Resolution and logarithmic resolution by weighted blowing up

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Work with Michael Tēmkin and Jarosław Włodarczyk
and work by Ming Hao Quek

Also parallel work by M. McQuillan

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How to resolve

To resolve a singular variety $X$ one wants to

1. find the worst singular locus $S \subset X$,
2. Hopefully $S$ is smooth - blow it up.
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Fact

This works for curves but not in general.
Example: Whitney’s umbrella

Consider $X = V(x^2 - y^2 z)$
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1. The worst singularity is the origin.
2. In the $z$ chart we get

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x = x'z, \quad y = y'z, \quad \text{giving}
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\[
x'^2z^2 - y'^2z^3 = 0, \quad \text{or} \quad z^2(x'^2 - y'^2z) = 0.
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   \[ x = x'z, \quad y = y'z, \quad \text{giving} \]
   \[ x'^2z^2 - y'^2z^3 = 0, \quad \text{or} \quad z^2(x'^2 - y'^2z) = 0. \]
   The first term is exceptional, the second is the same as \( X \).
Two theorems

Nevertheless:

**Theorem (ℵ-T-W, McQuillan, 2019, characteristic 0)**

There is a functor $F$ associating to a singular subvariety $X \subset Y$ of a smooth variety $Y$, a center $\bar{J}$ with stack theoretic weighted blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\text{maxinv}(X') < \text{maxinv}(X)$. In particular, for some $n$ the iterate $(X_n \subset Y_n) := F \circ_n (X \subset Y)$ of $F$ has $X_n$ smooth.

**Theorem (Quek, 2020, characteristic 0)**

There is a functor $F$ associating to a logarithmically singular subvariety $X \subset Y$ of a logarithmically smooth variety $Y$, a logarithmic center $\bar{J}$ with stack theoretic logarithmic blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\text{maxloginv}(X') < \text{maxloginv}(X)$. In particular, for some $n$ the iterate $(X_n \subset Y_n) := F \circ_n (X \subset Y)$ of $F$ has $X_n$ logarithmically smooth.
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**Theorem (K-Karu, 2000)**

There is a modification $X' \rightarrow B'$ which is logarithmically smooth.
Context: families

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Logarithmically smooth = toroidal:
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Logarithmically smooth = toroidal:

- A toric morphism $X \to B$ of toric varieties is a torus equivariant morphism.
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Logarithmically smooth = toroidal:

- A toric morphism $X \rightarrow B$ of toric varieties is a torus equivariant morphism.
- A toroidal embedding $U_X \subset X$ is an open embedding étale locally isomorphic to toric $T \subset V$.  

\[ \text{\LaTeX} \]
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- A toroidal morphism $X \rightarrow B$ of toroidal embeddings is étale locally isomorphic to a toric morphism.
Examples of toroidal morphisms

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- toric blowups
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Context: functoriality

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- For K–S–B or K-moduli want functoriality.

**Theorem (ℵ-T-W 2020)**

Given $X \to B$ there is a relatively functorial logarithmically smooth modification $X' \to B'$. 
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For K–S-B or K-moduli want functoriality.

**Theorem (Å-T-W 2020)**

*Given $X \to B$ there is a relatively functorial logarithmically smooth modification $X' \to B'$.*

- This respects $\text{Aut}_B X$.
- Does not modify log smooth fibers.
Following Hironaka, the above theorem is based on embedded methods:

**Theorem (\&-T-W 2020)**

Given $Y \to B$ logarithmically smooth and $I \subset \mathcal{O}_Y$, there is a relatively functorial logarithmically smooth modification $Y' \to B'$ such that $I\mathcal{O}_{Y'}$ is monomial.
Following Hironaka, the above theorem is based on embedded methods:

**Theorem (H-T-W 2020)**

Given $Y \to B$ logarithmically smooth and $\mathcal{I} \subset \mathcal{O}_Y$, there is a relatively functorial logarithmically smooth modification $Y' \to B'$ such that $\mathcal{I}\mathcal{O}_{Y'}$ is monomial.

