \Gamma_2)$$

is a chart, called the *product chart*. It is compatible with both the original charts.

Compatibility of charts is stable under base change:

**Proposition 4.11.** *Let $(U_1, Y_1 \to V_1, \Gamma_1)$ and $(U_2, Y_2 \to V_2, \Gamma_2)$ be two compatible charts for a family of generically fibered surfaces $X \to C \to S$. If $S' \to S$ is an arbitrary morphism, then*

$$(S' \times_S U_1, S' \times_S Y_1 \to S' \times_S V_1, \Gamma_1)$$

*and*

$$(S' \times_S U_2, S' \times_S Y_2 \to S' \times_S V_1, \Gamma_2)$$

*are compatible charts for the the family* $S' \times_S X \to S' \times_S C \to S'$.

This is easy.

We now come to the definition of our basic object:

**Definition 4.12.** A *family of fibered surfaces*

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathcal{C} \\ \downarrow \\ S \end{array}$$

is a family of generically fibered surfaces $X \to C \to S$ such that $C \to S$ is proper, together with a collection $\{(U_\alpha, Y_\alpha \to V_\alpha, \Gamma_\alpha)\}$ of mutually compatible charts, such that the images of the $U_\alpha$ cover $C$.

Such a collection of charts is called an *atlas*.

A family of fibered surfaces is called *balanced* if each chart in its atlas is balanced.

The family of generically fibered surfaces $X \to C \to S$ supporting the family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ will be called a family of *coarse fibered surfaces*.

**Lemma 4.13.** *If two charts for a family of fibered surfaces are compatible with all the charts in an atlas, they are mutually compatible.*

*Furthermore, if the family is balanced, then any chart which is compatible with every chart of the atlas is balanced.*

The proof is straightforward.

**Remark 4.14.** The lemma above allows us to define a family of fibered surfaces using a maximal atlas, if we want.

**Definition 4.15.** A *morphism of fibered surfaces* $\mathcal{X} \to \mathcal{C} \to S$ to $\mathcal{X}' \to \mathcal{C}' \to S'$ is a *cartesian* diagram of coarse fibered surfaces

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ C & \longrightarrow & C' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S', \end{array}$$

such that the pullback of the charts of an atlas of $\mathcal{X}' \to \mathcal{C}' \to S$ are all compatible with the atlas of $\mathcal{X} \to \mathcal{C} \to S$.

The composition of morphisms is the obvious one.

We will soon reinterpret this definition of a morphism.

### *4.16 Fibered surfaces as stacks*

Consider a family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ with an atlas $\{(U_\alpha, Y_\alpha \to V_\alpha, \Gamma_\alpha)\}$. For each pair of indices $(\alpha, \beta)$ let $R_{\alpha\beta}$ be the transition scheme from $(U_\alpha, Y_\alpha \to V_\alpha, \Gamma_\alpha)$ to $(U_\beta, Y_\beta \to V_\beta, \Gamma_\beta)$. Let $V$ be the disjoint union of the $V_\alpha$, and let $R$ be the disjoint union of the $R_{\alpha,\beta}$. These have the following structure:

- there are two projections $R \to V$, which are étale;
- there is a natural diagonal morphism $V \to R$ which sends each $V_\alpha$ to $R_{\alpha\alpha}$; and
- there is a product $R \times_V R \to R$, sending each $R_{\alpha\beta} \times_{V_\beta} R_{\beta\gamma}$ to $R_{\alpha\gamma}$ via composition of isomorphisms.

These various maps give $R \rightrightarrows V$ the structure of a groupoid. Since the diagonal map $R \to V \times_C V$ is unramified the groupoid defines a quotient Deligne–Mumford stack. In a slight abuse of notation, we call this stack $\mathcal{C}$ as well.

A similar groupoid structure can be formed using $Y_\alpha$ and their pullback to $R_{\alpha\beta}$, endowing $\mathcal{X}$ with the structure of a Deligne–Mumford stack as well. Note that $\mathcal{X} \to \mathcal{C}$ is representable, and the stack $\mathcal{X}$ is a family of stable $\nu$-pointed curves of genus $\gamma$ over the stack $\mathcal{C}$.

Let us list some properties of $\mathcal{X}$ and $\mathcal{C}$:

- There is a moduli morphism $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ associated to the family $\mathcal{X} \to \mathcal{C}$.
- The stack $\mathcal{C}$ is a proper nodal stack over $S$, namely, it has an étale cover, given by the schemes $V$ underlying the charts, which are nodal over S.
- Over the inverse image of $C_{\text{sm}}$ the scheme $R$ is isomorphic to the fibered product $V \times_C V$, so the quotient stack $\mathcal{C}_{\text{sm}}$ coincides with $C_{\text{sm}}$. A similar statement holds for $\mathcal{X}$.
- Since $U_\alpha$ is the schematic quotient of $V_\alpha$ by the action of $\Gamma_\alpha$, and $U \to C$ is étale, it is immediate that $C$ is the schematic quotient of the groupoid $R \rightrightarrows V$, in other words, the stack morphism $\mathcal{C} \to C$ exhibits $C$ as the coarse moduli scheme of $\mathcal{C}$. Similarly, $X$ is the coarse moduli scheme of $\mathcal{X}$.

As mentioned above, a fibered surface being *balanced* is tantamount to the existence of local smoothing of the fibered surface. It is interesting to interpret the requirement that the action of $\Gamma$ is *essential* within the language of stacks.

We claim that the action being essential is equivalent to the condition that the moduli morphism $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ be *representable*. This follows from the definition of essential action, using the following well known lemma:

**Lemma 4.17.** *Let $g : \mathcal{G} \to \mathcal{F}$ be a morphism of Deligne–Mumford stacks. The following two conditions are equivalent:*

1. *The morphism $g : \mathcal{G} \to \mathcal{F}$ is representable.*
2. *For any algebraically closed field $k$ and any $\xi \in \mathcal{G}(k)$, the natural group homomorphism $\mathrm{Aut}(\xi) \to \mathrm{Aut}(g(\xi))$ is a monomorphism.*

## *4.18 A stack-theoretic formulation of the category of fibered surfaces*

We will now formulate stack-theoretic data similar to that obtained above, and then compare them to the data of a fibered surface.

Let $S$ be a scheme over $\mathbf{Q}$. Consider a proper Deligne–Mumford stack $\mathcal{C} \to S$, such that its fibers are purely one-dimensional and geometrically connected, with nodal singularities. Call $C$ the moduli space of $\mathcal{C}$; this automatically exists as an algebraic space.

