INVARIANCE IN LOGARITHMIC GROMOV–WITTEN THEORY

DAN ABRAMOVICH AND JONATHAN WISE

ABSTRACT. Gromov–Witten invariants have been constructed to be deformation invariant, but their behavior under other transformations is subtle. In this note we show that *logarithmic* Gromov–Witten invariants are also invariant under appropriately defined *logarithmic modifications*.

sec:intro

1. INTRODUCTION

1.1. **Main result.** Consider two complex projective varieties $\underline{X}, \underline{Y}$ with a projective birational *toroidal morphism* $h : \underline{Y} \to \underline{X}$ as defined in [KKMSD73]. By the work [Kat89, Kat94] of Kato this means that these varieties are naturally endowed with *fine and saturated logarithmically smooth structures*, which we denote X and Y, and the morphism $Y \to X$ is *logarithmically étale*.

Following [GS13, Che10, AC11] there are algebraic stacks $\overline{\mathcal{M}}(Y)$ and $\overline{\mathcal{M}}(X)$, of stable logarithmic maps of curves to Y and X, admitting virtual fundamental classes $[\overline{\mathcal{M}}(Y)]^{\mathrm{vir}}$ and $[\overline{\mathcal{M}}(X)]^{\mathrm{vir}}$. Following [AMW12, Theorem B.6] we obtain a natural associated morphism of moduli stacks $\overline{\mathcal{M}}(h) : \overline{\mathcal{M}}(Y) \to \overline{\mathcal{M}}(X)$. The main result of this paper is as follows:

aintheorem **Theorem 1.1.1.** *With the assumptions above, we have*

$$\overline{\mathcal{M}}(h)_* \left([\overline{\mathcal{M}}(Y)]^{\mathrm{vir}}\right) = [\overline{\mathcal{M}}(X)]^{\mathrm{vir}}.$$

1.2. **Implication for logarithmic Gromov–Witten invariants.** The papers [GS13, AC11] provide technical conditions under which the stack $\overline{\mathcal{M}}(X)$ decomposes as an infinite disjoint union of open and closed substacks $\overline{\mathcal{M}}(X) = \coprod \overline{\mathcal{M}}_\Gamma(X)$ of finite type, each admitting a projective coarse moduli space. When these conditions hold for X, they automatically hold for the modification Y as well. Under those conditions one defines in [GS13, AC11] *logarithmic Gromov–Witten invariants* of X and Y. The theorem above implies that *the logarithmic Gromov–Witten invariants of X and Y coincide.* [1]

1.3. **Gromov–Witten invariants and birational invariance.** Algebraic Gromov–Witten invariants are virtual curve counts on a complex projective variety X, thus are biregular invariants. The formalism of virtual fundamental class shows that they are automatically deformation invariant: if X appears as a fiber of a smooth family, then its invariants coincide with the invariants of other smooth fibers. This is a key property in the study of Gromov–Witten invariants.

Unfortunately, the behavior of Gromov–Witten invariants under most other transformations, in particular a birational transformation $Y \to X$, is subtle. A number of authors

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[1]We hope that methods of the present paper will soon be used to prove the boundedness result of $\overline{\mathcal{M}}_\Gamma(X)$ in general, in which case the technical conditions of [GS13, AC11] become unnecessary.

have addresses this question, and found that good behavior can be proven in many special situations. Here is a non-exhaustive list:

(1) Gathmann [Gat01, Theorem 2.1] provided a procedure for calculating the behavior of genus-0 invariants under point blowing up.
(2) J. Hu [Hu00, Theorem 1.2] showed the birational invariance of genus ≤ 1 Gromov–Witten numbers under blowing up a point or a smooth curve, as well as arbitrary genus invariants when $\dim X \leq 3$.
(3) Lai [Lai09, Theorem 1.4] showed the birational invariance in genus 0 if $Y \to X$ is the blowing up of a smooth subvariety Z with convex normal bundle with enough sections, or if Z has contains no images of $\mathbf{P}^1$.
(4) Manolache [Man12, Proposition 5.14] showed birational invariance in genus 0 if Z is the transversal intersection of X with a smooth subvariety of an ambient homogeneous space.

A number of authors, including Maulik-Pandharipande [MP06] and J. Hu, T.J. Li and Y. Ruan [HLR08], considered the behavior of invariants under blowing up using the degeneration formula.

Theorem 1.1.1 shows that logarithmic Gromov–Witten invariants are well-suited to questions of birational invariance. It would be interesting to obtain comparison mechanisms between logarithmic and usual invariants similar to the results of [MP06].