- This is done by a sequence of logarithmic modifications,
- where in each step $E$ becomes part of the divisor $D_{Y'}$. 
Example 1

- $Y = \text{Spec } k[x, u]$; $D_Y = V(u)$; $B = \text{Spec } k$;
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- $Y = \text{Spec } k[x, u]$; $D_Y = V(u)$; $B = \text{Spec } k$; $I = (x^2, u^2)$.
- Blow up $J = (x, u)$
- $\mathcal{IO}_{Y'} = \mathcal{O}(-2E)$
Example 1/2

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3. $f : Y \to Y_0 \quad v = u^2 \quad \text{so} \quad \mathcal{I} = f^*\mathcal{I}_0$
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- $f: Y \rightarrow Y_0$; $v = u^2$ so $\mathcal{I} = f^*\mathcal{I}_0$
- By functoriality blow up $J_0$ so that $f^*J_0 = J = (x, u)$.
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- \( Y_0 = \text{Spec } k[x, v]; \quad D_{Y_0} = V(v); \quad \mathcal{I}_0 = (x^2, v), \)
- \( f : Y \to Y_0 \quad v = u^2 \quad \text{so} \quad \mathcal{I} = f^* \mathcal{I}_0 \)
- By functoriality blow up \( J_0 \) so that \( f^* J_0 = J = (x, u). \)
- Blow up \( J_0 = (x, \sqrt{v}) \)
- Whatever \( J_0 \) is, the blowup is a stack.
Example 1/2: charts

- **x chart:** \( v = v'x^2 \):
  \[
  (x^2, v) = (x^2, v'x^2) = (x^2)
  \]
  exceptional, so monomial.

- **\( \sqrt{v} \) chart:** \( v = w^2, x = x'w \), with \( \pm 1 \) action \((x', w) \mapsto (-x', -w)\):
  \[
  (x^2, v) = (x'^2w^2, w^2) = (w^2)
  \]
  exceptional, so monomial.

- The schematic quotient of the above is **not toroidal**.
Resolution again

**Theorem (H-T-W, McQuillan, characteristic 0)**

There is a functor $F$ associating to a singular subvariety $X \subset Y$ of a smooth variety $Y$, a center $\bar{J}$ with stack theoretic weighted blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\maxinv(X') < \maxinv(X)$. In particular, for some $n$ the iterate $(X_n \subset Y_n) := F^n(X \subset Y)$ of $F$ has $X_n$ smooth.

Example: For $X = V(x^2 - y^2 z)$ we have $\inv_p(X) = (2, 3, 3)$. We read it from the degrees of terms. The center is: $J = (x^2, y^3, z^3)$; $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$. 

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Abramovich
Resolution and logarithmic resolution
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Resolution again

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**Example**

For $X = V(x^2 - y^2z)$ we have $\text{inv}_p(X) = (2, 3, 3)$

We read it from the degrees of terms.

The center is:

$J = (x^2, y^3, z^3); \bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$. 
Example: blowing up Whitney’s umbrella \( x^2 = y^2 z \)

The blowing up \( Y' \to Y \) makes \( \bar{J} = (x^{1/3}, y^{1/2}, z^{1/2}) \) principal. Explicitly:

- **The z chart has** \( x = w^3 x_3, y = w^2 y_3, z = w^2 \) with chart

  \[
  Y' = \left[ \text{Spec} \mathbb{C}[x_3, y_3, w] / (\pm 1) \right],
  \]

with action of \( (\pm 1) \) given by \( (x_3, y_3, w) \mapsto (-x_3, y_3, -w) \).
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The blowing up $Y' \to Y$ makes $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly:

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The transformed equation is

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w^6 (x_3^2 - y_3^2),
$$
Example: blowing up Whitney’s umbrella $x^2 = y^2z$

The blowing up $Y' \to Y$ makes $\tilde{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly:

- The $z$ chart has $x = w^3x_3$, $y = w^2y_3$, $z = w^2$ with chart
  
  \[ Y' = [\text{Spec } \mathbb{C}[x_3, y_3, w] \ / (\pm 1)], \]

  with action of $(\pm 1)$ given by $(x_3, y_3, w) \mapsto (-x_3, y_3, -w)$.

  The transformed equation is
  
  \[ w^6(x_3^2 - y_3^2), \]

  and the invariant of the proper transform $(x_3^2 - y_3^2)$ is
  
  $(2, 2) < (2, 3, 3)$. 
Order (following Kollár’s book)

We fix $Y$ smooth and $\mathcal{I} \subset \mathcal{O}_Y$.

