Resolution and logarithmic resolution by weighted blowing up

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Algebraic geometry and Moduli Seminar

ETH Zürich, July 15, 2020

How to resolve

To resolve a singular variety X one wants to

(1) find the worst singular locus $S \subset X$,

(2) Hopefully S is smooth - blow it up.

Fact

This works for curves but not in general.

Example: Whitney's umbrella

Consider $X = V(x^2 - y^2 z)$

(1) The worst singularity is the origin.

(2) In the z chart we get
$x = x'z,\ y = y'z,$ giving
$x'^2 z^2 - y'^2 z^3 = 0, \quad \text{or} \quad z^2(x'^2 - y'^2 z) = 0.$
The first term is exceptional, the second is the same as X.

Two theorems

Nevertheless:

Theorem (ℵ-T-W, McQuillan, characteristic 0)

There is a functor F associating to a singular subvariety $X \subset Y$ embedded in a smooth variety Y, a center $\bar{J}$ with stack theoretic weighted blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\operatorname{maxinv}(X') < \operatorname{maxinv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.

Theorem (Quek, characteristic 0)

There is a functor F associating to a logarithmically singular subvariety $X \subset Y$ embedded in a logarithmically smooth variety Y, a logarithmic center $\bar{J}$ with stack theoretic logarithmic blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\operatorname{maxloginv}(X') < \operatorname{maxloginv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n logarithmically smooth.

Context: families

Hironaka's theorem resolves varieties. What can you do with families of varieties $X \to B$?

Theorem ($\aleph$-Karu, 2000)

There is a modification $X' \to B'$ which is logarithmically smooth.

Logarithmically smooth = toroidal:

- A toric morphism $X \to B$ of toric varieties is a torus equivariant morphism.
- A toroidal embedding $U_X \subset X$ is an open embedding étale locally isomorphic to toric $T \subset V$.
- A toroidal morphism $X \to B$ of toroidal embeddings is étale locally isomorphic to a toric morphism.

Examples of toroidal morphisms

A toric morphism $X \to B$ of toric varieties is a torus equivariant morphism.e.g.

- $$\operatorname{Spec}\mathbb{C}[x,y,z]/(xy-z^2) \to \operatorname{Spec}\mathbb{C},$$

- $$\operatorname{Spec}\mathbb{C}[x] \to \operatorname{Spec}\mathbb{C}[x^2],$$

- toric blowups

Context: functoriality

- Hironaka's theorem is functorial.
- [$\aleph$-Karu 2000] is not: relied on deJong's method.
- For K–S-B or K-moduli want functoriality.

Theorem ($\aleph$-T-W 2020)

Given $X \to B$ there is a relatively functorial logarithmically smooth modification $X' \to B'$.

- This respects $\mathrm{Aut}_B X$.
- Does not modify log smooth fibers.

Context: principalization

- Following Hironaka, the above theorem is based on embedded methods:

Theorem (ℵ-T-W 2020)

Given $Y \to B$ logarithmically smooth and $\mathcal{I} \subset \mathcal{O}_Y$, there is a relatively functorial logarithmically smooth modification $Y' \to B'$ such that $\mathcal{I}\mathcal{O}_{Y'}$ is monomial.

- This is done by a sequence of logarithmic modifications,
- where in each step E becomes part of the divisor $D_{Y'}$.

Example 1

- $Y = \operatorname{Spec} k[x, u]; \quad D_Y = V(u); \quad B = \operatorname{Spec} k; \quad \mathcal{I} = (x^2, u^2).$
- Blow up $J = (x, u)$
- $\mathcal{I}\mathcal{O}_{Y'} = \mathcal{O}(-2E)$

Example 1/2

- $Y = \operatorname{Spec} k[x, u]; \quad D_Y = V(u); \quad \mathcal{I} = (x^2, u^2)$
- $Y_0 = \operatorname{Spec} k[x, v]; \quad D_{Y_0} = V(v); \quad \mathcal{I}_0 = (x^2, v),$
- $f : Y \to Y_0 \quad v = u^2 \quad \text{so} \quad \mathcal{I} = f^*\mathcal{I}_0$
- By functoriality blow up J_0 so that $f^*J_0 = J = (x, u)$.
- Blow up $J_0 = (x, \sqrt{v})$
- Whatever J_0 is, the blowup is a stack.

Example 1/2: charts

- x chart: $v = v'x^2$:

$$(x^2, v) = (x^2, v'x^2) = (x^2)$$

  exceptional, so monomial.

- $\sqrt{v}$ chart: $v = w^2, x = x'w$, with ± 1 action $(x', w) \mapsto (-x', -w)$:

$$(x^2, v) = ({x'}^2 w^2, w^2) = (w^2)$$

  exceptional, so monomial.

- The schematic quotient of the above is not toroidal.

Resolution again

Theorem (ℵ-T-W, McQuillan, characteristic 0)

There is a functor F associating to a singular subvariety $X \subset Y$ embedded in a smooth variety Y, a center $\bar{J}$ with stack theoretic weighted blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\mathrm{maxinv}(X') < \mathrm{maxinv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.