1.4. **Setup.** As explained in [Ols03], a logarithmic variety X is logarithmically smooth if and only if it the associated map $\underline{X} \to \mathsf{Log}$ to the stack of logarithmic structures is smooth. As we recall in section 2 below, this map factors as $\underline{X} \to \underline{\mathcal{X}} \to \mathsf{Log}$ where $X \to \mathcal{X}$ is a strict smooth map to a "locally toric stack" $\mathcal{X}$, which has an étale cover by finitely many stacks of the form $[V/T]$, where V is a toric variety and T its torus. The stack $\mathcal{X}$ is logarithmically étale. The map $Y \to X$ is obtained as the pullback of a toric modification $\mathcal{Y} \to \mathcal{X}$. This means that $V' = \mathcal{Y} \times_{\mathcal{X}} V$ is a toric variety for the same torus T.

1.5. **Structure of proof.** The proof follows [AMW12] closely.

In Section 3.1 we construct moduli stacks of pre-stable logarithmic maps $\mathfrak{M}(\mathcal{Y})$ and $\mathfrak{M}(\mathcal{X})$ with maps $\psi_X : \overline{\mathcal{M}}(X) \to \mathfrak{M}(\mathcal{X})$ and $\psi_Y : \overline{\mathcal{M}}(Y) \to \mathfrak{M}(\mathcal{Y})$ constucted in Section 4. We show:

log-smooth **Proposition 1.5.1** (See Corollary 3.1.3). *The stacks $\mathfrak{M}(\mathcal{Y})$ and $\mathfrak{M}(\mathcal{X})$ are algebraic and log smooth.*

In order to compare the moduli spaces we construct in Section 3.2 a further stack $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ with a morphism $\psi'_Y : \overline{\mathcal{M}}(Y) \to \mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ constructed in Section 4 and $\alpha : \mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \to \mathfrak{M}(\mathcal{Y})$ such that $\psi_Y = \alpha \circ \psi'_Y$, see Section 5. We show:

Prop:M' **Proposition 1.5.2** (See Corollary 3.2.2 and Section 5.1). *The stack $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ is algebraic and the morphism α is étale and strict.*

We construct $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ with a morphism $\mathfrak{M}(h) : \mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \to \mathfrak{M}(\mathcal{X})$. We obtain a diagram

q:Costello (1)

$$\begin{tikzcd}
\overline{\mathcal{M}}(Y) \arrow[r, "\overline{\mathcal{M}}(h)"] \arrow[d, "\psi'_Y"'] & \overline{\mathcal{M}}(X) \arrow[d, "\psi_X"] \\
\mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \arrow[r, "\mathfrak{M}(h)"'] & \mathfrak{M}(\mathcal{X})
\end{tikzcd}$$

and prove

:cartesian **Proposition 1.5.3** (See Section 4). *The diagram 1 is cartesian.*

p:Costello **Proposition 1.5.4** (See Proposition 5.3.1). *The morphism $\mathfrak{M}(h)$ is of pure degree 1.*

We construct obstruction theories $\mathcal{E}_X$ relative to ψ_X and $\mathcal{E}_Y$ relative to ψ_Y and prove

bstruction **Proposition 1.5.5** (See Proposition 6.3.1). *We have*

$$[\overline{\mathcal{M}}(X)]^{\mathrm{vir}} = (\psi_X)^!_{\mathcal{E}_X}[\overline{\mathcal{M}}(\mathcal{X})] \qquad \textit{and} \qquad [\overline{\mathcal{M}}(Y)]^{\mathrm{vir}} = (\psi_Y)^!_{\mathcal{E}_Y}[\overline{\mathcal{M}}(\mathcal{Y})].$$

Proposition 1.5.2 implies that $\mathcal{E}_Y$ is an obstruction theory relative to ψ'_Y and

$$[\overline{\mathcal{M}}(Y)]^{\mathrm{vir}} = (\psi'_Y)^!_{\mathcal{E}_Y}[\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})].$$

Theorem 1.1.1 then follows from Costello's result [Cos06, Theorem 5.0.1]; see also [Man12, Proposition 5.29] and [Lai09, Proposition 3.15].

1.6. **Acknowledgements.** We thank Mark Gross who asked the question, and Steffen Marcus, with whom some of the techniques were developed in [AMW12].

ric-stacks

2. Construction of $\mathcal{X}$ and $\mathcal{Y}$

We treat the logarithmically smooth case. The general case is treated, along with a diagram involving Kato fans, polyhedral complexes and Berkovich analytic spaces, in a manuscript in preparation by M. Ulirsch.

2.1. **The stack $\mathcal{X}$.** We construct the stack $\mathcal{X}$ as a universal object depending on $\mathcal{X}$. First, there is a canonical morphism $\mathcal{X} \to \mathsf{Log}$; its image is an open substack of Log, but it is too coarse an object because different strata of X can map to the same point of Log. The idea is to correct this issue universally.

Let P be a fine, saturated sharp monoid, where *sharp* means that P has a unique invertible element $P^\times = \{0\}$. Define a functor $\mathscr{S}_P^{\log} : \mathsf{LogSch}^\circ \to \mathsf{Sets}$ by

$$\mathscr{S}_P^{\log}(X, M_X) = \operatorname{Hom}(P, \Gamma(X, \overline{M}_X)).$$

According to [Ols03, Proposition 5.17] this functor is representable by a logarithmically étale algebraic stack we denote $\mathscr{S}_P = \left[\operatorname{Spec} \mathbf{C}[P] \big/ \operatorname{Spec} \mathbf{C}[P^{\mathrm{gp}}]\right]$. The map $\mathscr{S}_P \to \mathsf{Log}$ is étale.