**Proposition 4.19.** *The morphism $C \to S$ is a flat family of nodal curves.*

**Proof.** First of all let us show that $C$ is flat over $S$. We may assume that $S$ is affine; call $R$ its ring. Fix a geometric point $c_0 \to C$, and call $C^{\mathrm{sh}}$ the strict henselization of $C$ at $c_0$. Let $U$ be an étale cover of $\mathcal{C}$, let $u_0$ be a geometric point of $U$ lying over $c_0$, and call $U^{\mathrm{sh}}$ the strict henselization of $U$ at $u_0$. If $\Gamma$ is the automorphism group of the object of $\mathcal{C}$ corresponding to $u_0$, then $\Gamma$ acts on $U^{\mathrm{sh}}$, and $C^{\mathrm{sh}}$ is the quotient $U^{\mathrm{sh}}/\Gamma$. Because the schemes are defined over $\mathbf{Q}$, the ring of $C^{\mathrm{sh}}$ is a direct summand of the ring of $Y^{\mathrm{sh}}$, as an $R$-module, so it is flat over $R$.

The fact that the fibers are nodal follows from the fact that, over an algebraically closed field, the quotient of a nodal curve by a group action is again a nodal curve. □

**Definition 4.20.** A stack-like family of curves of genus $g$ over a scheme $S$ consists of a proper stack $\mathcal{C} \to S$, whose fibers are purely one-dimensional and geometrically connected, with nodal singularities, such that:

1. the fibers of the morphism $C \to S$, where $C$ is the moduli space of $\mathcal{C}$, have genus $g$;
2. over the inverse image of the smooth locus $C_{\mathrm{sm}}$ of the map $C \to X$, the projection $\mathcal{C} \to C$ is an isomorphism.

If $\mathcal{C} \to S$ and $\mathcal{C}' \to S'$ are stack-like families of curves of genus $g$, a morphism $F\colon \mathcal{C} \to \mathcal{C}'$ consists of a cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & S' \end{array} .$$

The composition of morphisms of stack-like families of curves is defined in the obvious way. In this way stack-like families of curves of genus $g$ form a 2-category, 2-arrows being defined in the usual way.

**Proposition 4.21.** *The 2-category of stack-like families of curves of genus $g$ is equivalent (in the lax sense) to a 1-category.*

**Proof.** This is the same as saying that a 1-arrow in the category cannot have nontrivial automorphisms. The point here is that the stack $\mathcal{C}$ has an open subscheme which is stack-theoritically dense in $\mathcal{C}$, which is sufficient by the following lemma.

**Lemma 4.22.** *Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of Deligne–Mumford stacks over a scheme $S$. Assume that there exists an open representable substack (i.e. an algebraic space) $U \subseteq \mathcal{X}$ and a dense open representable substack $V \subseteq \mathcal{Y}$ such that $F$ maps $U$ into $V$. Further assume that the diagonal $\mathcal{Y} \to \mathcal{Y} \times_S \mathcal{Y}$ is separated. Then any automorphism of $F$ is trivial.*

**Definition 4.23.** A stack-like fibered surface $f\colon \mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ over a scheme $S$ is a stack-like family $\mathcal{C} \to S$ of curves of genus $g$ over $S$, with projective coarse moduli space $C \to S$, together with a representable morphism $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$.

As pointed out above, stack-like fibered surfaces form a 2-category. A 1-arrow $\Phi = (\phi, \alpha_\Phi)$ from $f\colon \mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ to $f'\colon \mathcal{C}' \to \overline{\mathcal{M}}_{\gamma,\nu}$ consists of a morphism of stack-like families of curves $\phi\colon \mathcal{C} \to \mathcal{C}'$, together with an isomorphism $\alpha_\Phi$ of the composition of $f' \circ \phi$ with $f$; the compostion of 1-arrows is defined in the obvious way. A 2-arrow from a 1-arrow $\Phi$ from $f\colon \mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ to $f'\colon \mathcal{C}' \to \overline{\mathcal{M}}_{\gamma,\nu}$ to another arrow $\Psi$ with the same domain and codomain consists of an isomorphism $\beta$ of the two functors $\phi$ and $\psi$, with the usual compatibility condition of the isomorphism $\beta$ with $\alpha_\Phi$ and $\alpha_\Psi$.

Because of the proposition, this category is equivalent to a nice unproblematic 1-category, which we can think of as the category of stack-like fibered surfaces, the arrows being given by isomorphism classes of morphisms.

We can now compare the two categories we have defined.

**Theorem 4.24.** *The category of stack-like fibered surfaces over schemes over $\mathbf{Q}$ is equivalent (in the lax sense) to the category of fibered surfaces.*

**Sketch of proof.** The construction in 4.16 shows that a fibered surface gives rise to a stack-like fibered surface in a functorial way.

To go from stack-like fibered surfaces to honest fibered surfaces, let us take a stack-like fibered surface $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$, and then take as a family of pointed nodal curves its coarse moduli space $C \to S$. The charts are given by étale morphism from schemes to $\mathcal{C}$. We leave the treatment of morphisms to the reader. □

From now on we will use *fibered surfaces* and *stack-like fibered surfaces* interchangeably.

## *4.25 Stable fibered surfaces*

We first define a moduli morphism on a coarse fibered surface:

**Lemma 4.26.** *Let $\mathcal{X} \to \mathcal{C} \to S$ be a family of fibered surfaces. Then the morphism $C_{\mathrm{sm}} \to \overline{\mathbf{M}}_{\gamma,\nu}$ induced by the restriction of $X \to C$ to $C_{\mathrm{sm}}$ extends uniquely to a morphism $C \to \overline{\mathbf{M}}_{\gamma,\nu}$.*

**Proof.** The unicity is clear from the fact that $\overline{\mathbf{M}}_{\gamma,\nu}$ is separated and $C_{\mathrm{sm}}$ is scheme-theoretically dense in $C$. To prove the existence of an extension is a local question in the étale topology; but if $\{(U_\alpha, Y_\alpha \to V_\alpha, \Gamma_\alpha)\}$ is an atlas then the families $Y_\alpha \to V_\alpha$ induce morphisms $V_\alpha \to \overline{\mathbf{M}}_{\gamma,\nu}$, which are $\Gamma_\alpha$-equivariant, yielding morphisms $U_\alpha \to \overline{\mathbf{M}}_{\gamma,\nu}$. These morphisms are extensions of the restriction to the $U_\alpha$ of the morphism $C_{\mathrm{sm}} \to \overline{\mathbf{M}}_{\gamma,\nu}$. Therefore they descend to $C$. □

We can use this lemma to define stable fibered surfaces:

**Definition 4.27.** A family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ is *stable* if the associated morphism $C \to \overline{\mathbf{M}}_{\gamma,\nu}$ is Kontsevich stable.

