FUNCTORIAL EMBEDDED RESOLUTION VIA WEIGHTED BLOWINGS UP

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Abstract. We provide a procedure for resolving, in characteristic 0, singularities of a variety X embedded in a smooth variety Y by repeatedly blowing up the worst singularities, in the sense of stack-theoretic weighted blowings up. No history, no exceptional divisors, and no logarithmic structures are necessary to carry this out.

A similar result was discovered independently by Marzo and McQuillan [MM19].

1. Introduction

1.1. **Statement of result.** We consider smooth variety Y of pure dimension n of finite type over a field k *of charactertistic* 0*,* and reduced closed subscheme $X \subset Y$ of pure codimension c; or more generally a closed substack X of a smooth Deligne–Mumford stack Y. Our goal is to resolve singularities of X embedded in Y, revisiting Hironaka's [Hir64, Main Theorem I].

Pairs $X \subset Y$ of possibly different dimensions form a category by considering *surjective* morphisms $(X_1 \subset Y_1) \to (X_2 \subset Y_2)$ of pairs where $f : Y_1 \to Y_2$ is *smooth* and $X_1 = X_2 \times_{Y_2} Y_1$ is the pullback of X_2. We in fact define a resolution functor on this category; it is functorial for all smooth morphisms, whether or not surjective, when interpreted appropriately. This follows principles of [Wło05, Kol07, BM08].

For a geometric point $p \in |X|$ we defined in [ATW17, §2.12.4] an upper-semi-continuous function, the *lexicographic order invariant*, which we rescale here and write as:

$$\operatorname{inv}_p(X) = (a_1(p), \ldots, a_k(p)) \quad \in \quad \mathbb{Q}_{\geq 0}^{\leq n} \quad := \quad \bigsqcup_{k \leq n} \mathbb{Q}_{\geq 0}^{k},$$

ordered lexicographically and taking values in a well-ordered subset. It detects singularities: the invariant is the sequence $\operatorname{inv}_p(X) = (1, \ldots, 1)$ of length c if and only if $p \in X$ is smooth, and otherwise it is bigger. Our invariant inv_x is compatible with smooth morphisms of pairs, whether or not surjective: $\operatorname{inv}_x(X_1) = \operatorname{inv}_{f(x)}(X_2)$.

We define

$$\operatorname{maxinv}(X) = \max_{p \in |X|} \operatorname{inv}_p(X).$$

This is compatible with *surjective* morphisms of pairs.

In Section 3 we introduce stack-theoretic weighted blowings up $Y' \to Y$ along centers locally of the form $\bar{J} = (x_1^{1/w_1}, \ldots x_k^{1/w_k})$, where$(\ell/w_1, \ldots, \ell/w_k) = \operatorname{maxinv}(X)$ for positive integers ℓ, w_i, and $x_1, \ldots x_n$ is a carefully chosen regular system of parameters.

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The aim of this paper is to prove the following:

Theorem 1.1.1. *There is a functor F associating to a* singular *pair $X \subset Y$ a center $\bar{J}$ with blowing up $Y' \to Y$ and proper transform $F(X \subset Y) = (X' \subset Y')$, such that* $\operatorname{maxinv}(X') < \operatorname{maxinv}(X)$. *In particular there is an integer n so that the iterated application $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.*

The stabilized functor $F^{\circ\infty}(X \subset Y)$ is functorial for all smooth morphisms of pairs, whether or not surjective.

We again mention that Theorem 1.1.1 was discovered independently by Marzo and McQuillan [MM19]. We thank Johannes Nicaise for bringing that to our attention.

1.2. **Invariants and parameters.** The notation for the present invariant $\operatorname{inv}_p(\mathcal{I})$ in [ATW17] was $a_1 \cdot \operatorname{inv}_{\mathcal{I}_X, a_1}(p)$, and extends to arbitrary ideal sheaves. Here it is applied solely when Y is smooth with trivial logarithmic structure.

This invariant is closely related to invariants developed in earlier papers on resolution of singularities, in particular Włodarczyk's [Wło05] and Bierstone and Milman's [BM97]. The local parameters $x_1, \ldots, x_k$ in the definition of J were already inrtoduced in [BM97, EV03, Wło05, ATW17] as a sequence of iterated hypersurfaces of maximal contact for appropriate coefficient ideals, see Section 4.3. In particular *each application of the resolution-step functor F is explicitly computable.*

In earlier work the ideal $(x_1, \ldots, x_k)$ was used to locally define the unique center of blowing up satisfying appropriate admissibility and functoriality properties for resolution using smooth blowings up. A central observation here is that the stack-theoretic weighted blowing up of $(x_1^{1/w_1}, \ldots x_k^{1/w_k})$ is also functorially associated to $X \subset Y$, see Theorem 4.7.1.

As we recall below, in general, after blowing up the reduced ideal $(x_1, \ldots, x_k)$, the invariant does not drop, and may increase. Earlier work replaced this invariant by an invariant including data of exceptional divisors and their history, or more recently, logarithmic structures. Another central observation here is that, with the use of weighted blowings up, no history, no exceptional divisors, and no logarithmic structures are necessary.

The present treatment requires the theory of Deligne–Mumford stacks. A careful application of Bergh's Destackification Theorem [Ber17, Theorem 1.2] or more directly its non-abelian generalization [BR19] allows one to replace $X_n \subset Y_n$ by a smooth embedded scheme $X' \subset Y'$ projective over $X \subset Y$.

The present treatment does not address logarithmic resolutions, a critical requirement of birational geometry. We will treat the necessary modifications in a follow-up paper.

We provide a proof of the theorem based on existing theory of resolution of singularities. We hope this will make it transparent to those familiar with the theory.

