Formal decomposition

Following Encinas–Villamayor, consider the algebra $G = \bigoplus G_i$ generated by $D^j(I)$ in degree $a_1 - j$, for $0 \leq j \leq a_1 - 1$. We have $G_{a_1!} = C(I, a_1)$. Writing formally $Y = \text{Spec } k[x_1, \ldots, x_n]$, with $H = V(x_1)$ maximal contact, we consider $\pi : Y \to H$.

Let $\tilde{G}_i = \pi^*(G_i|_H)$.

**Proposition (Formal decomposition)**

$$C(I, a_1) = (x_1^{a_1!}) + (x_1^{a_1!-1})\tilde{G}_1 + \cdots + (x_1)\tilde{G}_{a_1!-1} + \tilde{G}_{a_1!}.$$  

This is proven by decomposing into eigenspaces for $x_1\frac{\partial}{\partial x_1}$.

**Proposition ($D$-balanced property (Kollár))**

$$\tilde{G}_{a_1!-j} \subset \tilde{G}_{a_1!-j}.$$
The center is admissible

**Theorem**

\( J_\mathcal{I} \) is \( \mathcal{I} \)-admissible.

- This is equivalent to \( J^{(a_1-1)!} \) is \( \mathcal{C}(\mathcal{I}, a_1) \)-admissible.
- One checks that \( J^{(a_1-1)!} \) is admissible for each term in the formal decomposition.
- Hence it is admissible.
The unique admissibility theorem

\textbf{Theorem}

\[ J_I = (x_1^{a_1}, \ldots, x_k^{a_k}) \text{ is the unique admissible center of maximal invariant.} \]

- First if \( J' = (x_1^{b_1}, \ldots, x_m^{b_m}) \) is admissible one sees that \( b_1 \leq a_1 \), otherwise \( v_J(f) < 1 \) for \( f \in I \) of order \( a_1 \).
- Assume now \( (b_1, \ldots, b_m) > (a_1 \ldots, a_k) \). So \( a_1 = b_1 \).
- With a bit more work one may assume \( J' = (x_1^{a_1}, x_2^{b_2}, \ldots, x_m^{b_m}) \), with \( x_1' \in k[[x_2, \ldots, x_n]] \).
- Consider the formal completion. Induction gives \( (a_1 - 1)! (a_2, \ldots, a_k) \) is the maximal invariant of \( \tilde{G}_{a_1}! \), with unique center \( (x_2^{a_2}, \ldots, x_k^{a_k})(a_1-1)! \).
- On the other hand \( (x_1^{a_1}, x_2^{b_2}, \ldots, x_m^{b_m})(a_1-1)! \leq v(\tilde{G}_{a_1}) \), giving equality throughout.