## *4.28 Natural line bundles on a coarse fibered surface*

**Proposition 4.29.** *On each family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ there exists a canonically defined line bundle $\mathcal{L}_X$ on the coarse fibered surface $X$, which is relatively ample along the map $X \to C$. This line bundle satisfies:*

**1**. *For any morphism of fibered surfaces*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\phi} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

*there is an isomorphism of line bundles $\sigma_\phi \colon \mathcal{L}_{X'} \simeq \phi^*\mathcal{L}_X$.*

**2**. *These isomorphisms satisfy the cocycle condition, in the sense that $\sigma_{\mathrm{id}_X} = \mathrm{id}_{\mathcal{L}_X}$ for all families of fibered surfaces $X \to S$, and if we* *have two morphisms*

$$\begin{array}{ccccc} \mathcal{X}'' & \xrightarrow{\psi} & \mathcal{X}' & \xrightarrow{\phi} & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}'' & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \longrightarrow & S' & \longrightarrow & S \end{array}$$

*then*

$$\sigma_{\phi\psi} = \psi^*\sigma_\phi \circ \sigma_\psi \colon \mathcal{L}_{X''} \longrightarrow (\phi\psi)^*\mathcal{L}_X = \psi^*\phi^*\mathcal{L}_X\,.$$

**Proof.** Fix an integer $N$ which is divisible by the order of the automorphism group of any $\nu$-pointed stable curve of genus $\gamma$. The construction below will depend on the choice of the integer $N$. For each index $\alpha$ consider the line bundle $\omega_\alpha = \omega_{Y_\alpha/V_\alpha}(\Delta_\alpha)$, the relative dualizing sheaf of $Y_\alpha$ over $V_\alpha$ twisted by the divisor $\Delta_\alpha$ of marked points. This line bundle has a natural action of $\Gamma_\alpha$, and the stabilizers of all the geometric points act trivially on the fiber of $\omega_\alpha^{\otimes N}$; therefore $\omega_\alpha^{\otimes N}$ descends to a line bundle $\mathcal{L}_\alpha$ on $X \times_C U_\alpha$.

Given two indices $\alpha$ and $\beta$, let $R_{\alpha\beta}$ be the transition scheme of two corresponding charts and $Y_{\alpha\beta} \to R_{\alpha\beta}$ the family of pointed stable curves constructed above; there is a canonical isomorphism between the pullback of $\omega_\alpha$ to $Y_{\alpha\beta}$ and the analogous bundle $\omega_{\alpha\beta}$ on the family $Y_{\alpha\beta}$, so we can compose these to get isomorphisms between the pullbacks of $\omega_\alpha$ and $\omega_\beta$. These isomorphisms satisfy the cocycle condition and give the collection of the $\omega_\alpha$ the structure of a line bundle on the groupoid $R \rightrightarrows V$.

Passing to the $N$-th power, the isomorphisms above also descend to isomorphisms between the pullbacks of $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ on the fiber product $X \times_C (U_\alpha \times_C U_\beta)$ satisfying the cocycle condition, so the $\mathcal{L}_\alpha$ are the pullback of a well defined line bundle $\mathcal{L}_X$, which is the one we want. □

**Proposition 4.30.** *On each family of* stable *fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ there exists a canonically defined line bundle $\mathcal{A}_X$ on the coarse fibered surface $X$, which is relatively ample along the map $X \to S$. This line bundle satisfies the properties listed in Proposition 4.29.*

**Proof.** Fix $N$ as in the proof of Proposition 4.29. Fix an ample line bundle $H_0$ on $\overline{\mathbf{M}}_{\gamma,\nu}$, and let $H = H_0^{\otimes 3}$. Let $\mathcal{X} \to \mathcal{C} \to S$ be a stable fibered surface, and let $m_C : C \to \overline{\mathbf{M}}_{\gamma,\nu}$ be the associated moduli morphism. Consider the line bundle $A_C := (\omega_{C/S} \otimes m_C^*H)^{\otimes N}$ on $C$. By the theory of stable maps, the line bundle $A_C$ is ample relative to the morphism $C \to S$. It clearly satisfies the invariance condition in the proposition. Also, by Kollár's semipositivity lemma (see [Kollár1]) the line bundle $\mathcal{L}_X$ constructed above is nef. Set $\mathcal{A}_X = \mathcal{L}_X \otimes f^*A_C$. By Kleiman's criterion for ampleness we clearly have that $\mathcal{A}_X$ is ample. The invariance conditions follow by construction.

**Remark 4.31.** It is important to note that both $\mathcal{L}_X$ and $\mathcal{A}_X$ are invariants of the *coarse* fibered surfaces, independent of the given atlas. This is because, on the open subscheme $U \subset X$ where $X \to S$ is Gorenstein, these coincide with $\omega_{U/C}^N$ and $\omega_{U/S}^N$. Since $U$ has a complement of codimension $\geq 2$, the extensions to $X$ are unique.

# 5 The stack of fibered surfaces

Consider the stack $\overline{\mathcal{M}}_{\gamma,\nu}$ of stable $\nu$-pointed curves of genus $\gamma$ over $\mathbf{Q}$, and the associated moduli space $\overline{\mathbf{M}}_{\gamma,\nu}$; choose an ample line bundle $\mathcal{H}$ on $\overline{\mathbf{M}}_{\gamma,\nu}$. Fix two nonnegative integers $g$ and $d$.

We define a category $\mathcal{F}_g^d(\gamma,\nu)$, fibered over the category $\mathrm{Sch}/\mathbf{Q}$ of schemes over $\mathbf{Q}$, as follows. The objects are stable families $\mathcal{X} \to \mathcal{C} \to S$ such that for the associated morphism $f\colon C \to \overline{\mathbf{M}}_{\gamma,\nu}$, the degree of the line bundle $f^*\mathcal{H}$ on each fiber of $C$ over $S$ is $d$. The arrows are morphisms of fibered surfaces. We also have a subcategory $\mathcal{F}_g^d(\gamma,\nu)^{balanced}$ of stable *balanced* families.

There is an obvious morphism from $\mathcal{F}_g^d(\gamma,\nu)$ to the stack $\mathcal{K}_g^d\big(\overline{\mathbf{M}}_{\gamma,\nu}\big)$ of Kontsevich-stable maps of genus $g$ and degree $d$ into $\overline{\mathbf{M}}_{\gamma,\nu}$ which sends each stable family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ to the associated morphism $C \to \overline{\mathbf{M}}_{\gamma,\nu}$.

Our main result is:

**Theorem 5.1.** *The category $\mathcal{F}_g^d(\gamma,\nu)$ is a complete Deligne–Mumford stack, admitting a projective coarse moduli space $\mathbf{F}_g^d(\gamma,\nu)$. The subcategory $\mathcal{F}_g^d(\gamma,\nu)^{balanced}$ forms an open and closed substack.*

This is a very particular case of our general theorem of existence for the Deligne–Mumford stack of twisted stable maps from a curve into a general Deligne–Mumford stack with projective moduli space (see [ℵ-V1], [ℵ-V2]). A proof (a little sketchy at some technical points) can be found in the previous version of this paper. This is substantially simpler than in the general case, because it takes advantage of many simplifications afforded by dealing with families of algebraic curves, rather than by abstract objects in a Deligne–Mumford stack.