We also provide a direct construction, which may be more convincing to birational geometers not familiar with existing work. As we see below, the blowing up $Y' \to Y$ is obtained as the stack-theoretic blowing up $\mathcal{P}roj_Y(\mathcal{A})$, where the graded algebra $\mathcal{A}$ is canonically obtained from $\mathcal{I}_X$ using differential operators.

In future work (perhaps future revisions of this manuscript) we aim to reset the present treatment in the appropriate generality of qe schemes, and apply it to logarithmic schemes and families of schemes.

1.3. **Example: Whitney's umbrella revisited with weighted blowings up.**

1.3.1. *Blowing up without weights.* It is well-known that with smooth blowings up Theorem 1.1.1 is impossible, see [Kol07, Claim 3.6.3]. Consider the Whitney umbrella $x^2 = zy^2$. The origin seems to be the most singular point, and indeed, in characteristic 0, the theory of maximal contact and coefficient ideals leads to the center $\{x = y = z = 0\}$, but its blowing up leads to the Whitney umbrella occurring again on the z-chart: writing $x = x_1z$ and $y = y_1z$ we get, after clearing out z^2, the equation $x_1^2 = zy_1^2$.

Of course the Whitney umbrella can be resolved in one step by blowing up the line $x = y = 0$, but in characteristic $\neq 2$ this does not fit in any known embedded resolution algorithm.

A worse scenario appears with the singularity $x^2 + y^2 + z^mt^m = 0$, where after blowing up the origin the "worse" singularity $x_1^2 + y_1^2 + z^{2m-2}t_1^m = 0$ appears in the z-chart.

1.3.2. *Weighted blowing up.* A birational geometer knows that, in characteristic $\neq 2$, the Whitney umbrella $x^2 = y^2z$ asks to be resolved starting by blowing up (x^2, y^3, z^3). Similarly, $x^2 + y^2 + z^mt^m = 0$ asks for the blowing up of (x, y, z^m, t^m).

For the Whitney umbrella once again only the z chart is interesting, where the coordinates on the ambient stack are as follows:

$$X' = [\operatorname{Spec} \mathbb{C}[x_1, y_1, w]/(\pm 1)],$$

where $x_1 = x/w^3, y_1 = y/w^2$, and $z = w^2$, and the action of (± 1) given by $(x_1, y_1, w) \mapsto (-x_1, y_1, -w)$.

The equation $x^2 = zy^2$ translates to $w^6x_1^2 = w^6y_1^2$. Here $(w^6) = \mathcal{I}_E^6$ is the exceptional factor of the equation, and the proper transform is

$$x_1^2 = y_1^2.$$

In other words, with the weighted blowing up, the degrees $(2, 3, 3)$ immediately dropped to $(2, 2)$, with the spectre of infinite loops exorcised! One additional blowing up along $x_1 = y_1 = 0$ resolves the singularities.

1.3.3. *The second example.* The z chart of the weighted blowing up of the equation $x^2 + y^2 + z^mt^m = 0$ is in fact a scheme, with coordinates (x_1, y_1, z, t_1) satisfying $x = x_1z^m, y = y_1z^m$ and $t = zt_1$. After factoring z^{2m} we get $x_1^2 + y_1^2 + t_1^m = 0$, with lower degrees $(2, 2, m) < (2, 2, 2m, 2m)$. A single weighted blowing up resolves the singularities.

The x and y charts are smooth, though they do carry a nontrivial stack structure.

2. Valuative ideals, fractional ideals, and $\mathbb{Q}$-deals

Given an integral scheme Y we are interested in understanding ideals as they behave after arbitrary blowing up. It is thus natural to work with the Zariski-Riemann space $\mathbf{ZR}(Y)$ of Y, the projective limit of all projective birational transformations of Y, whose points consist of all valuation rings R of $K(Y)$ extending to a morphism $\operatorname{Spec} R \to Y$.

The space $\mathbf{ZR}(Y)$ carries a constant sheaf K, a subsheaf of rings $\mathcal{O}$ with stalk at v consisting of the valuation ring R_v, and a sheaf of ordered groups Γ such that

$v : K^* \to \Gamma$ is the valuation. The image $v(\mathcal{O}) =: \Gamma_+ \subset \Gamma$ is the valuation monoid consisting of non-negative sections of Γ.

We will freely use the analogous construction when Y is a DM stack and not necessarily integral: replacing Y by its normalization and taking the disjoint union of Zariski–Riemann spaces of components we reduce to the integral case, and if $\bar{Y}$ is the coarse moduli space we can take $\mathbf{ZR}(Y)$ to be the normalization of $Y \times_{\bar{Y}} ZR(Y)$.

By a *valuative ideal* on Y we mean a section $\gamma \in H^0(\mathbf{ZR}, \Gamma_+)$. Every ideal $\mathcal{I}$ on every birational model $Y' \to Y$, proper over Y, defines a valuative ideal we denote $v(\mathcal{I})$ by taking the minimal element of the image of $\mathcal{I}$ in Γ_+. Ideals with the same integral closure have the same valuative ideal. Every valuative ideal γ defines an ideal sheaf on every such Y' by taking $\mathcal{I}_\gamma := \{f \in \mathcal{O}_{Y'} | v(f) \geq \gamma_v \forall v\}$, which is automatically integrally closed.

By a *valuative fractional ideal* we mean a section $\gamma \in H^0(\mathbf{ZR}, \Gamma)$, not necessarily positive, with similar correspondenes. These do not figure in this paper.

The group $\Gamma_{\mathbb{Q}} = \Gamma \otimes \mathbb{Q}$ is also ordered. We denote the monoid of non-negative elements $\Gamma_{\mathbb{Q}+}$. By a *valuative $\mathbb{Q}$-ideal* we mean a section $\gamma \in H^0(\mathbf{ZR}, \Gamma_{\mathbb{Q}+})$. The definition of $\mathcal{I}_\gamma$ extends to this case. It is a convenient way to consider $\mathbb{Q}$-ideals. There is again a similar notion of a valuative fractional $\mathbb{Q}$-ideal.