**Definition**

For $p \in Y$ let $\text{ord}_p(\mathcal{I}) = \max\{a : \mathcal{I} \subseteq m_p^a\}$.

We denote by $D^a$ the sheaf of $a$-th order differential operators.

We note that $\text{ord}_p(\mathcal{I}) = \min\{a : D^a(\mathcal{I}) \neq 0\}$.

The invariant starts with $a_1 = \text{ord}_p(\mathcal{I})$.

**Proposition**

The order is upper semicontinuous.

**Proof.**

$V(D^a - I) = \{p : \text{ord}_p(\mathcal{I}) \geq a\}$.
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$V(\mathcal{D}^{a-1} - \mathcal{I}) = \{p : \text{ord}_p(\mathcal{I}) \geq a\}$. $\blacksquare$
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Maximal contact (following Kollár’s book)

Definition
A regular parameter $x_1 \in \mathcal{D}^{a_1-1}I_p$ is called a maximal contact element.

The center starts with $(x_1^{a_1}, \ldots)$. 
Maximal contact (following Kollár’s book)

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**Lemma (Hironaka, Giraud)**

*In characteristic 0 a maximal contact exists on an open neighborhood of $p$.*

Since $1 \in D^{a_1} I_p$ there is $x_1$ with derivative 1. This derivative is a unit in a neighborhood.

$$I = (x^p)$$
Maximal contact (following Kollár’s book)

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**Example**

For $I = (x^2 - y^2z)$ we have $\text{ord}_p I = 2$ with $x_1 = x$ (or $\alpha x + \text{h.o.t.}$ in $D(I)$).
Coefficient ideals (treated following Kollár)

We must restrict to $x_1 = 0$ the data of all

$I, DI, \ldots, D^{a_1-1}I$

with corresponding weights $a_1, a_1 - 1, \ldots, 1$. 

\[ \bigcirc \oplus D^{0}(I) T \bigoplus D^{a_1-2}(I) T^2 \]
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with corresponding weights $a_1, a_1 - 1, \ldots, 1$. We combine these in

$$C(I, a_1) := \sum f (I, DI, \ldots, D^{a_1-1}I),$$

where $f$ runs over monomials $f = t_0^{b_0} \cdots t_{a_1-1}^{b_{a_1-1}}$ with weights

$$\sum b_i(a_1 - i) \geq a_1!.$$

Define $\mathcal{I}[2] = C(I, a_1)|_{x_1=0}$. 
Defining $J_I$

Again $a_1 = \text{ord}_p \mathcal{I}$ and $x_1$ maximal contact. We denoted $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$ (with order $\geq a_1!$).
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**Definition**

Suppose $\mathcal{I}[2]$ has invariant $\text{inv}_p(\mathcal{I}[2])$ defined with parameters $\bar{x}_2, \ldots, \bar{x}_k$, with lifts $x_2, \ldots, x_k$. 

$(a_1, \ldots, a_k) = \ell(1/w_1, \ldots, 1/w_k)$ with $w_i, \ell \in \mathbb{N}$ and $\gcd(w_1, \ldots, w_k) = 1$. 

We set $\bar{\mathcal{I}} = (x_1/w_1, \ldots, x_k/w_k)$. 

Defining $J_\mathcal{I}$

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$$\text{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left(a_1, \frac{\text{inv}_p(\mathcal{I}[2])}{(a_1 - 1)!}\right)$$

and

$$J_\mathcal{I} = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

Note: need to consider rounds.
Defining $J_I$

Again $a_1 = \text{ord}_p I$ and $x_1$ maximal contact.

We denoted $I[2] = C(I, a_1)|_{x_1=0}$ (with order $\geq a_1!$).