# 6 Fibered surfaces and Alexeev stable maps

## 6.1 Semi-log-canonical surfaces

Semi-log-canonical surfaces are introduced in [K-SB] and further studied in [Al2] and [Al3]. We now review their definition. First, let us define log-canonical pairs.

**Definition 6.2.** Let $(X, D)$ be a pair consisting of a normal variety $X$ and an effective, reduced Weil divisor $D$. Denote by $X_{\rm sm}$ the nonsingular locus of $X$. We say that $(X, D)$ is a *log canonical pair* if the following conditions hold:

1. $(X, D)$ is log-$\mathbf{Q}$-Gorenstein, namely: for some positive integer $m$, we assume that the invertible sheaf $(\omega_{X_{\rm sm}}(D))^m$ extends to an invertible sheaf $(\omega_{X_{\rm sm}}(D))^{[m]}$ on $X$.
2. Let $r : Y \to X$ be a desingularization, such that the proper transform of D together with the exceptional locus of $Y \to X$ form a normal crossings divisor. Call this divisor, taken with reduced structure, $D'$. Then, we assume that the natural pullback of rational differentials gives a morphism of sheaves: $r^*(\omega_{X_{\rm sm}}(D))^{[m]} \to (\omega_Y(D'))^m$. That is, a logarithmic differential on $X$ pulls back to a logarithmic differential on $Y$.

A complete description of log-canonical singularities of surface pairs is given in [Al1].

Semi-log-canonical surfaces are a natural generalization of the above to the case of non-normal surfaces.

**Definition 6.3.** Let $X$ be a surface and $D$ a reduced, pure codimension 1 subscheme such that no component of $D$ lies in $Sing(X)$. We say that $(X, D)$ is a semi-log-canonical if the following conditions hold.

1. $X$ is Cohen Macaulay and normal crossings in codimension 1. Let $X_{nc}$ be the locus where $X$ is either nonsingular or normal crossings.
2. $(X, D)$ is log-$\mathbf{Q}$-Gorenstein, namely: for some positive integer $m$, we assume that the invertible sheaf $(\omega_{X_{nc}}(D))^m$ extends to an invertible sheaf on $X$.
3. Let $X' \to X$ be the normalization, $D'$ the Weil divisor corresponding of $D$ on $X'$, and $C \subset X'$ the conductor divisor, namely the divisor where $X' \to X$ is not one-to-one, taken with reduced structure. Then the pair $(X', D' + C)$ is log-canonical.

The unique locally free extension of $(\omega_{X_{nc}}(D))^m$ is denoted by $(\omega_X(D))^{[m]}$.

In case $D$ is empty, we just say that $X$ is semi-log-canonical (or "has semi-log-canonical singularities").

We will use the following lemma about quotients of semi-log-canonical surfaces:

**Lemma 6.4.** *Let $(Y, D)$ be a semi-log-canonical surface pair. Let $\Gamma \subset \mathrm{Aut}(Y, D)$ be a finite subgroup, and let $X = Y/\Gamma$, $D_X = D/\Gamma$. Then the following conditions are equivalent:*

1. *The quotient $(X, D_X)$ is semi-log-canonical;*

**2**. *The pair* $(X, D_X)$ *is log-*$\mathbf{Q}$*-Gorenstein.*

The proof is straightforward, see [Kollár2].

The reader is referred to [K-SB] for the refined notions of semismooth, semi-canonical and semi-log-terminal singularities.

## *6.5 Nodal families and semi-log-canonical singularities.*

A family of nodal curves over a nodal curve is always semi-log-canonical:

**Lemma 6.6.** *Let* $V$ *be a nodal curve and* $f : Y \to V$ *a nodal family. Then* $Y$ *has Gorenstein semi-log-canonical singularities. If, furthermore,* $D \subset Y$ *is a section which does not meet* $\mathrm{Sing}(f)$*, then* $(Y, D)$ *is a semi-log-canonical pair.*

**Proof.** Since any family of nodal curves is a Gorenstein morphism, we have that both $V \to \mathrm{Spec}\ k$ and $Y \to V$ are Gorenstein morphisms, therefore $Y$ is Gorenstein.

Let $y \in Y$ and $p = f(x) \in V$. It is convenient to replace $Y$ and $V$ by their respective formal completions at $y$ and $p$. The situation falls into one of the following cases:

**1**. Both $p \notin \mathrm{Sing}(V)$ and $y \notin Sing(f)$. Then $y \in Y_{ns}$, and there is nothing to prove.

**2**. $p \notin \mathrm{Sing}(V)$ and $y \in Sing(f)$. Choose a regular parameter $t$ at $p$. By the deformation theory of a node, we have the description

$$Y \simeq \mathrm{Spf}\ k[[u, v, t]]/(uv - h(t)),$$

with $h(0) = 0$. If $h(t) \not\equiv 0$ then we may write $uv = \mu t^k$ for some unit $\mu \in \mathcal{O}_{V,p}$, in which case $(Y, y)$ is a canonical singularity. If $h(t) \equiv 0$ then $(Y, y)$ is a normal crossing point, which is semismooth.

**3**. $p \in \mathrm{Sing}(V)$ and $y \notin Sing(f)$. Then $(Y, y)$ is a normal crossing point again.

**4**. $p \in \mathrm{Sing}(V)$ and $y \in Sing(f)$. We have

$$V \simeq \mathrm{Spf}\ k[[t_1, t_2]]/(t_1 t_2) \quad \text{and} \quad Y \simeq \mathrm{Spf}\ O_V[[u, v]]/(uv - h(t_1, t_2)).$$

We can write $h(t_1, t_2) = h_1(t_1) + h_2(t_2)$, with $h_i(0) = 0$.

(a) If neither $h_i$ is 0, write $h_i(t_i) = \mu_i t_i^{k_i}$, with $\mu_i$ units, then $(Y, y)$ is a degenerate cusp with exceptional locus a cycle of rational curves with exactly $\max(k_1 - 1, 1) + \max(k_2 - 1, 1)$ components.

(b) If, say, only $h_2 \equiv 0$, then we have a degenerate cusp with $\max(k_1 - 1, 1) + 2$ exceptional components.

(c) If $h_1 \equiv 0 \equiv h_2$, then we have a degenerate cusp with 4 components.

(See [K-SB] §4, for (4a)–(4c).)