By a *center* on X we mean the valuative $\mathbb{Q}$-ideal γ for which there is an affine covering $Y = \cup U_i$ and regular systems $(x_1^{(i)}, \ldots, x_k^{(i)}) = (x_1, \ldots, x_k)$ on U_i such that $\gamma_{U_i} = v(J)$ is the valuative $\mathbb{Q}$-ideal associated to $J := (x_1^{a_1}, \ldots, x_k^{a_k})$ for some $a_j \in \mathbb{Q}_{>0}$.

A center γ is *admissible* for a valuative $\mathbb{Q}$-ideal β if $\gamma_v \leq \beta_v$ for all v. A center is *admissible* for an ideal $\mathcal{I}$ if it is admissible for the associated vauative $\mathbb{Q}$-ideal $v(\mathcal{I})$.

We will use the notation $J = (x_1^{a_1}, \ldots, x_k^{a_k})$ to indicate the center, rather than the associated γ. The center is *reduced* if $w_i = 1/a_i$ are positive integers with $\gcd(w_1, \ldots, w_k) = 1$. For any center J we write $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ for the unique reduced center such that $\bar{J}^\ell = J$ for some $\ell \in \mathbb{Q}_{>0}$.

3. Weighted blowings up

Reid [Rei02] championed weighted blowings up in birational geometry. The paper [AH11] uses stack-theoretic projective spectra to study moduli spaces of varieties. Rydh's appendix [Ryd19] provides some foundations.

3.1. Graded algebras and their $\mathcal{P}roj$. Given a quasicoherent graded algebra $\mathcal{A} = \oplus_{m \geq 0} \mathcal{A}_m$ on Y with associated $\mathbb{G}_m$-action defined by $(t, s) \mapsto t^m s$ for $s \in \mathcal{A}_m$ we define its stack-theoretic projective spectrum to be

$$\mathcal{P}roj_Y \mathcal{A} := [(\mathrm{Spec}_{\mathcal{O}_Y} \mathcal{Y} \smallsetminus S_0)/\mathbb{G}_m],$$

where the vertex S_0 is the zero scheme of the ideal $\oplus_{m>0} \mathcal{A}_m$.

3.2. Rees algebras of ideals. If $\mathcal{I}$ is an ideal, its Rees algebra is $\mathcal{A}_{\mathcal{I}} := \oplus_{m \geq 0} \mathcal{I}^m$, and the blowing up of $\mathcal{I}$ is $Y' = Bl_Y(\mathcal{I}) := \mathcal{P}roj_Y(\mathcal{A}_{\mathcal{I}})$. It is the universal birational map making $\mathcal{I}\mathcal{O}_{Y'}$ invertible, in this case $Y' \to Y$ projective.

3.3. Rees algebras of valuative $\mathbb{Q}$-ideals. Now let γ be a valuative $\mathbb{Q}$-ideal, and define its Rees algebra to be

$$\mathcal{A}_\gamma := \bigoplus_{m \in \mathbb{N}} \mathcal{I}_{m\gamma}.$$

The blowing up of γ is defined to be $Y' = Bl_Y(\gamma) := \mathcal{P}roj_Y \mathcal{A}_\gamma$. Now $\mathcal{I}_{\gamma,Y'} = E \subset \mathcal{O}_{Y'}$ is an invertible *ideal*, and once again $Y' \to Y$ satisfies the corresponding universal property. [1] ←1

Note that if $Y_1 \to Y$ is smooth and $Y_1' = Bl_Y(\gamma\mathcal{O}_{Y_1})$ then $Y_1' = Y' \times_Y Y_1$.

3.4. **Weighted blowings up: local equations.** Now consider the situation where γ is a center of the special form $J = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$, with $w_i \in \mathbb{N}$. In this case the algebra $\mathcal{A}_\gamma = \bigoplus_{m\in\mathbb{N}} \mathcal{I}_{m\gamma}$, with $\mathcal{I}_{m\gamma} = (x_i^{b_1} \cdots x_n^{b_n} | \sum w_i b_i \geq m)$ is the integral closure of the simpler algebra with generators (x_i) in degree w_i. We can therefore describe $Bl_Y(J) = Bl_Y(\gamma)$, which deserves to be called a stack-theoretic weighted blowing up, explicitly in local coordinates.

The chart associated to x_1 has local variables $u, x_2', \ldots, x_n'$, where

- $x = u^{w_1}$,
- $x_i' = x_i/u^{w_i}$ for $2 \leq i \leq k$, and
- $x_j' = x_j$ for $j > k$.

The group $\boldsymbol{\mu}_{w_1}$ acts through

$$(u, x_2', \ldots, x_k') \quad \mapsto \quad (\zeta_{w_1} u,\ \zeta_{w_1}^{-w_2} x_2',\ \ldots,\ \zeta_{w_1}^{-w_k} x_k')$$

and trivially on $x_j', j > k$, giving an étale local isomorphism of the chart with $[\operatorname{Spec} \mathbb{C}[u, x_2', \ldots, x_n']/\boldsymbol{\mu}_{w_1}]$. It is easy to see that these charts glue to a stack-theoretic modification $Y' \to Y$ with a smooth Y' and its coarse space is the classical (singular) weighted blowing up.

We sometimes, but not always, insist on $\gcd(w_1, \ldots, w_k) = 1$, in which case the center is *reduced*. The relationshop is summarized by the following immediate lemma:

Lemma 3.4.1. *If $J' = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ and $J'' = (x_1^{1/cw_1}, \ldots, x_k^{1/cw_k})$ with w_i, c positive integers, if $Y', Y'' \to Y$ are the corresponding blowings up, with E', E'' the exceptional divisors, then $Y'' = Y'(\sqrt[c]{E'})$ is the root stack of Y' along E'.*

3.5. **Weighted blowings up: local toric description.** Again working locally, assume that $Y = \operatorname{Spec} k[x_1, \ldots, x_n]$. It is the affine toric variety associated to the monoid $\mathbb{N}^n \subset \sigma = \mathbb{R}^n_{\geq 0}$. Here the generator e_i of $\mathbb{N}^n$ corresponds to the monomial valuation v_i associated to the divisor $x_i = 0$, namely $v_i(x_j) = \delta_{ij}$.