:no-covers **Lemma 2.1.1.** *Every étale cover of $\mathscr{S}_P$ admits a section.*

Proof. An étale cover of $\mathscr{S}_P$ induces an étale cover of $\operatorname{Spec} \mathbf{C}[P]$ and by restriction, an étale cover of $\operatorname{Spec} \mathbf{C}[P^{\mathrm{gp}}]$. But $\operatorname{Spec} \mathbf{C}[P^{\mathrm{gp}}]$ is a torus and its connected étale covers all arise from surjective homomorphisms of tori. Since P is sharp, the closed orbit of $\operatorname{Spec} \mathbf{C}[P]$ is a point, and any surjective homomorphism of tori which extend to an étale map over the closed orbit of $\operatorname{Spec} \mathbf{C}[P]$ is an isomorphism. $\square$

Proposition 2.1.2. *Let (X, M_X) be a logarithmic scheme. Then there is an algebraic stack $\mathcal{X}$ and factorization $X \to \mathcal{X} \to \mathsf{Log}$ of the morphism $X \to \mathsf{Log}$, such that*

(1) $\mathcal{X} \to \mathsf{Log}$ is representable and étale, and
(2) whenever $X \to \mathcal{X} \to \mathsf{Log}$ factors as $X \to \mathcal{Y} \to \mathsf{Log}$ with $\mathcal{Y} \to \mathsf{Log}$ representable and étale, there is a unique arrow $\mathcal{X} \to \mathcal{Y}$ whose composition with $\mathcal{Y} \to \mathsf{Log}$ is the given arrow $\mathcal{X} \to \mathsf{Log}$.

In other words, $X \to \mathcal{X}$ is an initial map from X to a logarithmic stack $\mathcal{X}$ that is representable and étale over Log.

Proof. First we work étale locally in X and assume that there is a point x of X such that $\overline{M}_{X,x}$ lifts to a chart. Let $\mathcal{X} = \mathscr{S}_{\overline{M}_{X,x}}$. We show that $\mathcal{X}$ is initial among all maps to étale sheaves over Log. Suppose that we had another factorization $X \to \mathcal{Y} \to \mathsf{Log}$. Then $\mathcal{Y} \times_{\mathsf{Log}} \mathcal{X} \to \mathcal{X}$ is étale. It is also surjective since it contains the image of x in $\mathcal{X}$, which is the unique closed point. Letting $\mathcal{Y}'$ be the connected component of $\mathcal{Y} \times_{\mathsf{Log}} \mathcal{X}$ containing the image of x, it follows that $\mathcal{Y}' \to \mathcal{X}$ is an isomorphism, since $\mathcal{X}$ has no nontrivial étale covers by Lemma 2.1.1. Composing the inverse of this isomorphism with the projection $\mathcal{Y}' \to \mathcal{Y}$ gives a map $\mathcal{X} \to \mathcal{Y}$. This is clearly unique.

To treat the general case, consider the collection of all étale spaces $\widetilde{X} \to X$ admitting a factorization $\widetilde{X} \to \widetilde{\mathcal{X}} \to \mathsf{Log}$ satisfying conditions (1),(2) of the proposition. This collection contains all sufficiently small neighborhoods of all geometric points of X. Since the collection of étale sheaves over Log is closed under colimits, this collection is also closed under colimits. Hence it is all of $\text{ét}(X)$, in particular in includes X. □

2.2. **Functoriality.** Note that by construction the map $X \to \mathcal{X}$ is strict.

The universal property characterizing $\mathcal{X}$ implies that it is functorial in X with respect to strict morphisms.

Now suppose that Y is an algebraic stack equipped with *two* logarithmic structures M_Y and M'_Y, as well as a morphism $M'_Y \to M_Y$. Working étale-locally in Y, we see that there is a map $\mathscr{S}_{M_Y} \to \mathscr{S}_{M'_Y}$, which evidently glues to give a global map.

This shows that the formation of $X \to \mathcal{X}$ is in fact functorial with respect to *all* logarithmic morphisms $Y \to X$. Of course, in this generality the map $\mathcal{Y} \to \mathcal{X}$ does not commute with the étale projections to Log.