**Definition**

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$$\bar{J}_I = (x_1^{1/w_1}, \ldots, x_k^{1/w_k}).$$
Examples of $J_I$

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Example (0) for $X = V(x^2 + y^2z)$

$$C(I_2) \mid_{x=0} = \begin{pmatrix} x^2 & y^3 & 3 \\ x & y^3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ 1 & 2 \end{pmatrix}$$
Examples of $J_{\mathcal{I}}$

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Example

(0) for $X = V(x^2 + y^2 z)$ we have $\mathcal{I}[2] = (y^2 z)$, leading to

$J_{\mathcal{I}} = (x^2, y^3, z^3), \quad \overline{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}, z^{1/2})$

(1) for $X = V(x^5 + x^3 y^3 + y^8)$

\[
\begin{align*}
\mathcal{I}[2] &= (y^{180}) \quad (5, 1)!
\end{align*}
\]
Examples of $J_\mathcal{I}$

$\text{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left( a_1, \frac{\text{inv}_p(\mathcal{I}[2])}{(a_1 - 1)!} \right)$, with $J_\mathcal{I} = (x_1^{a_1}, \ldots, x_k^{a_k})$.

**Example**

(0) for $X = V(x^2 + y^2 z)$ we have $\mathcal{I}[2] = (y^2 z)$, leading to $J_\mathcal{I} = (x^2, y^3, z^3)$, $\bar{J}_\mathcal{I} = (x^{1/3}, y^{1/2}, z^{1/2})$.

(1) for $X = V(x^5 + x^3 y^3 + y^8)$ we have $\mathcal{I}[2] = (y)^{180}$, so $J_\mathcal{I} = (x^5, y^{180/24}) = (x^5, y^{15/2})$, $\bar{J}_\mathcal{I} = (x^{1/3}, y^{1/2})$.

(2) for $X = V(x^5 + x^3 y^3 + y^7)$
Examples of $J_I$

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Example

(0) for $X = V(x^2 + y^2 z)$ we have $I[2] = (y^2 z)$, leading to $J_I = (x^2, y^3, z^3)$, $\bar{J}_I = (x^{1/3}, y^{1/2}, z^{1/2})$

(1) for $X = V(x^5 + x^3 y^3 + y^8)$ we have $I[2] = (y)^{180}$, so $J_I = (x^5, y^{180/24}) = (x^5, y^{15/2})$, $\bar{J}_I = (x^{1/3}, y^{1/2})$.

(2) for $X = V(x^5 + x^3 y^3 + y^7)$ we have $I[2] = (y)^{7 \cdot 24}$, so $J_I = (x^5, y^7)$, $\bar{J}_I = (x^{1/7}, y^{1/5})$.

Implementation: Jonghyun Lee, Anne Frühbis-Krüger.
Properties of the invariant

**Proposition**

- $\text{inv}_p$ is well-defined.
- $\text{inv}_p$ is lexicographically upper-semi-continuous.
- $\text{inv}_p$ is functorial.
- $\text{inv}_p$ takes values in a well-ordered set.

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**Theorem (MC-invariance [Włodarczyk, Kollár])**

*Given maximal contacts $x_1, x_1'$ there are étale $\pi, \pi' : \tilde{Y} \rightarrow Y$ such that $\pi^* x_1 = \pi'^* x_1' \ldots$ and $\pi^* C(\mathcal{I}, a_1) = \pi'^* C(\mathcal{I}, a_1)$.*
Definition of $Y' \to Y$

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

$$A_{\bar{J}} \subset O_Y[T]$$

as the integral closure of the image of

$$O_Y[Y_1, \ldots, Y_n] \to O_Y[T]$$

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Y_i \quad \longrightarrow \quad x_i T^{w_i}.
$$

Let

$$
S_0 \subset \text{Spec}_Y \bar{A}_{\bar{J}}, \quad S_0 = V((\bar{A}_{\bar{J}})_{>0}).
$$

Then

$$
\text{Bl}_{\bar{J}}(Y) := \text{Proj}_Y \bar{A}_{\bar{J}} := \left[ (\text{Spec} \bar{A}_{\bar{J}} \setminus S_0) / \mathbb{G}_m \right].
$$
Description of $Y' \rightarrow Y$

- **Charts:** The $x_1$-chart is

\[
[\text{Spec } k[u, x'_2, \ldots, x'_n] / \mu_{w_1}],
\]

with $x_1 = u^{w_1}$ and $x_i = u^{w_i}x'_i$ for $2 \leq i \leq k$, and induced action:

\[
(u, x'_2, \ldots, x'_n) \mapsto (\zeta u, \zeta^{-w_2}x'_2, \ldots, \zeta^{-w_k}x'_k, x'_{k+1}, \ldots, x'_n).
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- **Toric stack:** Consider $\text{Spec } k[x_1, \ldots, x_n, T]$ with $\mathbb{G}_m$ action with weights $(w_1, \ldots, w_n, -1)$. Let $U$ be the open set where one of the $x_i$ is a unit. Then $Y' = [U/\mathbb{G}_m]$.