The monomial x_i^{1/w_i} defines the linear function on σ whose value on $(b_1, \ldots, b_n)$ is its valuation b_i/w_i. The ideal $(x_1^{1/w_1}, \ldots, x_k^{1/w_k})$ thus defines the piecewise linear function $\min_i\{b_i/w_i\}$, which becomes linear precisely on the star subdivision $\Sigma = v_{\bar{J}} \star \sigma$ with

$$v_{\bar{J}} = (w_1, \ldots, w_k, 0, \ldots, 0).$$

This defines the scheme theoretic weighted blowing up $\bar{Y}'$. Note that this cocharacter $v_{\bar{J}}$ is a multiple of the valuation associated to the exceptional divisor of the center.

Since $v_{\bar{J}}$ is assumed integral, we can apply the theory of toric stacks [BCS05, FMN10, GS15a, GS15b, GM15]. We have a smooth toric stack $Y' \to \bar{Y}'$ associated to the same fan Σ with the cone $\sigma_i = \langle v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n\rangle$ endowed with the sublattice $N_i \subset N$ generated by the elements $v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n$, for all $i = 1, \ldots, k$. This toric stack is precisely the stack theoretic weighted blowing up $Y' \to Y$.

[1] (Dan) Refer to Rydh's writeup or else prove

4. Centers, invariants, and blowing up

4.1. The principalization theorem.

Theorem 4.1.1 (Principalization)**.** *There is a functor F associating to $\mathcal{I} \subset \mathcal{O}_Y, \mathcal{I} \neq (0), (1)$ a center $\bar{J}$ with blowing up $Y' \to Y$ and proper transform $\mathcal{I}' \subset \mathcal{O}'_Y$ such that $\operatorname{maxinv}(\mathcal{I}') < \operatorname{maxinv}(\mathcal{I})$. In particular there is an integer n so that the iterated application $(\mathcal{I}_n \subset \mathcal{O}_{Y_n}) := F^{\circ n}(\mathcal{I} \subset \mathcal{O}_Y)$ of F has $\mathcal{I}_1 = (1)$.*

The stabilized functor $F^{\circ\infty}(\mathcal{I} \subset \mathcal{O}_Y)$ is functorial for all smooth morphisms, whether or not surjective.

Theorem 1.1.1 follows from Theorem 4.1.1 by stopping at the point where $\operatorname{maxinv}(\mathcal{I}) = (1, \dots, 1)$, the sequence of length c.

4.2. Coefficient ideals. We rely on [ATW17], except that we use the saturated coefficient ideal as in [Kol07, ATW18]:

Definition 4.2.1. Let $\mathcal{I} \subset \mathcal{O}_Y$ and $a > 0$ an integer. Then

$$C(\mathcal{I}, a) = \sum f(\mathcal{I}, \mathcal{D}^{\leq 1}\mathcal{I}, \dots, \mathcal{D}^{\leq a-1}\mathcal{I}),$$

where $f(t_0, \dots, t_{a-1})$ runs over all monomials $t_0^{b_0} \cdots t_{a_1}^{b_{a-1}}$ of weighted degree

$$\sum_{i=0}^{a-1} (a-i) \cdot b_i \quad \geq \quad a!.$$

The formation of $C(\mathcal{I}, a)$ is functorial for smooth morphisms: if $Y_1 \to Y$ is smooth then $C(\mathcal{I}, a)\mathcal{O}_{Y_1} = C(\mathcal{I}\mathcal{O}_{Y_1}, a)$.

Now consider $\mathcal{I} \subset \mathcal{O}_Y$ and assume $x \in \mathcal{D}^{\leq a-1}\mathcal{I}$ is a maximal contact element at $p \in Y$.

Lemma 4.2.2 ([Kol07, BM08, ATW18])**.** *After passing to completions we may write*

$$C(\mathcal{I}, a) = (x_1^{a!}) + (x_1^{a!-1}\tilde{\mathcal{I}}_{a!-1}) + \cdots + (x_1\tilde{\mathcal{I}}_1) + \tilde{\mathcal{I}}_0,$$

where

$$\mathcal{I}_0 \subset (x_2, \dots, x_n)^{a!} \subset k[\![x_2, \dots, x_n]\!],$$

where $\mathcal{I}_{j+1} := \mathcal{D}^{\leq 1}(\mathcal{I}_j)$ satisfy $\mathcal{I}_{a!-k}\mathcal{I}_{a!-l} \subset \mathcal{I}_{a!-(k+l)}$, and $\tilde{\mathcal{I}}_j = \mathcal{I}_j k[\![x_1, \dots, x_n]\!]$. In fact $\mathcal{I}_j$ is the restriction to $x_1 = 0$ of the similarly formed ideal

$$\sum_{\sum_{i=0}^{a-1}(a-i)\cdot b_i \ \geq \ a!-j} f(\mathcal{I}, \mathcal{D}^{\leq 1}\mathcal{I}, \dots, \mathcal{D}^{\leq a-1}\mathcal{I}).$$

4.3. Existence of invariants and centers. Fix an ideal $\mathcal{I} = \mathcal{I}[1] \neq 0$. We define a finite sequence of integers b_i, rational numbers a_i, and parameters x_i.