2.3. **Modifications.** We want to show that a toroidal modification $\underline{Y} \to \underline{X}$, namely a logarithmically étale modificaiton $Y \to X$, is induced by a logarithmically étale modification $\mathcal{Y} \to \mathcal{X}$. By functoriality we have a commutative diagram

eqn:4 (2)
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{X} \end{tikzcd}$$

and we need to show that the diagram is cartesian and $\mathcal{Y} \to \mathcal{X}$ is a modification. This can be checked étale locally over X, so we may assume $\underline{X}$ is a toroidal embedding without self-intersections, equivalently a Zariski logarithmically smooth scheme. In this case Kato constructed in [Kat94, Proposition 10.1] a morphsim of monoidal spaces $(\underline{X}, \overline{M}_X) \to F(X)$ to the *Kato fan* $F(X)$ of X; in [Kat94, Proposition 9.9 (2)] it is shown that toroidal modifications $Y \to X$ are equivalent to subdivisions $F(Y)$ of $F(X)$. We note that $F(\mathcal{X}) = F(X)$, hence the subdivision $F(Y)$ of $F(X)$ corresponds to a modification $\mathcal{Y} \to \mathcal{X}$ as required.

3. Algebraicity

3.1. The stacks $\mathfrak{M}(\mathcal{X})$ and $\mathfrak{M}(\mathcal{Y})$.

Proposition 3.1.1. *Let $\mathcal{Y} \to \mathcal{X}$ be a logarithmically étale S-morphism of algebraic stacks that are locally of finite presentation over S. Define $\mathfrak{M}(\mathcal{Y}/S)$ and $\mathfrak{M}(\mathcal{X}/S)$ to be the stacks of minimal logarithmic morphisms from logarithmically smooth curves into the fibers of $\mathcal{Y}$ and $\mathcal{X}$ over S. Then the map $\mathfrak{M}(\mathcal{Y}/S) \to \mathfrak{M}(\mathcal{X}/S)$ is representable by logarithmically étale morphisms of algebraic stacks.*

Proof. We factor the map $\mathfrak{M}(\mathcal{Y}/S) \to \mathfrak{M}(\mathcal{X}/S)$ as

$$\mathfrak{M}(\mathcal{Y}/S) \to \mathsf{Log}(\mathfrak{M}(\mathcal{Y}/S)) \to \mathsf{Log}(\mathfrak{M}(\mathcal{X}/S)) \to \mathfrak{M}(\mathcal{X}/S).$$

The first of these maps is an open embedding, as minimality is an open condition. The fibers of the last map are known to be algebraic stacks by the main result of [Ols03]. The algebraicity will therefore follow if we show that $\mathsf{Log}(\mathfrak{M}(\mathcal{Y}/S)) \to \mathsf{Log}(\mathfrak{M}(\mathcal{X}/S))$ is representable by algebraic spaces. But given a diagram of solid lines

$$\begin{tikzcd}
 & & \mathcal{Y} \arrow[d] \\
C_T \arrow[r] \arrow[d, "\pi'"'] \arrow[urr, dashed] & C \arrow[r] \arrow[d, "\pi"'] & \mathcal{X} \arrow[dl] \\
T \arrow[r] & S, &
\end{tikzcd}$$

the lifts of the map $C \to \mathcal{X}$ can be identified with sections of the étale sheaf $\mathsf{Log}(\mathcal{Y}) \times_{\mathsf{Log}(\mathcal{X})} C_T$ over C_T. In other words, lifts of the map $T \to \mathsf{Log}(\mathfrak{M}(\mathcal{X}/S))$ to $\mathsf{Log}(\mathfrak{M}(\mathcal{Y}/S))$ can be identified with sections over T of the étale sheaf $\pi'_*(\mathsf{Log}(\mathcal{Y}) \times_{\mathsf{Log}(\mathcal{X})} C_T)$. As C is proper over S and pushforward of étale sheaves commutes with base change, it follows that the fiber of $\mathsf{Log}(\mathfrak{M}(\mathcal{Y}/S))$ may be identified here with $\pi_*(\mathsf{Log}(\mathcal{Y}) \times_{\mathsf{Log}(\mathcal{X})} C)$. This is an étale sheaf and every étale sheaf is representable by algebraic spaces.

For later reference, we prove that $\mathfrak{M}(\mathcal{Y}/S) \to \mathfrak{M}(\mathcal{X}/S)$ is logarithmically étale as a lemma:

Lemma 3.1.2. *Assume that $\mathcal{Y} \to \mathcal{X}$ is a logarithmically étale S-morphism. Then $\mathfrak{M}(\mathcal{Y}/S) \to \mathfrak{M}(\mathcal{X}/S)$ is logarithmically étale.*

Consider the logarithmic lifting problem

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & \mathfrak{M}(\mathcal{Y}/S) \arrow[d] \\
T' \arrow[r] \arrow[ur, dashed] & \mathfrak{M}(\mathcal{X}/S)
\end{tikzcd}$$

where T' is a strict, square-zero extension of T. This corresponds to

$$\begin{tikzcd}
 & & \mathcal{Y} \arrow[d] \\
C \arrow[r] \arrow[d] \arrow[urr, bend left] & C' \arrow[r] \arrow[d] \arrow[ur, dashed] & \mathcal{X} \arrow[d] \\
T \arrow[r] & T' \arrow[r] & S.
\end{tikzcd}$$

Then the extension exists and is unique because $\mathcal{Y} \to \mathcal{X}$ is logarithmically étale and $C \to C'$ is a strict infinitesimal extension.