The statement about the pair $(Y, D)$ follows easily, since $D$ meets $\mathrm{Sing}(Y)$ transversally at normal crossing points: locally in the étale topology it is isomorphic to the pair

$$(\mathrm{Spec}\ k[t_1, t_2, u]/(t_1t_2), \mathrm{Spec}\ k[t_1, t_2, u]/(t_1t_2, u)),$$

which is semi-log-canonical. □

For the following lemma assume that the base field is algebraically closed. Here we are interested in balanced quotients of families of nodal curves over a nodal base. Studying the resulting singularities becomes easy when we show that the situation can be deformed, which is shown in this lemma.

**Lemma 6.7.** *Let $V$ be the formal completion of a nodal curve and $f : Y \to V$ the formal completion of a nodal family at a closed point. Let $y \in Y$ be the closed point and assume $f(y) = p$ is a node. Suppose a finite cyclic group $\Gamma_y \subset \mathrm{Aut}(Y \to V)$ of order $r$ fixes $y$ and acts faithfully on $V$ stabilizing the two branches of $V$ at $p$ with complementary eigenvalues (in other words, the action is balanced). Then there exists a smoothing*

$$\begin{array}{ccc} Y & \subset & Y' \\ \downarrow & & \downarrow \\ V & \subset & V' \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ k & \subset & \mathrm{Spf}\ k[[s]] \end{array}$$

*and a lifting of the action of $\Gamma_y$ to $Y' \to V'$.*

**Proof.**

**1**. Suppose $y \notin Sing(f)$. We may choose a parameter $u$ along the fiber so that $Y_y = \mathrm{Spf}\ k[[t_1, t_2, u]]/(t_1t_2)$. It is not hard to choose $u$ as an eigenvector for $\Gamma_y$. The action is

$$(t_1, t_2, u) \mapsto (\zeta t_1, \zeta^{-1} t_2, \zeta^a u).$$

This clearly lifts to the family given by $t_1t_2 = s$.

**2**. Suppose now $y \in Sing(f)$. We have the local equation $uv = h_1(t_1) + h_2(t_2)$. It is easy to choose $u, v$ so that $uv$ is an eigenvector.

(a) Suppose neither $h_i$ is 0. After a change of coordinates we may assume our local equation is $uv = t_1^{k_1} + t_2^{k_2}, t_1t_2 = 0$ (so $r$ divides $k_1 + k_2$). We can analyse the action of $\Gamma_y$ via its action on the fiber $t_1 = t_2 = 0$. Depending upon whether $\Gamma_y$ stabilizes the branches of $u$ and $v$ or switches them, the action is either $(u, v) \mapsto (\zeta^a u, \zeta^{k_1-a} v)$ or $(u, v) \mapsto (v, \zeta^{k_1} u)$. In either case this action lifts to the deformation $uv = t_1^{k_1} + t_2^{k_2}, t_1 t_2 = s$.

(b) Suppose only $h_2 \equiv 0$. We have $uv = t_1^{k_1}, t_1 t_2 = 0$. Again, the possible actions are either $(u, v) \mapsto (\zeta^a u, \zeta^{k_1-a} v)$ or $(u, v) \mapsto (v, \zeta^{k_1} u)$. As before, the action lifts to the deformation.

(c) If we have $uv = t_1 t_2 = 0$, the possible actions are either $(u, v) \mapsto (\zeta^a u, \zeta^{k-a} v)$ or $(u, v) \mapsto (v, \zeta^k u)$ for some integers $k$ and $a$. For any choice of positive $k_1, k_2$ such that $k_1 - k_2 \equiv k \mod r$ we have that this action lifts to the family $uv = t_1^{k_1} t_2^{k_2}; t_1 t_2 = s$.

Now we look at quotients. First the case of a fixed point which lies in the smooth locus of a fiber:

**Lemma 6.8.** *Let $V$ be a nodal curve and $f : Y \to V$ a nodal family, and let $y \in Y, y \notin \mathrm{Sing}(f)$. Let $\Gamma \subset \mathrm{Aut} Y$ be a balanced finite subgroup, fixing $y$. Let $q : Y/\Gamma \to X$ be the quotient map and let $x = q(y)$. Then $X$ has a semi-log-terminal singularity at $x$. If, furthermore, $D \subset Y$ is a section through $y$ which is $\Gamma$-stable, then $(X, D/\Gamma)$ is a semi-log-canonical pair.*

The fact that $X$ has a semi-log-terminal singularity is in [K-SB] 4.23(iii). The statement with $D$ follows as in Lemma 6.4.

We are left to deal with a fixed point which is a node:

**Lemma 6.9.** *Let $V$ be a nodal curve and $f : Y \to V$ a nodal family, and let $y \in Y, y \in \mathrm{Sing}(f)$. Let $\Gamma \subset \mathrm{Aut} Y$ be a balanced finite subgroup, fixing $y$. Let $q : Y/\Gamma \to X$ be the quotient map and let $x = q(y)$. Then $X$ has a Gorenstein semi-log-canonical singularity at $x$.*

By Lemma 6.4 it is enough to show that $X$ is Gorenstein, and for this it suffices to show that the quotient of the smoothing $Y'$ in Lemma 6.7 is Gorenstein. The quotient variety $Y'/\Gamma$ is clearly Cohen–Macaulay; therefore it suffices to show that its canonical divisor class is Cartier. We are in case (2) of Lemma 6.7. In the cases (2a) and (2b) (respectively (2c)), the sheaf $\omega_{Y'}$ is generated at $y$ by $\frac{du \wedge dv \wedge dt_2}{t_1^{k_1-1}}$ (respectively, $\frac{du \wedge dv \wedge dt_2}{t_1^{k_1-1} t_2^{k_2}}$). The generator is easily seen to be $\Gamma_p$-invariant. □

We have thus obtained:

**Proposition 6.10.** *Let $X \to C$ with sections $s_i : C \to X$ be a coarse fibered surface, $S_i = Im(s_i)$ and $D = \sum S_i$. Then $(X, D)$ is a semi-log-canonical pair.*

## *6.11 Alexeev stable maps*

In [Al3], V. Alexeev defined *surface stable maps*, for which he constructed complete moduli spaces. Our goal here is to compare our moduli of balanced fibered surfaces and coarse balanced fibered surfaces with Alexeev's moduli spaces. In particular, we would like to construct a stack of stable balanced coarse fibered surfaces.

Let $X$ be a reduced, connected projective surface and $D \subset X$ a reduced subscheme of codimension 1. Let $M \subset \mathbf{P}^r$ be a projective scheme.

A morphism $f : X \to M$ is called a *stable map* of the pair $(X, D)$ to $M$, if

**1**. the pair $(X, D)$ has only semi-log-canonical singularities; in particular, for some integer $m > 0$ the sheaf $(\omega_X(D))^{[m]}$ is invertible.

