$$(f_1^{a_1},\ldots,f_k^{a_k}) \;:=\; (\min\{a_i\cdot v(f_i)\})_v \quad\in\quad H^0(\mathbf{ZR}(Y),\Gamma_{\mathbb{Q}+}) \tag{1}$$

**Lemma 0.0.1.** *After passing to completions we may write*

$$C(\mathcal{I},a)=(x_1^{a!})+(x_1^{a!-1}\tilde{\mathcal{C}}_1)+\cdots+(x_1\tilde{\mathcal{C}}_{a!-1})+\tilde{\mathcal{C}}_{a!}.$$

**Theorem 0.0.2.** *The invariant* $\mathrm{inv}_p$ *is independent of the choices. It is upper-semi-continuous. It is functorial for smooth morphisms: if* $f:Y_1\to Y$ *is smooth and* $p'\in Y'$ *then* $\mathrm{inv}_{p'}(\mathcal{I}\mathcal{O}_{Y_1})=\mathrm{inv}_{f(p')}(\mathcal{I})$.

*Proof.* The integer $a_1=\mathrm{ord}_p(\mathcal{I})=\max\{a:\mathcal{I}_p\subseteq\mathfrak{m}_p^a\}$ requires no choices. Given a regular sequence $(x_1,\ldots,x_n)$ extending $(x_1,\ldots,x_k)$, and given another maximal contact element $x_1'$, we may choose constants $t_i$, and replace $x_2,\ldots x_n$ by $x_2+t_2x_1,\ldots x_n+t_nx_1$ so that also $(x_1',x_2,\ldots,x_n)$ is a regular sequence. We can now write $x_1'=\alpha x_1+f$ with $\alpha\neq 0$ and $f\in\tilde{\mathcal{C}}_1$, and the ideal $\mathcal{I}[2]=\bar{\mathcal{C}}_{a_1!}$ remains unchanged. By induction $a_2,\ldots,a_k$ are independent of choices. Hence $(a_1,\ldots,a_k)$ is independent of choices.

Since the closed subscheme $V(\mathcal{D}^{\leq a}\mathcal{I})$ is the locus where $\mathrm{ord}_p(\mathcal{I})\geq a$, the order is upper-semi-continuous. The subscheme $V(\mathcal{D}^{\leq a_1}\mathcal{I})$ is contained in $V(x_1)$ on which $\mathrm{inv}_p(\mathcal{I}[2])$ is upper-semi-continuous by induction, hence $\mathrm{inv}_p(\mathcal{I})$ is upper-semi-continuous.

Since both $\mathrm{ord}_p(\mathcal{I})$ and the formation of coefficient ideals are functorial for smooth morphisms, the invariant is functorial for smooth morphisms. ♣

**Lemma 0.0.3.** *If* $x_1'$ *is another maximal contact element such that* $(x_1',x_2,\ldots,x_n)$ *is a regular sequence, then* $J=({x_1'}^{a_1},x_2^{a_2}\ldots,x_k^{a_k})$ *is also a center associated to* $\mathcal{I}$ *at* $p$.

Again $x_1'=\alpha x_1+f$ with $\alpha\neq 0$ and $f\in\tilde{\mathcal{C}}_1$, and the ideal $\mathcal{I}[2]=\tilde{\mathcal{C}}_{a_1!}$ remains unchanged.

0.0.4. *Basic properties.* The description of the monomial valuation of $J$ immediately provides the following lemmas:

**Lemma 0.0.5.** *If* $J$ *is both* $\mathcal{I}_1$*-admissible and* $\mathcal{I}_2$*-admissible then* $J$ *is* $\mathcal{I}_1+\mathcal{I}_2$*-admissible. If* $J$ *is* $\mathcal{I}$*-admissible then* $J^k$ *is* $\mathcal{I}^k$*-admissible. More generally if* $J^{c_j}$ *is* $\mathcal{I}_j$*-admissible then* $J^{\sum c_j}$ *is* $\prod\mathcal{I}_j$*-admissible.*

Indeed if $v_J(f)\geq 1$ and $v_J(g)\geq 1$ then $v_J(f+g)\geq 1$ and $v_J(f^{c_1}+g^{c_2})\geq c_1+c_2$, etc.