∎

over-point **Corollary 3.1.3.** *If $\mathcal{X}$ is a logarithmic algebraic stack that is logarithmically étale over a point then $\mathfrak{M}(\mathcal{X})$ is a logarithmically smooth algebraic stack.*

Proof. We know that the stack of pre-stable logarithmic curves is algebraic, and the proposition shows that $\mathfrak{M}(\mathcal{X}) \to \mathfrak{M}$ is relatively algebraic and logarithmically étale. Since $\mathfrak{M}$ is logarithmically smooth, it follows that $\mathfrak{M}(\mathcal{X})$ is as well. ∎

Since the stacks $\mathcal{Y}$ and $\mathcal{X}$ in Proposition 1.5.1 are logarithmically étale over a point, we may apply Corollary 3.1.3 to prove the proposition.

Sec:M'

3.2. **The stack $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$.** Our arguments require a further stack $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$, the moduli space of diagrams

Eq:M' (3)

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \overline{C} \arrow[r] & \mathcal{X} \end{tikzcd}$$

in which C and $\overline{C}$ are pre-stable logarithmic curves, $C \to \overline{C}$ is logarithmically étale. We write $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ for the open substack of $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$ where the automorphism group
2→ of (3) relative to its image $\overline{C} \to \mathcal{X}$ in $\mathfrak{M}(\mathcal{X})$ is finite.[2]

-log-etale **Corollary 3.2.1.** *Assume that $\mathcal{Y} \to \mathcal{X}$ is a logarithmically étale morphism of logarithmic algebraic stacks over S. Let $\mathsf{Log}(\mathfrak{F})$ be the stack of logarithmically commutative diagrams*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \overline{C} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

in which C and $\overline{C}$ are logarithmically smooth curves over T. Then $\mathfrak{F}$ is representable by algebraic stacks relative to $\mathfrak{M}(\mathcal{X}/S)$.

Proof. Working relative to $\mathfrak{M}(\mathcal{X}/S)$ we may assume that a diagram

$$\begin{tikzcd} \overline{C} \arrow[r] \arrow[d] & \mathcal{X} \\ S & \end{tikzcd}$$

[2] Jonathan: added definition of $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$

is given. Then we are to prove that the stack of all logarithmically commutative diagrams

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{Y} \arrow[d] \\ & \overline{C} \arrow[r] \arrow[d] & \mathcal{X} \arrow[dl] \\ T \arrow[r] & S & \end{tikzcd}$$

is algebraic. We may identify this as the space of pre-stable logarithmic maps $\mathfrak{M}(\mathcal{Y} \times_{\mathcal{X}} \overline{C}/S)$. But $\mathcal{Y} \times_{\mathcal{X}} \overline{C} \to \overline{C}$ is logarithmically étale, so the proposition implies it is sufficient to show $\mathfrak{M}(\overline{C}/S)$ is algebraic. However, $\overline{C}$ is locally projective over S so pre-stable maps (not necessarily logarithmic) to $\overline{C}/S$ are well known to be algebraic. By [Che10, GS13],[3] it follows that pre-stable logarithmic maps to $\overline{C}/S$ form an algebraic stack as well. ←3 □

-algebraic **Corollary 3.2.2.** *Let $\mathcal{Y} \to \mathcal{X}$ be a logarithmic morphism between algebraic stacks that are logarithmically étale over a point. Then the stack $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ is algebraic.*

Proof. By Corollary 3.1.3, we know that $\mathfrak{M}(\mathcal{X})$ is algebraic. By Corollary 3.2.1, we deduce that $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$ is relatively algebraic over $\mathfrak{M}(\mathcal{X})$, hence is algebraic. But the stability condition defining $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ inside $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$ is open, so it now follows that $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ is algebraic. □

This gives the algebraicity statement of Proposition 1.5.2.

:cartesian

4. The cartesian diagram

Our assumptions provide us with a cartesian diagram of logarithmic stacks (2) in which

(1) $\mathcal{X}$ and $\mathcal{Y}$ are logarithmically étale over a point,
(2) the morphism of algebraic stacks underlying $\mathcal{Y} \to \mathcal{X}$ is representable by algebraic spaces,[4] ←4
(3) the vertical arrows are strict, and
(4) X and Y are proper logarithmic schemes.

We consider the following diagram

eqn:1 (4)
$$\begin{tikzcd} \overline{\mathcal{M}}(Y) \arrow[r] \arrow[d] & \overline{\mathcal{M}}(X) \arrow[d] \\ \mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \arrow[r] & \mathfrak{M}(\mathcal{X}) \end{tikzcd}$$

with the following definitions:

(1) $\overline{\mathcal{M}}(X)$ and $\overline{\mathcal{M}}(Y)$ are, respectively, the moduli stacks of stable logarithmic maps into X and Y,
(2) $\mathfrak{M}(\mathcal{X})$ is the moduli space of pre-stable logarithmic maps into $\mathcal{X}$, and
(3) $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ is the moduli space of diagrams (3) described in section 3 above.