It is an example of a *fantastack* [Geraschenko-Satriano], the stack quotient of a Cox construction.
What is $J$?

Definition

Consider the Zariski-Riemann space $\mathbb{ZR}(Y)$ with its sheaf of ordered groups $\Gamma$, and associated sheaf of rational ordered group $\Gamma \otimes \mathbb{Q}$.

- A valuative $\mathbb{Q}$-ideal is

$$\gamma \in H^0 (\mathbb{ZR}(Y), (\Gamma \otimes \mathbb{Q})_{\geq 0}).$$
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A center is in particular a valuative $\mathbb{Q}$-ideal. It is also an idealistic exponent or graded sequence of ideals.
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\[ v\left( (x, y)^2 \right) \leq v \left( x^2, y^2 \right) \]
Admissibility and coefficient ideals

**Definition**

$J$ is $\mathcal{I}$-admissible if $J \leq v(\mathcal{I})$. 
Admissibility and coefficient ideals

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$J$ is $\mathcal{I}$-admissible if $J \leq \nu(\mathcal{I})$.

**Lemma**

This is equivalent to $\mathcal{O}_{Y'} = E^\ell \mathcal{I}'$, with $J = \overline{J}^\ell$ and $\mathcal{I}'$ an ideal.

Indeed, on $Y'$ the center $J$ becomes $E^\ell$, in particular principal. This is more subtle in Quek’s theorem!
Admissibility and coefficient ideals

**Definition**

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Indeed, on $Y'$ the center $J$ becomes $E^\ell$, in particular principal. This is more subtle in Quek’s theorem!

**Proposition**

A center $J$ is $\mathcal{I}$-admissible if and only if $J^{(a_1-1)!}$ is $C(\mathcal{I}, a_1)$-admissible.
The key theorems

Theorem

$\text{inv}_p(I)$ is the maximal invariant of an $I$-admissible center.

Theorem

$J_I$ is well-defined: it is the unique admissible center of maximal invariant.

$x, \frac{2}{\Delta x} \text{ are in } C(I, \alpha)$
The key theorems

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**Theorem**

\[ C(\mathcal{I}, a_1) \mathcal{O}_{Y'} = E_{\ell'} C' \text{ with } \text{inv}_p C' < \text{inv}_p(\mathcal{C}(\mathcal{I}, a_1)). \]
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**Theorem**

\[
\mathcal{I} \mathcal{O}_{Y'} = E^{\ell} \mathcal{I}' \quad \text{with} \quad \text{inv}_{p'} \mathcal{I}' < \text{inv}_p(\mathcal{I}).
\]

This is a consequence of Kollár’s \( D \)-balanced property of \( C(I, a_1) \).
Quek’s theorem is necessary

\[ \mathcal{I} = (x^2yz + yz^4) \subset \mathbb{C}[x, y, z]. \]
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- \( \mathcal{I} = (x^2yz + yz^4) \subset \mathbb{C}[x, y, z] \).
- Then \( \text{maxinv}(\mathcal{I}) = (4, 4, 4) \) with center \( J = (x^4, y^4, z^4) \), a usual blowup.
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The \( z \)-chart has \( \mathcal{I}' = (y^f(x^2 + z)) \). The new invariant is \((2, 2)\) with reduced center \((y, x^2 + z)\), which is tangent to the exceptional \( z = 0 \).
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- Instead work with logarithmic derivative in \( z \).
- The logarithmic invariant is \( (3, 3, \infty) \) with center \( (y^3, x^3, z^{3/2}) \) and reduced logarithmic center \( (y, x, z^{1/2}) \).
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This reduces logarithmic invariants respecting logarithmic, hence exceptional, divisors.
The end

Thank you for your attention