**2**. For a sufficiently large integer $n$, the sheaf $(\omega_X(D))^{[m]} \otimes f^*\mathcal{O}_M(mn)$ is ample.

It is easy to see that the property of a morphism $f : X \to M$ being stable is independent of the choice of the projective embedding of $M$.

Note that, given a stable map $f : X \to M$, one has a well-defined triple of rational numbers:

$$A = c_1(\omega_X(D))^2; \quad B = c_1(\omega_X(D)) \cdot c_1(f^*\mathcal{O}_M(1)); \quad C = c_1(f^*\mathcal{O}_M(1))^2.$$

One can define a functor of families of stable maps with fixed invariants $A, B$ and $C$. This is somewhat subtle, since the sheaf $\omega_X(D)$ is not invertible, and saturation does not commute with base change. This can be resolved either by restricting to "allowable" deformations on which the saturation $(\omega_X(D))^{[m]}$ does commute with base change (this is discussed in an unpublished work by Kollár), or by endowing the surface $X$ with the structure of a Deligne–Mumford stack using the log-Gorenstein covers (this has not been carried out in the literature). Once the functor is defined, one can look for a moduli space. The following result of Alexeev gives the answer:

**Theorem 6.12. (Alexeev)** *Given rationals $A, B$ and $C$, there is a Deligne–Mumford stack $\mathcal{A}l_{A,B,C}(M)$ admitting a projective coarse moduli space $\mathbf{A}l_{A,B,C}(M)$ for surface stable maps $f : X \to M$ with invariants $A, B, C$.*

Now let $\mathcal{X} \to \mathcal{C} \to S$ be a balanced fibered surface and $X \to C \to S$ the associated coarse fibered surface. We have a morphism $C \to \overline{\mathbf{M}}_{\gamma,\nu}$, which we can compose with $X \to C$ and obtain a morphism $f : X \to \overline{\mathbf{M}}_{\gamma,\nu}$. In addition to that, we have $\nu$ sections $C \to X$. The union of the images of these sections gives rise to a divisor $D \subset X$. We have already seen that $(X, D)$ is a semi-log-canonical pair. We now claim:

**Proposition 6.13.** *The morphism $f : X \to \overline{\mathbf{M}}_{\gamma,\nu}$ is a stable map of the pair $(X, D)$ to $\overline{\mathbf{M}}_{\gamma,\nu}$.*

**Proof.** Fix a projective embedding $\overline{\mathbf{M}}_{\gamma,\nu} \in \mathbf{P}^r$. Since $C \to \overline{\mathbf{M}}_{\gamma,\nu}$ is a stable map, we have that $\omega_{C/S} \otimes f^*\mathcal{O}_{\overline{\mathbf{M}}_{\gamma,\nu}}(n)$ is ample for $n \geq 3$. Moreover, the line bundle $\mathcal{L} = \omega_{X/C}^{[m]}$ defined in Lemma 4.30 is nef and relatively ample for $X \to C$. Therefore $\omega_{X/S}^{[m]} \otimes f^*\mathcal{O}_{\overline{\mathbf{M}}_{\gamma,\nu}}(mn)$ is an invertible sheaf, which is relatively ample for $X \to S$. □

It is now easy to see that we have a finite morphism

$$\mathcal{F}_g^d(\gamma,\nu)^{balanced} \to \mathcal{A}l_{A,B,C}(\overline{\mathbf{M}}_{\gamma,\nu})$$

for a suitable $A, B, C$. The image can be viewed as the stack parametrizing the underlying surfaces $X$ of the coarse balanced fibered surfaces, along with the map to $\overline{\mathbf{M}}_{\gamma,\nu}$. In many cases one can recover the structure map $X \to C$, but not always: consider two non-isomorphic smooth stable curves $C_1, C_2$ over a field $K$ which become isomorphic over $\bar{K}$, and set $X = C_1 \times C_2$. The surface $X$ has two structures of a stable coarse fibered surface, defined by $X \to C_1$ and $X \to C_2$, which both give rise to a the same constant map to $\overline{\mathbf{M}}_{\gamma,\nu}$.

One can avoid this phenomenon in the following manner: the stable map $C \to \overline{\mathbf{M}}_{\gamma,\nu}$ naturally induces a map $C \to \mathbf{K}_{g,1}^d(\overline{\mathbf{M}}_{\gamma,\nu})$ (onto the fiber over the point in $\mathbf{K}_g^d(\overline{\mathbf{M}}_{\gamma,\nu})$ corresponding to $C \to \overline{\mathbf{M}}_{\gamma,\nu}$). The family $\mathcal{X} \to \mathcal{C}$ induces a map to the "universal curve" $X \to \overline{\mathbf{M}}_{\gamma,\nu+1}$. One can replace the morphism $X \to \overline{\mathbf{M}}_{\gamma,\nu}$ in the previous consideration by the morphism $X \to \mathbf{K}_{g,1}^d(\overline{\mathbf{M}}_{\gamma,\nu}) \times \overline{\mathbf{M}}_{\gamma,\nu+1}$. One still needs to show that the morphism $X \to C$ is determined by the map, which is not entirely trivial.

Perhaps the most natural approach is to introduce the morphism $X \to C$ into the definition. Consider a pair of objects $((X \to S, X \to \overline{\mathbf{M}}_{\gamma,\nu}),$ $(C \to S, C \to \overline{\mathbf{M}}_{\gamma,\nu}))$ in $\mathcal{A}l_{A,B,C}(\overline{\mathbf{M}}_{\gamma,\nu}) \times \mathcal{K}_g^d\big(\overline{\mathbf{M}}_{\gamma,\nu}\big)(S)$. By the theory of Hilbert schemes, the functor

$$\begin{array}{rcl} \mathrm{Sch}/S & \to & \mathrm{Sets} \\ T & \mapsto & Hom_T(X \times_S T, C \times_S T) \end{array}$$

is representable by a scheme with quasi-projective connected components, whose formation commutes with base changes. It follows by étale descent that there is a separated Deligne-Mumford stack $\mathcal{T}$ of triples $\big((X \to S, X \to \overline{\mathbf{M}}_{\gamma,\nu}),\ (C \to S, C \to \overline{\mathbf{M}}_{\gamma,\nu}),\ (X \to C)\big)$. There is a morphism $\mathcal{F}_g^d(\gamma,\nu)^{balanced} \to \mathcal{T}$, and we call the image $\mathcal{C}F_g^d(\gamma,\nu)^{balanced}$ the *stack of stable balanced coarse fibered surfaces.*

Since the proof of Alexeev's theorem is quite involved, especially showing boundedness of the stable maps, and since a good resolution of the issue of the "right" deformation space is not available in the literature, it is worthwhile to see that in this particular case we can deduce the existence of the space of stable coarse fibered surfaces from our work so far.