[3] Jonathan: precise reference
[4] Jonathan: is this necessary?

The map $\overline{\mathcal{M}}(X) \to \mathfrak{M}(\mathcal{X})$ is defined by composition of $C \to X$ with $X \to \mathcal{X}$. The map $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \to \mathfrak{M}(\mathcal{X})$ is obtained by sending a diagram (3) to the map $\overline{C} \to \mathcal{X}$. The map $\overline{\mathcal{M}}(Y) \to \overline{\mathcal{M}}(X)$ is defined by [AMW12, B.6]: an object $C \to Y$ of $\overline{\mathcal{M}}(Y)$ induces a stable
5→ map[5] $\overline{C} \to X$; it comes along with a comutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Y \arrow[d] \\ \overline{C} \arrow[r] & X. \end{tikzcd}$$

Since $Y = X \times_{\mathcal{X}} \mathcal{Y}$ this extends uniquely to

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \overline{C} \arrow[r] & X \arrow[r] & \mathcal{X}. \end{tikzcd}$$

This gives us the map $\overline{\mathcal{M}}(Y) \to \mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$.

Proof of Proposition 1.5.3. We verify that diagram (4) is logarithmically cartesian. As its
6→ vertical arrows are strict,[6] this will imply that the underlying diagram of algebraic stacks is cartesian as well.

Suppose that we are given maps $S \to \mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ and $S \to \overline{\mathcal{M}}(X)$ along with an isomorphism between the induced maps $S \to \mathfrak{M}(X)$. These data correspond to a diagram of solid lines

eqn:3

$$\begin{tikzcd} C \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & Y_S \arrow[r] \arrow[d] & \mathcal{Y}_S \arrow[d] \\ \overline{C} \arrow[r] & X_S \arrow[r] & \mathcal{X}_S \end{tikzcd} \tag{5}$$

of logarithmic algebraic stacks over S. We obtain a map $C \to Y_S$ completing the commutative diagram by the universal property of the fiber product.

It remains only to verify that the stability condition of $\overline{\mathcal{M}}(Y)$ holds in (5) if and only if the stability condition of $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ does. Let G be the automorphism group of the image of (5) in $\overline{\mathcal{M}}(Y)$, let G'' be the automorphism group of the image in $\overline{\mathcal{M}}(X)$, and let G' be the kernel of $G \to G''$. We note that G'' is finite, so G' is finite if and only if G is.

On the other hand G' may be identified with the kernel of the map $\mathrm{Aut}_{\mathfrak{M}'(\mathcal{Y}\to\mathcal{X})} \to \mathrm{Aut}_{\mathfrak{M}(\mathcal{X})}$. Therefore the finiteness of G' is precisely the stability condition of $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ and the finteness of G is the stability condition of $\overline{\mathcal{M}}(Y)$. $\square$

:universal

5. The universal logarithmically étale modification

Let $\mathcal{Y} \to \mathcal{X}$ be a morphism of logarithmic algebraic stacks. We obtain a correspondence

$$\begin{tikzcd} & \mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \arrow[dl] \arrow[dr] & \\ \mathfrak{M}(\mathcal{Y}) & & \mathfrak{M}(\mathcal{X}) \end{tikzcd}$$

[5]Jonathan: stable map

[6]Jonathan: verify?

where $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$ is the moduli space of minimal logarithmic diagrams

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \overline{C} \arrow[r] & \mathcal{X} \end{tikzcd}$$

such that $C \to \overline{C}$ is logarithmically étale.

Sec:M'toM

5.1. The arrow $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \to \mathfrak{M}(\mathcal{Y})$.

Proof of Proposition 1.5.2. [7] Algebraicity was shown in Corollary 3.2.2. A logarithmic infinitesimal lifting problem ←7

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & \mathfrak{M}(\mathcal{Y}) \end{tikzcd}$$

corresponds to a logarithmic extension problem

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, dashed] \arrow[dd, bend right] & \mathcal{Y} \arrow[d] \\ \overline{C} \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & \overline{C}' \arrow[r, dashed] \arrow[d, dashed] & \mathcal{X} \\ S \arrow[r] & S'. \end{tikzcd}$$

Now, $C \to \overline{C}$ contracts chains of semistable components[8] so we may apply [AMW12, Appendix B] to obtain $\overline{C}'$ (uniquely). All that is left is to produce the map $\overline{C}' \to \mathcal{X}$ and show it is unique. This follows from the two lemmas below. □ ←8

Lemma 5.1.1. *$\overline{C}'$ is the pushout of the maps $C \to \overline{C}$ and $C \to C'$ in the category of logarithmic schemes.*

Proof. The underlying space of $\overline{C}'$ is the pushout of the maps underlying $C \to \overline{C}$ and $C \to C'$. Also the logarithmic structure on $\overline{C}'$ is the push-forward of the logarithmic structure on C'. This implies the result.[9] □ ←9