**Lemma 0.0.6.** *If* $J$ *is* $\mathcal{I}$*-admissible then* $J'=J^{\frac{a_1-1}{a_1}}$ *is* $\mathcal{D}(\mathcal{I})$*-admissible. If* $a_1>1$ *and* $J^{\frac{a_1-1}{a_1}}$ *is* $\mathcal{I}$*-admissible then* $J$ *is* $x_1\mathcal{I}$*-admissible.*

*Proof.* For the first statement note that if $\sum_{i=1}^k\alpha_i/a_i\geq 1$ and $\alpha_j\geq 1$ then

$$v_J\left(\frac{\partial(x_1^{\alpha_1}\cdots x_n^{\alpha_n})}{\partial x_j}\right)\quad=\quad\sum_{i=1}^k\alpha_i/a_i\;-\;1/a_j\quad\geq\quad 1\;-\;1/a_1,$$

so

$$v_{J'}\left(\frac{\partial(x_1^{\alpha_1}\cdots x_n^{\alpha_n})}{\partial x_j}\right)\quad\geq 1,$$

as needed. The other statement is similar. ♣

**Lemma 0.0.7.** *For* $\mathcal{I}_0\subset k[x_2,\ldots,x_n]$ *write* $\tilde{\mathcal{I}}_0=\mathcal{I}_0k[x_1,\ldots,x_n]$*. Assume* $a_1\leq a_2$ *and* $(x_2^{a_2},\ldots x_k^{a_k})$ *is* $\mathcal{I}_0$*-admissible. Then* $(x_1^{a_1},\ldots x_k^{a_k})$ *is* $\tilde{\mathcal{I}}_0$*-admissible.*

Here for generators of $\mathcal{I}_0$ we have $\sum_{i=1}^k\alpha_i/a_i=\sum_{i=2}^k\alpha_i/a_i$.

**Lemma 0.0.8.** $J$ *is* $\mathcal{I}$*-admissible if and only if* $J^{(a_1-1)!}$ *is* $C(\mathcal{I},a_1)$*-admissible.*

This combines Lemmas 0.0.5 and 0.0.6 for the terms defining $C(\mathcal{I},a_1)$.

**Theorem 0.0.9.** *If* $(a_1,\ldots,a_k)=\mathrm{inv}_p(\mathcal{I})$*, with corresponding parameters* $x_1,\ldots,x_k$*, and* $J=(x_1^{a_1},\ldots,x_k^{a_k})$ *a corresponding center, then* $J$ *is* $\mathcal{I}$*-admissible.*

*Proof.* Applying Lemma 0.0.8, we replace $\mathcal{I}$ by $C(\mathcal{I}, a_1)$, rescale the invariant up to $a_1!$ and work on formal completion. We may therefore write

$$\mathcal{I} = (x_1^{a_1}) + (x_1^{a_1-1}\tilde{\mathcal{I}}_1) + \cdots + (x_1\tilde{\mathcal{I}}_{a_1-1}) + \tilde{\mathcal{I}}_{a_1}$$

as in Lemma 0.0.1.

The inductive hypothesis implies $(x_2^{a_2}, \ldots, x_k^{a_k})$ is $\bar{\mathcal{I}}_{a_1}$-admissible. By Lemma 0.0.7 $J$ is $\tilde{\mathcal{I}}_{a_1}$-admissible. By Lemma 0.0.6 $J$ is $(x_1^{a_1-j}\tilde{\mathcal{I}}_j)$-admissible, So by Lemma 0.0.5 $J$ is $\mathcal{I}$-admissible, as needed. ♣

**Theorem 0.0.10.** *The center $J$ associated to $\mathcal{I}$ is unique.*

*Proof.* Rescaling, we amy assume $a_i$ are integers and centers are represented by ideals. The problem is local, and can be verified on formal completions at a point $p \in Y$, so that again we may write using the technical proposition

$$\mathcal{I} = (x_1^{a_1}) + (x_1^{a_1-1}\tilde{\mathcal{I}}_1) + \cdots + (x_1\tilde{\mathcal{I}}_{a_1-1}) + \tilde{\mathcal{I}}_{a_1}.$$

Let $J = (x_1^{a_1}, \ldots, x_k^{a_k})$ and $J' = ({x'_1}^{a_1}, {x'_2}^{a_2}, \ldots, {x'_k}^{a_k})$ be centers associated to $\mathcal{I}$.