Example

For $X = V(x^2 - y^2 z)$ we have $inv_p(X) = (2, 3, 3)$
We read it from the degrees of terms.
The center is:
$J = (x^2, y^3, z^3); \bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$.

Example: blowing up Whitney's umbrella $x^2 = y^2 z$

The blowing up $Y' \to Y$ makes $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly:

- The z chart has $x = w^3 x_3, y = w^2 y_3, z = w^2$ with chart

  $$Y' \;=\; [\,\mathrm{Spec}\,\mathbb{C}[x_3, y_3, w] \;/\; (\pm 1)\,],$$

  with action of (± 1) given by $(x_3, y_3, w) \mapsto (-x_3, y_3, -w)$.
  The transformed equation is

  $$w^6(x_3^2 - y_3^2),$$

  and the invariant of the proper transform $(x_3^2 - y_3^2)$ is $(2,2) < (2,3,3)$.

Order

We fix Y smooth and $\mathcal{I} \subset \mathcal{O}_Y$.

Definition

For $p \in Y$ let $\operatorname{ord}_p(\mathcal{I}) = \max\{a : \mathcal{I} \subseteq \mathfrak{m}_p^a\}$.

- We denote by $\mathcal{D}^a$ the sheaf of a-th order operators.
- We note that $\operatorname{ord}_p(\mathcal{I}) = \min\{a : \mathcal{D}^a \mathcal{I}_p\} = (1)$.
- The invariant starts with $a_1 = \operatorname{ord}_p(\mathcal{I})$.

Proposition

The order is upper semicontinuous.

Proof.

$V(\mathcal{D}^{a-1}\mathcal{I}) = \{p : \operatorname{ord}_p(\mathcal{I}) \geq a\}$. ♠

Maximal contact

Definition

A regular parameter $x_1 \in \mathcal{D}^{a_1-1}\mathcal{I}_p$ is called a maximal contact element.

The center starts with $(x_1^{a_1}, \ldots)$.

Lemma

In characteristic 0 a maximal contact exists on an open neighborhood of p.

Since $1 \in \mathcal{D}^{a_1}\mathcal{I}_p$ there is x_1 with derivative 1. This derivative is a unit in a neighborhood.

Example

For $\mathcal{I} = (x^2 - y^2 z)$ we have $\mathrm{ord}_p \mathcal{I} = 2$ with $x_1 = x$ (or $x + y^2 + yz + h.o.t.$).

Coefficient ideals

We must restrict to $x_1 = 0$ the data of all

$$\mathcal{I},\ \mathcal{D}\mathcal{I},\ \ldots,\ \mathcal{D}^{a_1-1}\mathcal{I}$$

with corresponding weights $a_1,\ a_1-1,\ \ldots,\ 1$.
We combine these in

$$C(\mathcal{I}, a_1)\ :=\ \sum f\left(\mathcal{I}, \mathcal{D}\mathcal{I}, \ldots, \mathcal{D}^{a_1-1}\mathcal{I}\right),$$

where f runs over monomials $f = t_0^{b_0} \cdots t_{a_1-1}^{b_{a_1-1}}$ with weights

$$\sum b_i(a_1 - i) \geq a_1!.$$

Define $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$.

Defining $J_{\mathcal{I}}$

Again $a_1 = \mathrm{ord}_p \mathcal{I}$ and x_1 maximal contact.
We denoted $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$.

Definition

Suppose $\mathcal{I}[2]$ has invariant $\mathrm{inv}_p(\mathcal{I}[2])$ defined with parameters $\bar{x}_2, \ldots, \bar{x}_k$, with lifts $x_2, \ldots, x_k$. Set

$$\mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left(a_1, \frac{\mathrm{inv}_p(\mathcal{I}[2])}{(a_1 - 1)!}\right)$$

and

$$J_{\mathcal{I}} = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

Write $(a_1, \ldots, a_k) = \ell(1/w_1, \ldots, 1/w_k)$ with $w_i, \ell \in \mathbb{N}$ and $\gcd(w_1, \ldots, w_k) = 1$. We set

$$\bar{J}_{\mathcal{I}} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k}).$$

Examples of $J_{\mathcal{I}}$

$$\mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left(a_1, \frac{\mathrm{inv}_p(\mathcal{I}[2])}{(a_1-1)!}\right), \quad \text{with} \quad J_{\mathcal{I}} = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

Example

(0) for $X = V(x^2 + y^2 z)$ we have $\mathcal{I}[2] = (y^2 z)$, leading to $J_{\mathcal{I}} = (x^2, y^3, z^3), \quad \bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}, z^{1/2})$

(1) for $X = V(x^5 + x^3y^3 + y^8)$ we have $\mathcal{I}[2] = (y)^{180}$, so $J_{\mathcal{I}} = (x^5, y^{180/24}) = (x^5, y^{15/2}),\ \bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2})$.

(2) for $X = V(x^5 + x^3y^3 + y^7)$ we have $\mathcal{I}[2] = (y)^{7\cdot 24}$, so $J_{\mathcal{I}} = (x^5, y^7), \quad \bar{J}_{\mathcal{I}} = (x^{1/7}, y^{1/5})$.