Lemma 5.1.2. *$\overline{C}'$ is the pushout of the maps $C \to \overline{C}$ and $C \to C'$ in the 2-category of logarithmic stacks.*

Proof. The construction of $\overline{C}'$ is local in the étale topology of $\overline{C}$, so we may work étale locally in $\overline{C}$. We may therefore assume that given maps $\overline{C} \to \mathcal{X}$ and $C' \to \mathcal{X}$ factor through a smooth, strict chart $X \to \mathcal{X}$. But then these maps extend uniquely in a compatible way to $\overline{C}' \to X$ by the previous lemma. The uniqueness of this extension guarantees that the induced map $\overline{C}' \to \mathcal{X}$ is independent of the chart. □

[7] Jonathan: argument below doesn't address strictness

[8] Jonathan: justification?

[9] Dan: How to write this without atrocious notation?

5.2. **The arrow $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \to \mathfrak{M}(\mathcal{X})$.**

Proposition 5.2.1. *Assume that $\mathcal{Y} \to \mathcal{X}$ is logarithmically étale. The map $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \to \mathfrak{M}(\mathcal{X})$ is logarithmically étale.*

Proof. Consider a logarithmic lifting problem

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \arrow[d] \\ S' \arrow[r] \arrow[ur, dashed] & \mathfrak{M}(\mathcal{X}) \end{tikzcd}$$

with $S \subset S'$ a strict infinitesimal extension. This corresponds to the lifting problem,

$$\begin{tikzcd} C \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & C' \arrow[r, dashed] \arrow[d, dashed] & \mathcal{Y} \arrow[d] \\ \overline{C} \arrow[r] \arrow[d] & \overline{C}' \arrow[r] \arrow[d] & \mathcal{X} \\ S \arrow[r] & S'. \end{tikzcd}$$

Because $C \to \overline{C}$ is logarithmically étale and $\overline{C} \subset \overline{C}'$ is a strict infintesimal extension, it follows that there is an extension $C' \to \overline{C}'$ of $C \to \overline{C}$ and that this is unique up to unique isomorphism. The only thing remaining is to construct the map $C' \to \mathcal{Y}$. But this amounts to lifting the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ C' \arrow[r] \arrow[ur, dashed] & \mathcal{X}, \end{tikzcd}$$

which we can do uniquely, since $\mathcal{Y} \to \mathcal{X}$ is logarithmically étale. ☐

5.3. **Birationality: proof of Proposition 1.5.4.**

prop:univ **Proposition 5.3.1.** *Suppose that $\mathcal{X} \to \mathcal{Y}$ is logarithmically étale, proper, and birational. Then the maps $\mathfrak{M}(\mathcal{Y} \to \mathcal{X}) \to \mathfrak{M}(\mathcal{X})$ and $\mathfrak{M}(\mathcal{Y} \to \mathcal{X}) \to \mathfrak{M}(\mathcal{Y})$ are birational.*

Proof. All of the stacks $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$, $\mathfrak{M}(\mathcal{X})$, and $\mathfrak{M}(\mathcal{Y})$ are logarithmically smooth. Therefore they have dense open substacks where their logarithmic structures are trivial. We show that these dense open substacks are isomorphic.

Consider an S-point of $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$, where S has the trivial logarithmic structure. We have a commutative diagram

eqn:2 (6)
$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \overline{C} \arrow[r] & \mathcal{Y} \end{tikzcd}$$

with $C \to \overline{C}$ logarithmically étale and both C and $\overline{C}$ logarithmically smooth over S. This implies first that the underlying curves of C and $\overline{C}$ are smooth, and second that the map of schemes underlying $C \to \overline{C}$ is a branched cover. But the stabilization of $C \to \overline{C}$ must also be an isomorphism, so its degree must be 1 and therefore $C \to \overline{C}$ is an isomorphism. This proves that $\mathfrak{M}(\mathcal{Y} \to \mathcal{X}) \to \mathfrak{M}(\mathcal{Y})$ is an isomorphism over the loci with trivial logarithmic structures.

Now consider an S-point $\overline{C} \to \mathcal{X}$ of $\mathfrak{M}(\mathcal{X})$. We may obtain a point of $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$ by taking $C = \overline{C} \times_{\mathcal{X}} \mathcal{Y}$. Note that the map $C \to \overline{C}$ is proper and logarithmically étale, so C is a proper, logarithmically smooth curve over S. It is immediate that this is a section of $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$ over $\mathfrak{M}(\mathcal{X})$ and remains only to verify that if (6) is an S-point of $\mathfrak{M}(\mathcal{Y} \to \mathcal{X})$ then $C = \overline{C} \times_{\mathcal{X}} \mathcal{Y}$. However, this follows from the fact that $C \to \overline{C}$ is an isomorphism, as we saw above. $\square$