Indeed, the existence of $\mathcal{F}_g^d(\gamma,\nu)^{balanced}$ implies that there is a scheme $Z$ and a finite surjective morphism $Z \to \mathcal{F}_g^d(\gamma,\nu)^{balanced}$. Over $Z$ we have a family of fibered surfaces, and in particular a family of coarse fibered surfaces $X \to C \to Z$. We have constructed a relatively ample line bundle $L$ on $X$; we may replace $L$ by a suitable power and thus assume that $L$ is relatively ample. After passing to the frame bundle $Z'$ of sections of $L$, we may assume that the pushforward of $L$ to $Z$ is a free sheaf $V$. Since $X \to Z$ is flat and embedded in $\mathbf{P}(V)$, we have a morphism $Z \to Hilb$ to a suitable Hilbert scheme. The projective linear group acts with finite stabilizers, therefore the quotient is a Deligne–Mumford stack, parametrizing the surfaces $X$ underlying our coarse fibered surfaces. We can now apply the construction above to define $\mathcal{C}F_g^d(\gamma,\nu)^{balanced}$.

It is interesting to look for properties of the morphism

$$\mathcal{F}_g^d(\gamma,\nu)^{balanced} \to \mathcal{C}F_g^d(\gamma,\nu)^{balanced}.$$

For instance, it is clearly birational on the closure of the locus of normal fibered surfaces. As it turns out, there is little more that one can say: this morphism fails in general to be one to one, and moreover, it may be ramified, even on the closure of the locus of normal fibered surfaces. To show this, one simply needs to produce examples of (deformable) coarse fibered surfaces admitting incompatible atlases. The following examples are local, but can be easily globalized.

**Example 6.14.** Here we give an example of two incompatible balanced charts for a coarse fibered surface. Fix an algebraically closed field $k$ of characteristic 0, and set $S = \operatorname{Spec} k$. Take a smooth projective curve $W$ over $k$ with an automorphism $s$ of order 2 with a fixed point $p \in W(k)$, and let $Y_0$ be the curve obtained by attaching two copies of $W$ at $p$. Consider the action of a cyclic group $\Gamma$ of order 2 on $X_0$ where a generator acts like $s$ on each copy of $W$; also let $f$ be the equivariant automorphism of $Y_0$ which acts like $s$ on one copy of $W$ and as the identity on the other. The point is that $f$ commutes with $s$, and the automorphism of $Y_0/\Gamma$ induced by $f$ is the identity.

Let $L$ be a universal deformation space of $X_0$; $\Gamma$ acts on $L$. Let $V'$ be a small étale neighborhood of 0 in $\mathbf{A}^1$, and let $\Gamma$ act on $V'$, such that a generator sends $t$ to $-t$. Choose a non-constant $\Gamma$-equivariant map $V' \to L$; this yields a family $Y'$ of stable curves on $V'$ whose fiber over 0 is exactly $X_0$; $\Gamma$ also acts on $Y$. Let $V$ be the union of two copies of $V'$ glued at 0; there are two families $Y_1$ and $Y_2$ on $V$ obtained by attaching two copies of $Y'$ at $X_0$, one using the identity, the other using $f$. These will not be isomorphic in general, not after an automorphism of $V$, and not even after going to an étale neighborhood of 0 in $V$.

Set $U = V/\Gamma$ and call $X$ the union of two copies of $Y'/\Gamma$ along $X_0/\Gamma$; we claim that the two generically fibered surfaces $X_1 = Y_1/\Gamma \to V/\Gamma = U$ and $X_2 = Y_2/\Gamma \to V/\Gamma = U$ are canonically isomorphic to $X \to U$. In fact the structure sheaf $O_{X_i'}$ fits into an exact sequence

$$0 \longrightarrow O_{X_i'} \longrightarrow O_{Y'} \times O_{Y'} \longrightarrow O_{X_0} \longrightarrow 0,$$

where the map $O_{Y'} \times O_{Y'} \longrightarrow O_{X_0}$ is the difference of the two projections for $i = 1$, and the difference of one projection with the other projection twisted by $f$ for $i = 2$. Now take invariants; we get an exact sequence

$$0 \longrightarrow O_{X_i'/\Gamma} \longrightarrow O_{Y'/\Gamma} \times O_{Y'/\Gamma} \longrightarrow O_{X_0/\Gamma} \longrightarrow 0;$$

where the two maps $O_{Y_1/\Gamma} \times O_{Y_2/\Gamma} \to O_{X_0/\Gamma}$ are equal, because $f$ induces the identity on $O_{X_0/\Gamma}$.

So $(U, Y_1 \to V, \Gamma)$ and $(U, Y_2 \to V, \Gamma)$ are incompatible charts for the surface $X \to U$. If we call $\overline{U}$ the union of two disjoint copies of $U$ and $\overline{V}$ the union of two disjoint copies of $V$ and $\overline{Y} \to V$ the family of stable curves which coincides with $Y_1$ on one copy and with $Y_2$ on the other one, we even obtain an example $(\overline{U}, \overline{Y} \to V, \Gamma)$ of a chart that is not compatible with itself. This is not surprising, because we are using the étale topology, so $U$ is not in general embedded in $C$; this could not happen if we were working over $\mathbf{C}$ with the analytic topology.

**Example 6.15.** We now give an example of two incompatible charts on a fibered surface over $S = \text{Spec } k[\epsilon]/(\epsilon^2)$, which coincide modulo $\epsilon$. We then globalize this example to a complete curve. This implies, in particular, that the map $\mathcal{F}_g^d(\gamma, \nu) \to \mathcal{C}F_g^d(\gamma, \nu)$ is not always unramified.

Let $F = \text{Spec } k[u, w]/(uw)$. We will use $F$ as a constant fiber in charts $Y \to V$, so that $Y = F \times V$. Specifically, consider the following two curves: $V_0 = \text{Spec } k[z, t, \epsilon]/(zt, \epsilon^2)$ and $V_\epsilon = \text{Spec } k[z, t, \epsilon]/(zt - \epsilon, \epsilon^2)$. Define $Y_0 = F \times V_0$ and $Y_\epsilon = F \times V_\epsilon$. We have obvious morphisms $Y_0 \to V_0 \to S$ and $Y_\epsilon \to V_\epsilon \to S$. The fibers over Spec $k \subset B$ are clearly isomorphic, but $Y_0$ and $Y_\epsilon$ are clearly non-isomorphic.

We now define an action of $C_6$, the cyclic group of order 6, on these schemes. Let $\zeta$ be a primitive sixth root of 1. We choose a generator of $C_6$ and call it $\zeta$ as well; define its action as follows:

$$(u, w, z, t, \epsilon) \mapsto (\zeta^3 u, \zeta^2 w, \zeta z, \zeta^{-1} t, \epsilon)$$

This clearly defines an action of $C_6$ on both $Y_0 \to V_0 \to S$ and $Y_\epsilon \to V_\epsilon \to S$.

