A tale of Algebra and Geometry

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Intersection theory on algebraic stacks and their moduli spaces [Inv. 1989]

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- This is wonderful: it tells you something about varieties, which you can use even if you know nothing about the tools, namely stacks.
- Angelo’s thesis leads to many explicit computations, numerous theses, and further work at the foundation of enumerative geometry (180 citations).

Theorem [Invent. Math. 1998]

Assume that \(\kappa\) has characteristic \(\neq 2\) and \(3\). Then

\[ A^*(\mathcal{M}_2) = \mathbb{Z}[\lambda_1, \lambda_2]/(10\lambda_1, 2\lambda_1^2 - 24\lambda_2) \]
“But you can’t really ignore the automorphisms, can you?”

- The setting is Harvard, possibly around 1990, a course on moduli of curves, by the great conjurer of families of curves. There is a discussion of moduli functors, properties, tangent spaces, etc.
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- The next class was given by Angelo, a career-changing event, the first proper introduction to algebraic stacks for many.
- He is basically telling students and professor alike how to seriously think about families and moduli.
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However . . .
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The student is scared and would have weaseled out.

But the professor has two bodyguards on both sides, nodding, smiling. Angelo has this towering figure, and there is no way the student would escape!
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- The next morning Angelo reveals a beautiful Lemma.
The Purity Lemma [JAMS 2002]

Let $\mathcal{M} \to \mathbf{M}$ be the coarse moduli space of a separated Deligne-Mumford stack, $X$ a separated $S_2$ surface, $P$ a closed point. Assume that the local fundamental group of $U = X \setminus P$ around $P$ is trivial.
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Let $f : X \to \mathbf{M}$ be a morphism. Suppose there is a lifting $\tilde{f}_U : U \to \mathcal{M}$:

\[
\begin{array}{ccc}
  U & \rightarrow & X \quad f \\
  \downarrow & & \downarrow \\
  \mathbf{M} & \rightarrow & \mathcal{M}
\end{array}
\quad \rightarrow \quad
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Then the lifting extends to $X$:

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The purity lemma: localization and lifting on $U$

- The problem is étale local, so we may pass to strict henselization.
- We can thus assume $U$ simply connected,
- and $\mathcal{M} = [V/\Gamma]$, with $V \to \mathcal{M}$ finite étale.
- Consider $V_U = U \times_\mathcal{M} V$.

Since $V_U \to U$ is finite étale and $U$ simply connected there is a section $U \to V_U$ composing to a morphism $U \to V$. 
The purity lemma: end of proof

- Consider the closure $Y$ of $U$ in $X \times_M V$:

\[
\begin{array}{ccc}
Y & \hookrightarrow & X \times_M V \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & M
\end{array}
\]

- As $V \rightarrow M$ is finite, $Y \rightarrow X$ is finite.
- As $U \rightarrow X$ is birational and isomorphism away from codimension 2, $Y \rightarrow X$ is also.
- As $X$ is $S_2$, we have $Y \rightarrow X$ an isomorphism.
- $X \rightarrow Y \rightarrow \cdots \rightarrow M$ is the needed lifting.
Random gems

**Theorem [Invent. Math. 1998]**

Assume that $\kappa$ has characteristic $\neq 2$ and $3$. Then $\mathcal{M}_2 = [X/GL_2]$, where $X$ is the space of smooth degree 6 binary forms (and the action is twisted!).
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**Theorem 1 [Kresch-Vistoli, BLMS 2004]**

Let $X$ be a Deligne–Mumford quotient stack over a field having a quasiprojective coarse moduli space. Then $X$ has a finite flat lci cover $Z \to X$ by a quasiprojective scheme which is as smooth as $X$. 
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Theorem 1.2 [Brosnan-Reichstein-Vistoli 2009]
The essential dimension of $\mathcal{M}_2$ is 5.
This is an interim report.

More to come!