**Case 1: $x_1 = x'_1$.** We may assume by induction $x'_i \equiv x_i \mod x_1$. Formula (1) shows that $J = J'$ as valuative $\mathbb{Q}$-ideals.

**Case 2: $x_i = x'_i$ for $i > 1$.** Write $x'_1 = x_1 + f$, where $f \in \tilde{\mathcal{I}}_1$. We may write $J' = ((x'_1)^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k})$.

The basic lemmas imply that $J$ is admissible for each term in $J'$ hence $J$ is admissible for the *ideal* $J'$. Reversing the roles we have that $J'$ is admissible for the *ideal* $J$. This implies that $J = J'$ as valuative $\mathbb{Q}$-ideals.

**Case 3: $J'$ is general but $(x'_1, x_2, \ldots, x_n)$ is a regular sequence.** By Lemma 0.0.3 the center $J'_1 := ((x'_1)^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k})$ is associated to $\mathcal{I}$ as well. By Case 2 $J = J'_1$ as valuative $\mathbb{Q}$-ideals. By Case 1 $J'_1 = J'$ as valuative $\mathbb{Q}$-ideals, so $J = J'$ as valuative $\mathbb{Q}$-ideals, as needed.

**Case 4: the general case.** Since $(x_1, \ldots, x_n)$ is a regular sequence there are constants $t_i$ so that, setting $x''_i = x_i + t_i x_1$, both $(x_1, x''_2, \ldots, x''_n)$ and $(x'_1, x''_2, \ldots, x''_n)$ are regular sequences. By Case 1, $J = (x_1^{a_1}, {x''_2}^{a_2}, \ldots, {x''_n}^{a_n})$ as valuative $\mathbb{Q}$-ideals. By Case 3, $(x_1^{a_1}, {x''_2}^{a_2}, \ldots, {x''_n}^{a_n}) = J'$ as valuative $\mathbb{Q}$-ideals, so $J = J'$ as valuative $\mathbb{Q}$-ideals, as needed.

♣

**Theorem 0.0.11.** *Assume $\mathcal{I}_p \neq (1)$, and let $(a_1, \ldots, a_k) = \mathrm{inv}_p(\mathcal{I})$, with corresponding parameters $x_1, \ldots, x_k$, and $J = (x_1^{a_1}, \ldots, x_k^{a_k})$. For $c \in \mathbb{N}_{>0}$ write $Y'_c \to Y$ for the blowing up of the rescaled center $\bar{J}^{1/c} := (x_1^{1/(w_1 c)}, \ldots, x_k^{1/(w_k c)})$, with corresponding factorization $\mathcal{I}\mathcal{O}_{Y'_c} = E^{a_1 w_1 c}\mathcal{I}'$. Then for every point $p'$ over $p$ we have $\mathrm{inv}_{p'}(\mathcal{I}') < \mathrm{inv}_p(\mathcal{I})$.*

*Proof.* If $k = 0$ the ideal is $(0)$ and there is nothing to prove. When $k = 1$ the ideal is $(x_1^{a_1})$, which becomes exceptional with proper transform $(1)$. We now assume $k > 1$.

Again using Lemma 0.0.1, we choose formal coordinates, work with $\tilde{\mathcal{C}} := C(\mathcal{I}, a_1)$, and write

$$\tilde{\mathcal{C}} = (x_1^{a_1!}) + (x_1^{a_1!-1}\tilde{\mathcal{C}}_1) + \cdots + (x_1\tilde{\mathcal{C}}_1) + \tilde{\mathcal{C}}_{a_1!}.$$

Writing $\tilde{\mathcal{C}}\mathcal{O}_{Y'_c} = E^{a_1! w_1 c}\tilde{\mathcal{C}}'$, we will first show that $\mathrm{inv}_{p'}(\tilde{\mathcal{C}}') < (a_1-1)! \cdot (a_1, a_2, \ldots, a_k)$ for all points $p'$ over $p$.