Set $a_1 = b_1 := \operatorname{ord}_p(\mathcal{I}[1])$, and take the parameter x_1 to be a maximal contact element at p. Inductively one writes $\mathcal{I}[i+1] = C(\mathcal{I}[i], b_i)|_{V(x_1,\dots,x_i)}$, the restricted coefficient ideal, with order $\operatorname{ord}_p(\mathcal{I}[i+1]) = b_{i+1}$, one sets $a_{i+1} = b_{i+1}/(b_i - 1)!$, and one takes x_{i+1} a lifting to Y of the maximal contact element for $\mathcal{I}[i+1]$.

Equivalently, $\operatorname{inv}_p(\mathcal{I}[1]) = (a_1, \operatorname{inv}_p(\mathcal{I}[2])/(a_1 - 1)!)$ the concatenation, and $x_2, \dots$ are lifts of the parameters for $\mathcal{I}[2]$.

The invariant takes values in the well-ordered subset

$$a_1 \quad \in \quad \mathbb{N} \qquad \text{and} \qquad a_{i+1} \quad \in \quad \mathbb{N} \cdot \frac{1}{(b_i - 1)!}.$$

It was shown in [ATW17] that this invariant is functorial for smooth morphisms: if $Y_1 \to Y$ is smooth and $p' \in Y'$ then $\mathrm{inv}_{p'}(\mathcal{I}\mathcal{O}_{Y_1}) = \mathrm{inv}_p(\mathcal{I})$.

4.4. **Admissibility of centers.** As in earlier work on resolution of singularities, *admissibility* allows flexibility in studying the behavior of ideals under blowings up of centers. This becomes important when an ideal is related to the sum of ideals with different invariants of their own, but all admitting a common admissible center.

We focus on sequences $(a_1, \ldots, a_k)$ which occur as invariants, in particular a_1 a positive integer and $a_i \leq a_{i+1}$.

4.4.1. *Admissibility and blowing up.* We say that a center $J = (x_1^{a_1}, \ldots, x_k^{a_k})$ is $\mathcal{I}$-admissible at p if the inequality $v(x_1^{a_1}, \ldots, x_k^{a_k}) \leq v(\mathcal{I})$ of valuative $\mathbb{Q}$-ideals is satisfied. This can be described in terms of the weighted blowing up $Y' \to Y$ of the reduced center $\bar{J} := (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$, with w_i integers with $\gcd(w_1, \ldots, w_k) = 1$ as follows: let $E = \bar{J}\mathcal{O}_{Y'}$, which is an invertible ideal sheaf. Note that since $a_1 w_1$ is an integer also $J\mathcal{O}_{Y'} = E^{a_1 w_1}$ is an invertible ideal sheaf. Therefore $J = (x_1^{a_1}, \ldots, x_k^{a_k})$ is $\mathcal{I}$-admissible if and only if $E^{a_1 w_1}$ is $\mathcal{I}\mathcal{O}_{Y'}$ admissible, if and only if $\mathcal{I}\mathcal{O}_{Y'} = E^{a_1 w_1}\mathcal{I}'$, with $\mathcal{I}'$ an ideal.

In terms of its monomial valuation, J is admissible for $\mathcal{I}$ if and only if $v_J(f) \geq 1$ for all $f \in \mathcal{I}$. This means that if $f = \sum c_{\bar{\alpha}} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ then $\sum_{i=1}^k \alpha_i / a_i \geq 1$ whenever $c_{\bar{\alpha}} \neq 0$.

If $Y_1 \to Y$ is smooth and J is $\mathcal{I}$-admissible then $J\mathcal{O}_{Y_1}$ is $\mathcal{I}\mathcal{O}_{Y_1}$-admissible, with the converse holding when $Y_1 \to Y$ is surjective.

4.4.2. *Working with rescaled centers.* For induction to work in the arguments below, it is worthwhile to consider blowings up of centers of the form

$$\bar{J}^{1/c} := (x_1^{1/(w_1 c)}, \ldots, x_k^{1/(w_k c)})$$

for a positive integer c. We note that this does change the notion of admisibility of J. We also use the notation $J^\alpha := (x_1^{a_1 \alpha}, \ldots, x_k^{a_k \alpha})$ throughout - the equality should be understood in terms of valuative $\mathbb{Q}$-ideals.

4.4.3. *Basic propoerties.* The description of the monomial valuation of J immediately provides the following lemmas:

Lemma 4.4.4. *If J is both $\mathcal{I}_1$-admissible and $\mathcal{I}_2$-admissible then J is $\mathcal{I}_1 + \mathcal{I}_2$-admissible. If J is $\mathcal{I}$-admissible then J^k is $\mathcal{I}^k$-admissible. More generally if J^{c_j} is $\mathcal{I}_j$-admissible then $J^{\sum c_j}$ is $\prod \mathcal{I}_j$-admissible.*

Indeed if $v_J(f) \geq 1$ and $v_J(g) \geq 1$ then $v_J(f+g) \geq 1$, etc.

Lemma 4.4.5. *If J is $\mathcal{I}$-admissible then $J' = J^{\frac{a_1 - 1}{a_1}}$ is $\mathcal{D}(\mathcal{I})$-admissible. If $a_1 > 1$ and $J^{\frac{a_1-1}{a_1}}$ is $\mathcal{I}$-admissible then J is $x_1\mathcal{I}$-admissible.*

Proof. For the first statement note that if $\sum_{i=1}^k \alpha_i / a_i \geq 1$ and $\alpha_j \geq 1$ then

$$v_J\left(\frac{\partial(x_1^{\alpha_1} \cdots x_n^{\alpha_n})}{\partial x_j}\right) \quad = \quad \sum_{i=1}^k \alpha_i / a_i \ - \ 1/a_j \quad \geq \quad 1 \ - \ 1/a_1,$$

so

$$v_{J'}\left(\frac{\partial(x_1^{\alpha_1} \cdots x_n^{\alpha_n})}{\partial x_j}\right) \quad \geq 1,$$

as needed. The other statement is similar. ♣

Lemma 4.4.6. *For $\mathcal{I}_0 \subset k[x_2, \ldots, x_n]$ write $\tilde{\mathcal{I}}_0 = \mathcal{I}_0 k[x_1, \ldots, x_n]$. Assume $a_1 \leq a_2$ and $(x_2^{a_2}, \ldots x_k^{a_k})$ is $\mathcal{I}_0$-admissble. Then $(x_1^{a_1}, \ldots x_k^{a_k})$ is $\tilde{\mathcal{I}}_0$-admissble.*

Here for generators of $\mathcal{I}_0$ we have $\sum_{i=1}^k \alpha_i/a_i = \sum_{i=2}^k \alpha_i/a_i$.