6. Obstruction theories

6.1. **The arrow $\overline{\mathcal{M}}(X) \to \mathfrak{M}(\mathcal{X})$.** First we show that the natural obstruction theory for $\overline{\mathcal{M}}(X)$ over $\mathfrak{M}(\mathcal{X})$ agrees with the one over $\mathfrak{M}$ defined in [AC11, Che10, GS13]. Let $S \subset S'$ be a strict square-zero extension over $\mathfrak{M}(\mathcal{X})$ with ideal J and assume given an S-point of $\overline{\mathcal{M}}(X)$. We have a diagram of solid lines

eqn:5 (7)
$$\begin{tikzcd} & & X \arrow[d] \\ C \arrow[r] \arrow[d] \arrow[rru, bend left, "f"] & C' \arrow[r] \arrow[d] \arrow[ru, dashed] & \mathcal{X} \\ S \arrow[r] & S'. & \end{tikzcd}$$

Note that because $\mathcal{X}$ is étale over Log, lifts of this diagram are precisely the same as lifts of the diagram

$$\begin{tikzcd} & & X \arrow[d] \\ C \arrow[r] \arrow[d] \arrow[rru, bend left, "f"] & C' \arrow[r] \arrow[d] \arrow[ru, dashed] & \mathsf{Log} \\ S \arrow[r] & S'. & \end{tikzcd}$$

Since X is smooth over $\mathcal{X}$, the logarithmic lifts of either of these diagrams form a torsor on C under the sheaf of abelian groups $f^* T_{X/\mathcal{X}} \otimes J = f^* T_X^{\log} \otimes J$. Therefore if we define $\mathscr{E}(J)$ to be the stack on S of $f^* T_X^{\log} \otimes J$-torsors on C we obtain an obstruction theory in the sense of [Wis11] for $\overline{\mathcal{M}}(X)$ over $\mathsf{Log}(\mathfrak{M}(\mathcal{X}))$ or over $\mathsf{Log}(\mathfrak{M})$. The latter of these is the one defined in [AC11, Che10, GS13].

6.2. **The arrow $\overline{\mathcal{M}}(Y) \to \mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$.** A similar argument will apply to demonstrate that this obstruction theory pulls back to an obstruction theory for $\overline{\mathcal{M}}(Y)$ over $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$. A lifting problem

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \overline{\mathcal{M}}(Y) \arrow[d] \\ S' \arrow[r] \arrow[ru, dashed] & \mathfrak{M}'(\mathcal{Y} \to \mathcal{X}) \end{tikzcd}$$

corresponds to the following lifting problem:

eqn:6 (8)

$$\begin{tikzcd}
 & & Y \arrow[d] \\
C \arrow[r] \arrow[d] \arrow[urr, bend left, "g"] & C' \arrow[r] \arrow[d] \arrow[ur, dashed] & \mathcal{Y} \arrow[d] \\
\overline{C} \arrow[r] \arrow[d] & \overline{C}' \arrow[r] \arrow[d] & \mathcal{X} \\
S \arrow[r] & S'. &
\end{tikzcd}$$

As before, the lifts form a torsor under $g^*T_{Y/\mathcal{Y}} = g^*T_Y^{\log}$. But in view of the cartesian diagram (2), this torsor is precisely the same as the torsor of lifts of the induced diagram (7). On the other hand, the third row of (8) is obviously irrelevant to the lifting question, so the obstruction theory is the same as the one for $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$ over $\mathfrak{M}(\mathcal{Y})$.

6.3. **Conclusion.** We have therefore proved the following precise restatement of Proposition 1.5.5:

prop:obs **Proposition 6.3.1.** *Let $\mathscr{E}$ denote the perfect relative obstruction theory for $\overline{\mathcal{M}}(X)$ over $\mathsf{Log}(\mathfrak{M})$ and let $\mathscr{E}'$ denote the perfect relative obstruction theory for $\overline{\mathcal{M}}(Y)$ over $\mathsf{Log}(\mathfrak{M})$. Then*

(1) $\mathscr{E}$ is also a perfect relative obstruction theory for $\overline{\mathcal{M}}(X)$ over $\mathfrak{M}(\mathcal{X})$,
(2) $\mathscr{E}'$ is also a perfect relative obstruction theory for $\overline{\mathcal{M}}(Y)$ over $\mathfrak{M}'(\mathcal{Y} \to \mathcal{X})$, and
(3) $\Phi^*\mathscr{E} = \mathscr{E}'$.

We may now combine Propositions 5.3.1 and 6.3.1 with Costello's theorem [Cos06, Theorem 5.0.1] to deduce $\Phi_*[\overline{\mathcal{M}}(X)]^{\mathrm{vir}} = [\overline{\mathcal{M}}(Y)]^{\mathrm{vir}}$.

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(Abramovich) Department of Mathematics, Brown University, Box 1917, Providence, RI 02912, U.S.A.
E-mail address: abrmovic@math.brown.edu

(Wise) University of Colorado, Boulder, Boulder, Colorado 80309-0395, USA
E-mail address: jonathan.wise@math.colorado.edu