Let $C_0 = V_0/C_6$. Explicitly,

$$C_0 = \text{Spec } k[z^6, t^6, \epsilon]/(z^6 t^6, \epsilon^2).$$

Similarly let $C_\epsilon = V_\epsilon/C_6$. It is easy to see that we have

$$C_\epsilon = \text{Spec } k[z^6, t^6, \epsilon]/(z^6 t^6, \epsilon^2).$$

Thus we can use the notation

$$C = V_0/C_6 = V_\epsilon/C_6.$$

We denote $X_0 = Y_0/C_6$ and $X_\epsilon = Y_\epsilon/C_6$. Clearly $X_0 \to C \to S$ and $X_\epsilon \to C \to S$ are coarse fibered surfaces over $S$. Our main claim is that they are isomorphic.

We will produce an isomorphism of coarse fibered surfaces by choosing isomorphisms over suitable open sets, showing that these glue together over a large open set, and arguing that an isomorphism on the large open set must extend. Let us first work out the open sets.

We denote by $U_0$ the localization of $Y_0$ at $u, \ldots$ , and by $T_\epsilon$ the localization of $Y_\epsilon$ at $t$. Explicitly,

$$
\begin{aligned}
U_0 &= \text{Spec } k[u, u^{-1}, z, t, \epsilon]/(zt, \epsilon^2) \\
W_0 &= \text{Spec } k[w, w^{-1}, z, t, \epsilon]/(zt, \epsilon^2) \\
Z_0 &= \text{Spec } k[u, w, z, z^{-1}, \epsilon]/(uw, \epsilon^2) \\
T_0 &= \text{Spec } k[u, w, t, t^{-1}, \epsilon]/(uw, \epsilon^2) \\
U_\epsilon &= \text{Spec } k[u, u^{-1}, z, t, \epsilon]/(zt - \epsilon, \epsilon^2) \\
W_\epsilon &= \text{Spec } k[w, w^{-1}, z, t, \epsilon]/(zt - \epsilon, \epsilon^2) \\
Z_\epsilon &= \text{Spec } k[u, w, z, z^{-1}, \epsilon]/(uw, \epsilon^2) \\
T_\epsilon &= \text{Spec } k[u, w, t, t^{-1}, \epsilon]/(uw, \epsilon^2)
\end{aligned}
$$

These open sets are clearly stable under the action of $C_6$.

The expressions above immediately give isomorphisms $Z_0 \simeq Z_\epsilon$ and $T_0 \simeq T_\epsilon$. These canonically induce isomorphisms $Z_0/C_6 \simeq Z_\epsilon/C_6$ and $Z_0/C_6 \simeq Z_\epsilon/C_6$.

Let us address the open sets $U_0$ and $U_\epsilon$. These are clearly nonisomorphic, but we claim that their quotients by the subgroup of two elemets are isomorphic. Indeed, note that $\zeta^3$ acts trivially on $u, u^{-1}$. Therefore

$$
\begin{aligned}
U_\epsilon/C_2 &\simeq \text{Spec } k[u, u^{-1}] \times V_\epsilon/C_2 \\
&= \text{Spec } k[u, u^{-1}] \times \text{Spec } k[z^2, t^2, zt, \epsilon]/(zt - \epsilon, \epsilon^2) \\
&= \text{Spec } k[u, u^{-1}] \times \text{Spec } k[z^2, t^2, \epsilon]/(\epsilon^2) \\
&\simeq \text{Spec } k[u, u^{-1}] \times V_0/C_2 \\
&\simeq U_0.
\end{aligned}
$$

Taking the quotient by the residual $C_3$ action, this isomorphism clearly induces an isomorphism

$$U_\epsilon/C_6 \simeq U_0/C_6.$$

Note that on the intersections $U_\epsilon/C_2 \cap Z_\epsilon/C_2$ and $U_0/C_2 \cap Z_0/C_2$ the isomorphism above coincides with the restriction of the isomorphism $Z_\epsilon/C_2 \simeq Z_0/C_2$. Indeed, both are given by identifiying the variables $u, z^2$ and $\epsilon$. Therefore these isomorphisms coincide on $U_\epsilon/C_6 \cap Z_\epsilon/C_6$ as well. A similar situation is obtained by replacing $Z_\epsilon$ by $T_\epsilon$.

We now address the open sets $W_\epsilon$ and $W_0$ in a similar manner, noting that this time $C_3$ acts trivially on the variable $w$. Thus

$$\begin{aligned} W_\epsilon/C_3 &\simeq \text{Spec } k[w,w^{-1}] \times V_\epsilon/C_3 \\ &= \text{Spec } k[w,w^{-1}] \times \text{Spec } k[z^3,t^3,zt,\epsilon]/(zt-\epsilon,\epsilon^2) \\ &= \text{Spec } k[w,w^{-1}] \times \text{Spec } k[z^3,t^3,\epsilon]/(\epsilon^2) \\ &\simeq \text{Spec } k[w,w^{-1}] \times V_0/C_3 \\ &\simeq W_0. \end{aligned}$$

Again, it is easy to check that this isomorphism agrees on the intersections with the open sets $Z_\epsilon$ and $T_\epsilon$. We note that all these isomorphisms are isomorphisms over the curve $C$, which coincide with the identity modulo $\epsilon$.

Let $X'_\epsilon$ be the open subscheme which is the complement of the image of the point $u = w = z = t = 0$ on $Y_\epsilon$; define $X'_0$ in an analogous way. Thus far we have obtained an isomorphism $\phi' : X'_\epsilon \to X'_0$. The closure $\Sigma \subset X_\epsilon \times_C X_0$ of the graph of $\phi'$ is a subscheme supported along the graph of the identity $(X_\epsilon)_{\text{Spec } k} = (X_0)_{\text{Spec } k}$. Therefore the projections $p_\epsilon : \Sigma \to X_\epsilon$ and $p_0 : \Sigma \to X_0$ are finite. Note also $p_\epsilon$ and $p_0$ are isomorphic along a dense open set. Since the schemes $X_0$ and $X_\epsilon$ satisfy Serre's condition $S_2$, these projection admit sections, giving rise to a morphism $\phi : X_\epsilon \to X_0$, which by the same reason is an isomorphism.

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Department of Mathematics
Boston University
111 Cummington Street
Boston, MA 02215
USA
e-mail: abrmovic@@math.bu.edu

Dipartimento di Matematica
Università di Bologna
Piazza di Porta San Donato 5
40127 Bologna, Italy
e-mail: vistoli@dm.unibo.it