Lemma 4.4.7. *J is $\mathcal{I}$-admissible if and only if $J^{(a_1-1)!}$ is $C(\mathcal{I}, a_1)$-admissible.*

This combines Lemmas 4.4.4 and 4.4.5 for the terms defining $C(\mathcal{I}, a_1)$.

4.5. **Our chosen center is admissible.**

Theorem 4.5.1. *If $(a_1, \ldots, a_k) = \mathrm{inv}_p(\mathcal{I})$, with corresponding parameters $x_1, \ldots, x_k$, and $J = (x_1^{a_1}, \ldots, x_k^{a_k})$, then J is $\mathcal{I}$ admissible.*

Proof. Applying Lemma 4.4.7, we replace $\mathcal{I}$ by $C(\mathcal{I}, a_1)$, rescale the invariant appropriately and work on formal completion. We may therefore write

$$\mathcal{I} = (x_1^{a_1}) + (x_1^{a_1-1}\tilde{\mathcal{I}}_{a_1-1}) + \cdots + (x_1\tilde{\mathcal{I}}_1) + \tilde{\mathcal{I}}_0$$

as in Lemma 4.2.2.

The inductive hypothesis implies $(x_2^{a_2}, \ldots, x_k^{a_k})$ is $\mathcal{I}_0$-admissible. By Lemma 4.4.6 J is $\tilde{\mathcal{I}}_0$-admissible. By Lemma 4.4.5 J is $(x_1^{a_1-j}\tilde{\mathcal{I}}_{a_1-j})$-admissible, implying that J is $\mathcal{I}$-admissible, as needed. ♣

As an example for the added flexibility provided by admissibility, the center (x_1^6, x_2^6) is $(x_1^3 x_2^3)$-admissible because this is the corresponding invariant, but also $(x_1^5, x_2^{15/2})$ is admissible. This second center becomes important when one considers instead the ideal $(x_1^5 + x_1^3 x_2^3)$, or even $(x_1^5 + x_1^3 x_2^3 + x_2^8)$, whose invariant is $(5, 15/2)$, as described in Section 5 below.

4.6. **The invariant drops.** With admissibility of the center we can now analyze the behavior of the invariant:

Theorem 4.6.1. *Let $(a_1, \ldots, a_k) = \mathrm{inv}_p(\mathcal{I})$, with corresponding parameters $x_1, \ldots, x_k$, and $J = (x_1^{a_1}, \ldots, x_k^{a_k})$. For $c \in \mathbb{N}_{>0}$ write $Y'_c \to Y$ for the blowing up of the rescaled center $\bar{J}^{1/c} := \left(x_1^{1/(w_1 c)}, \ldots, x_k^{1/(w_k c)}\right)$, with corresponding factorization $\mathcal{I}\mathcal{O}_{Y'_c} = E^{a_1 w_1 c}\mathcal{I}'$. Then for every point p' over p we have $\mathrm{inv}_{p'}(\mathcal{I}') < \mathrm{inv}_p(\mathcal{I})$.*

Proof. If $k = 0$ the ideal is (0) and there is nothing to prove. When $k = 1$ the ideal is $(x_1^{a_1})$, which becomes exceptional with proper transform (1). We now assume $k > 1$.

Again using Lemma 4.2.2, we choose formal coordinates, work with $\mathcal{C} := C(\mathcal{I}, a_1)$, rescale the invariant by $(a_1 - 1)!$, and write

$$\mathcal{C} = (x_1^{a_1!}) + (x_1^{a_1!-1}\tilde{\mathcal{C}}_{a_1!-1}) + \cdots + (x_1\tilde{\mathcal{C}}_1) + \tilde{\mathcal{C}}_0.$$

Writing $\mathcal{C}\mathcal{O}_{Y'_c} = E^{a_1! w_1 c}\mathcal{C}'$, we will first show that $\mathrm{inv}_{p'}(\mathcal{C}') < (a_1-1)! \cdot (a_1, a_2, \ldots, a_k)$ for all points p' over p.

Write $H = \{x_1 = 0\}$, and $H' \to H$ the blowing up of the reduced center associated to $J_H := (x_2^{a_2}, \ldots, x_k^{a_k})$. By Lemma 3.4.1 the proper transform $\tilde{H}' \to H$ of H via the blowing up of J is the root stack $H'(\sqrt[cc']{E_H})$ of H' along $E_H \subset H'$, where $c' = \mathrm{lcm}(w_1, \ldots, w_k)/\mathrm{lcm}(w_2, \ldots, w_k)$.

We now inspect the behavior on different charts. On the x_1-chart the first term becomes $(x_1^{a_1!}) = E^{a_1! w_1 c} \cdot (1)$ so $\mathrm{inv}_{p'}\mathcal{C}' = \mathrm{inv}(1) = 0$.[2] This implies that on all

[2]This reflects the fact that before passing to the coefficient ideal $\mathrm{ord}(\mathcal{I}') < a_1$ on this chart - it need not become a unit ideal in general!

other charts it suffices to consider $p' \in H' \cap E$. By the inductive assumption, for such points we have

$$\mathrm{inv}_{p'}(\mathcal{C}_0)' < (a_1 - 1)! \cdot (a_2, \dots, a_k).$$

Note that the term $(x_1^{a_1!})$ in $\mathcal{C}$ is transformed to a term of the same form $({x'}_1^{a_1!})$. It follows that $\mathrm{ord}_{p'}(\mathcal{C}') \leq a_1!$, and if $\mathrm{ord}_{p'}(\mathcal{C}') < a_1!$ then a fortiori $\mathrm{inv}_{p'}(\mathcal{C}') < \mathrm{inv}_p(\mathcal{C})$.