Properties of the invariant

Proposition

- inv_p *is well-defined.*
- inv_p *is lexicographically upper-semi-continuous.*
- inv_p *is functorial.*
- inv_p *takes values in a well-ordered set.*

We define $\text{maxinv}(X) = \max_p \text{inv}_p(X)$.
The invariant is well defined because of the MC-invariance property of $C(\mathcal{I}, a_1)$. The rest is induction!

Theorem (MC-invariance [Włodarczyk, Kollár])

Given maximal contacts x_1, x_1' *there are étale* $\pi, \pi' : \tilde{Y} \rightrightarrows Y$ *such that* $\pi^* x_1 = \pi'^* x_1' \ldots$ *and* $\pi^* C(\mathcal{I}, a_1) = \pi'^* C(\mathcal{I}, a_1)$.

Definition of $Y' \to Y$

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

$$\mathcal{A}_{\bar{J}} \quad \subset \quad \mathcal{O}_Y[T]$$

as the integral closure of the image of

$$\begin{array}{ccc} \mathcal{O}_Y[Y_1, \ldots, Y_n] & \longrightarrow & \mathcal{O}_Y[T] \\ Y_i & \longmapsto & x_i T^{w_i}. \end{array}$$

Let

$$S_0 \subset \operatorname{Spec}_Y \mathcal{A}_{\bar{J}}, \quad S_0 = V((\mathcal{A}_{\bar{J}})_{>0}).$$

Then

$$Bl_{\bar{J}}(Y) \ := \ \mathcal{P}roj_Y \mathcal{A}_{\bar{J}} \ := \quad \left[(\operatorname{Spec} \mathcal{A}_{\bar{J}} \smallsetminus S_0) \ / \ \mathbb{G}_m \right].$$

Description of $Y' \to Y$

- **Charts:** The x_1-chart is

$$[\operatorname{Spec} k[u, x_2', \ldots, x_n'] \ / \ \boldsymbol{\mu}_{w_1}],$$

with $x_1 = u^{w_1}$ and $x_i = u^{w_i} x_i'$ for $2 \leq i \leq k$, and induced action:

$$(u, x_2', \ldots, x_n') \ \mapsto \ (\zeta u \, , \, \zeta^{-w_2} x_2' \, , \, \ldots \, , \, \zeta^{-w_k} x_k', \ x_{k+1}', \ldots, x_n').$$

- **Toric stack:** Consider $\operatorname{Spec} k[x_1, \ldots, x_n, T]$ with $\mathbb{G}_m$ action with weights $(w_1, \ldots, w_n, -1)$. Let U be the open set where one of the x_i is a unit. Then $Y' = [U/\mathbb{G}_m]$.
  It is an example of a *fantastack* [Geraschenko-Satriano], the stack quotient of a Cox construction.

What is J?

Definition

Consider the Zariski-Riemann space $\mathbf{ZR}(Y)$ with its sheaf of ordered groups Γ, and associated sheaf of rational ordered group $\Gamma \otimes \mathbb{Q}$.

- A valuative $\mathbb{Q}$-ideal is

$$\gamma \in H^0\left(\mathbf{ZR}(Y), (\Gamma \otimes \mathbb{Q})_{\geq 0}\right).$$

- $\mathcal{I}_\gamma := \{f \in \mathcal{O}_Y : v(f) \geq \gamma_v \forall v\}$.
- $v(\mathcal{I}) := (\min v(f) : f \in \mathcal{I})_v$.

A center is in particular a valuative $\mathbb{Q}$-ideal. It is also an idealistic exponent or graded sequence of ideals.

Admissibility and coefficient ideals

Definition

J is $\mathcal{I}$-admissible if $v(J) \leq v(\mathcal{I})$.

Lemma

This is equivalent to $\mathcal{I}\mathcal{O}_{Y'} = E^\ell \mathcal{I}'$, with $J = \bar{J}^\ell$ and $\mathcal{I}'$ an ideal.

Indeed, on Y' the center J becomes E^ℓ, in particular principal.
This is more subtle in Quek's theorem!

Proposition

A center J is $\mathcal{I}$-admissible if and only if $J^{(a_1-1)!}$ is $C(\mathcal{I}, a_1)$- admissible.

This is a consequence of Kollár's $\mathcal{D}$-balanced property of $C(\mathcal{I}, a_1)$.

The key theorems

Theorem

$\mathrm{inv}_p(\mathcal{I})$ *is the maximal invariant of an $\mathcal{I}$-admissible center.*

Theorem

$I_{\mathcal{I}}$ *is well-defined: it is the unique admissible center of maximal invariant.*

Theorem

$C(\mathcal{I}, a_1)\mathcal{O}_{Y'} \ = \ E^{\ell'} C'$ *with* $\mathrm{inv}_{p'} C' \ < \ \mathrm{inv}_p(C(\mathcal{I}, a_1)).$

Theorem

$\mathcal{I}\mathcal{O}_{Y'} = E^{\ell}\mathcal{I}'$ *with* $\mathrm{inv}_{p'}\mathcal{I}' \ < \ \mathrm{inv}_p(\mathcal{I}).$

The end

Thank you for your attention