Write $H = \{x_1 = 0\}$, and $H' \to H$ the blowing up of the reduced center $\bar{J}_H$ associated to $J_H := (x_2^{a_2}, \ldots, x_k^{a_k})$. By Lemma **??** the proper transform $\tilde{H}' \to H$ of $H$ via the blowing up of $\bar{J}$ is the root stack $H'(\sqrt[(cc')]{E_H})$ of $H'$ along $E_H \subset H'$, where $c' = \frac{\mathrm{lcm}(w_1,\ldots,w_k)}{\mathrm{lcm}(w_2,\ldots,w_k)}$. Therefore $\tilde{H}'$ is the blowing up of $\bar{J}_H^{1/(cc')}$, allowing for induction.

We now inspect the behavior on different charts. On the $x_1$-chart we have $x_1 = u^{w_1 c}$ so the first term becomes $(x_1^{a_1!}) = E^{a_1! w_1 c} \cdot (1)$ and $\mathrm{inv}_{p'}\tilde{\mathcal{C}}' = \mathrm{inv}(1) = 0.$[1] This implies that on all other charts it suffices to consider $p' \in \tilde{H}' \cap E$, as all other points belong to the $x_1$-chart. By the inductive assumption, for such points we have

$$\mathrm{inv}_{p'}(\bar{\mathcal{C}}'_{a_1!}) < (a_1-1)! \cdot (a_2, \ldots, a_k).$$

[1]This reflects the fact that before passing to the coefficient ideal $\mathrm{ord}(\mathcal{I}') < a_1$ on this chart - it need not become a unit ideal in general!

Note that the term $(x_1^{a_1!})$ in $\tilde{\mathcal{C}}$ is transformed, via $x_1 = u^{w_1 c} x_1'$ to the form $E^{a_1! w_1 c}({x_1'}^{a_1!})$. It follows that $\mathrm{ord}_{p'}(\tilde{\mathcal{C}}') \leq a_1!$, and if $\mathrm{ord}_{p'}(\tilde{\mathcal{C}}') < a_1!$ then a fortiori $\mathrm{inv}_{p'}(\tilde{\mathcal{C}}') < \mathrm{inv}_p(\tilde{\mathcal{C}})$.

If on the other hand $\mathrm{ord}_{p'}(\tilde{\mathcal{C}}') = a_1!$ then the variable $x_1'$ is a maximal contact element. Using the inductive assumption we compute

$$\mathrm{inv}_{p'}(({x'}_1^{a_1!}) + \tilde{\mathcal{C}}'_{a_1!}) = \left(a_1!, \mathrm{inv}_{p'}(\bar{\mathcal{C}}'_{a_1!})\right) < (a_1!, \mathrm{inv}_{p'}(\bar{\mathcal{C}}_{a_1!})) = (a_1 - 1)!(a_1, \ldots, a_k).$$

Since $\tilde{\mathcal{C}}'$ includes this ideal, we obtain again $\mathrm{inv}_{p'}(\tilde{\mathcal{C}}') < \mathrm{inv}_p(\tilde{\mathcal{C}})$, as claimed.

We deduce that $\mathrm{inv}_{p'}(\mathcal{I}') < \mathrm{inv}_p(\mathcal{I})$ as well: Once again we may assume $x_1'$ is a maximal contact element and $\mathrm{ord}_{p'}(\mathcal{I}') = a_1$. We have the standard inclusions $\mathcal{I}'^{(a_1-1)!} \subset \tilde{\mathcal{C}}' \subset C(\mathcal{I}', a_1)$, hence

$$\mathrm{inv}_{p'}(\mathcal{I}'^{(a_1-1)!}) \geq \mathrm{inv}_{p'}(\tilde{\mathcal{C}}') \geq \mathrm{inv}_{p'}(C(\mathcal{I}', a_1)).$$

Since $\mathrm{inv}_{p'}(\mathcal{I}'^{(a_1-1)!}) = \mathrm{inv}_{p'}(C(\mathcal{I}', a_1))$ we have equalities throughout, hence

$$\mathrm{inv}_{p'}(\mathcal{I}') = \frac{1}{(a_1-1)!}\mathrm{inv}_{p'}(\tilde{\mathcal{C}}') < \frac{1}{(a_1-1)!}\mathrm{inv}_p(\tilde{\mathcal{C}}) = \mathrm{inv}_p(\mathcal{I}),$$

as needed.

♣