If on the other hand $\mathrm{ord}_{p'}(\mathcal{C}') = a_1$ then the variable x_1' is a maximal contact element. Using the inductive assumption we compute

$$\mathrm{inv}_{p'}({x'}_1^{a_1}, \mathcal{C}_0') = \big(a_1, \mathrm{inv}(\mathcal{C}_j')\big) < (a_1, \mathrm{inv}(\mathcal{C}_j)) = (a_1, \dots, a_k).$$

Since $\mathcal{C}'$ includes this ideal, we obtain again $\mathrm{inv}_{p'}(\mathcal{C}') < \mathrm{inv}_p(\mathcal{C})$, as claimed.

We note that as in [ATW18], in this case the invariant of the original ideal drops as well. Indeed since the invariant of $\mathcal{C}'$ drops, the invariant of at least one of the terms $(\mathcal{C}_{a_1!-j})'$ drops, implying that the invariant of at least one of the corresponding monomials $(f(\mathcal{I}, \mathcal{D}^{\leq 1}\mathcal{I}, \dots, \mathcal{D}^{\leq a-1}\mathcal{I}))'$ drops below its expected value, implying in turn that the invariant of $(\mathcal{D}^{\leq j}\mathcal{I})'$ drops, which in particular implies that the invariant of $\mathcal{I}'$ drops by Lemma 4.4.5, as needed. ♣

4.7. **Uniqueness of centers.** The definition of the center J involved an iterated choice of maximal contact elenments x_i, which are in general not unique. However,

Theorem 4.7.1. *The center J is unique.*

Proof. We may pass to completions, replace ideals by coefficient ideals and rescale, so that again we may assume

$$\mathcal{I} = (x_1^{a_1}) + (x_1^{a_1-1}\tilde{\mathcal{I}}_{a_1-1}) + \cdots + (x_1\tilde{\mathcal{I}}_1) + \tilde{\mathcal{I}}_0.$$

If $J = (x_1^{a_1}, \cdots, x_k^{a_k})$ and J_1 is another center, we may assume by induction that $J_1 = ((x_1+f)^{a_1}, x_2^{a_2}, \cdots, x_k^{a_k})$, where $f \in \tilde{\mathcal{I}}_{a_1-1}$. The lemmas imply that J is admissible for each term in J_1 hence J is admissible for the ideal J_1. Reversing the roles we have that J_1 is admissible for the ideal J. This implies that as centers they agree. ♣

Proof of Theorem 4.1.1. Theorem 4.1.1 follows from Theorems 4.6.1 and 4.7.1. ♣

4.8. **Deriving the graded algebra $\mathcal{A}$ of $\bar{J}$ from $\mathcal{I}$.** More than uniqueness, we have a canonical way to derive the graded algebra $\mathcal{A}_{\bar{J}}$ associated to $\bar{J}$ from the ideal $\mathcal{I}$.

Let $\mathcal{A}^{(0)} = \mathcal{O} \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \cdots$ be the Rees algebra of $\mathcal{I}$.

We define $\mathcal{A}^{(1)}$ to be the algebra graded in $N_1 := (1/a_1)\mathbb{Z}$, generated by $\mathcal{D}^{\leq j}\mathcal{A}^{(0)}_m$ placed in degree $m - (j/a_1)$, for $j = 0, \dots a_1 - 1$. In particular $\mathcal{A}^{(1)}_{1/a_1} = \mathcal{D}^{\leq a_1 - 1}\mathcal{I}$ is the maximal contact ideal. This is the differential-closed algebra associated to $\mathcal{I}$ in [EV07].

For any maximal contact element x_1 consider the sheaf $\mathcal{D}_{\log x_1}$ of differential operators preserving the ideal (x_1). Let $N_2 := N_1 + (1/a_2)\mathbb{Z}$. Define $\mathcal{A}^{(2)}$ to be the algebra graded in N_2, generated by $\mathcal{D}^{\leq j}_{\log x_1}\mathcal{A}^{(1)}_m$ placed in degree $m - (j/a_2)$, for $j = 0, \dots a_2 - 1$.

Inductively, for any element x_i of $\mathcal{A}^{(i)}_{1/a_i}$ of order 1 which is nonzero modulo $(x_1, \dots, x_{i-1})$ consider the sheaf $\mathcal{D}_{\log(x_1 \cdots x_i)}$ of differential operators preserving the ideal $(x_1 \cdots x_i)$. Let $N_{i+1} := N_i + (1/a_{i+1})\mathbb{Z}$. Define $\mathcal{A}^{(i+1)}$ to be the algebra graded in N_{i+1}, generated by $\mathcal{D}^{\leq j}_{\log(x_1\cdots x_i)}\mathcal{A}^{(i)}_m$ placed in degree $m - (j/a_{i+1})$, for $j = 0, \ldots a_{i+1} - 1$.

Lemma 4.8.1. *The algebra $\mathcal{A}^{(i+1)}$ is independent of the choice of $x_1, \ldots, x_i$.*

Proof. Sketch of proof. Indeed, if for instance we replace x_i by $x_i + f(x_{i+1}, \ldots, x_n)$ then for $j > i$ the operator $\nabla_j := \partial/\partial x_j$ is replaced by $\nabla'_j := \nabla_j - (\partial f/\partial x_j)\nabla_i$. Note that

$$(\nabla_j - \nabla'_j)\mathcal{A}^{(i)}_m \;=\; (\partial f/\partial x_j)\nabla_i\mathcal{A}^{(i)}_m \quad\subset\quad \mathcal{A}^{(i+1)}_{1/a_i - 1/a_j}\mathcal{A}^{(i+1)}_{m-1/a_i} \;\subset\; \mathcal{A}^{(i+1)}_{m-1/a_j},$$

as needed. An identical computation shows that if we replace the lifted element x_i to $x_i + f(x_1, \ldots, x_{i-1})$ the algebra $\mathcal{A}^{(i+1)}$ does not change. ♣

Theorem 4.8.2. *$\mathcal{A}^{(k)} = \mathcal{A}_{\bar{J}}$ as graded algebras.*

Proof. Up to rescaling, we may replace $\mathcal{I}$ by $\mathcal{C}(\mathcal{I}, a_1)$, work on formal completions, and write

$$\mathcal{I} = (x_1^{a_1}) + (x_1^{a_1-1}\tilde{\mathcal{I}}_{a_1-1}) + \cdots + (x_1\tilde{\mathcal{I}}_1) + \tilde{\mathcal{I}}_0.$$

Writing $J_0 = (x_1^{a_2}, \ldots, x_k^{a_k})$ we may form the graded algebra $\mathcal{B}^{(k-1)}$ associated to $\mathcal{I}_0$. By induction we have $\mathcal{B}^{(k-1)} = \mathcal{A}_{\bar{J}_0}$. Taking into account the rescaling factor $c' = \mathrm{lcm}(w_1, \ldots, w_k)/\mathrm{lcm}(w_2, \ldots, w_k)$ the result follows.

♣

4.9. **Interpretations.**

(1) (Newton polyhedron) Given the coordinates x_i, the center describes the lowest facet of the newton polyhedron of $\mathcal{I}$, and the invariant is its slope.
(2) (Tropicalization) Given the coordinates, the center is the monomial valuation described by the barycenter of the top facet of $\mathrm{Trop}(\mathcal{I})$.
(3) (Nonarchimedean geometry) as the tropicalization with respect to the coordinates x_i embeds in the Berkovich space, the center once again is the barycenter of the top facet of Y^{an} associated to the piecewise linear function determined by $\mathcal{I}$.

5. Two examples

Consider the plane curve

$$X = V(x^5 + x^3y^3 + y^k)$$

with $k \geq 5$. Its resolution depends on whether or not $k \geq 8$.

5.1. **The case $k \geq 8$.** This curve is singular at the origin p. We have $a_1 = \mathrm{ord}_p(\mathcal{I}_X) = 5$. Since $\mathcal{D}^{\leq 4}\mathcal{I} = (x, y^2)$ we may take $x_1 = x$ and $H = V(x)$. A direct computation provides the coefficient ideal

$$C(\mathcal{I}_X, 5)|_H = (\mathcal{D}^{\leq 3}(\mathcal{I}_X)|_H)^{120/2} = (y^{180}),$$

with $b_2 = 180$ and $a_2 = 180/(4!) = 15/2$. Rescaling, we need to take the weighted blowup of $\bar{J} = (x^{1/3}, y^{1/2})$.

- In the x-chart we have $x = u^3, y = u^2y'$, giving
$$Y'_x = [\operatorname{Spec} k[u, y']/\boldsymbol{\mu}_3],$$
the action given by $(u, y') \mapsto (\zeta_3 u, \zeta_3 y')$. The equation of X becomes
$$u^{15}(1 + {y'}^3 + u^{2k-15}{y'}^k),$$
with proper transform $X'_x = V(1 + {y'}^3 + u^{2k-15}{y'}^k)$ smooth.
- In the y-chart we have $y = v^2, x = v^3x'$, giving
$$Y'_y = [\operatorname{Spec} k[x', v]/\boldsymbol{\mu}_2],$$
the action given by $(x', v) \mapsto (-x', -v)$. The equation of X becomes $v^{15}({x'}^5 + {x'}^3 + v^{2k-15})$, with proper transform $X'_y = V({x'}^5 + {x'}^3 + v^{2k-15})$.
Note that X'_y is smooth when $k = 8$. Otherwise it is singular at the origin with invariant $(3, 2k-15)$, which is lexicographically strictly smaller than $(5, 15/2)$; A single weighted blowing up resolves the singularity.

5.2. **The case $k \leq 7$.** Consider now the same equation with $k = 7$ (the cases $k = 5, 6$ being similar). We still take $a_1 = 5, x_1 = x$ and $H = V(x)$. This time
$$C(\mathcal{I}_X)|_H = ((\mathcal{I}_X)|_H)^{120/5} = (y^{168}),$$
with $b_2 = 7 \cdot (4!)$ and $a_2 = 7$. We take the weighted blowup of $J = (x^{1/7}, y^{1/5})$.

- In the x-chart we have $x = u^7, y = u^5y'$, giving
$$Y'_x = [\operatorname{Spec} k[u, y']/\boldsymbol{\mu}_7],$$
the action given by $(u, y') \mapsto (\zeta_7 u, \zeta_7^{-5} y')$. The equation of X becomes
$$u^{35}(1 + u{y'}^3 + {y'}^7),$$
with proper transform $X'_x = V(1 + u{y'}^3 + {y'}^7)$ smooth.
- In the y-chart we have $y = v^5, x = v^7x'$, giving
$$Y'_y = [\operatorname{Spec} k[x', v]/\boldsymbol{\mu}_5],$$
the action given by $(x', v) \mapsto (\zeta_5^{-7} x', \zeta_5 v)$. The equation of X becomes $v^{35}({x'}^5 + v{x'}^3 + 1)$, with smooth proper transform $X'_y = V({x'}^5 + v{x'}^3 + 1)$.

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