WEAK TOROIDALIZATION OVER NON-CLOSED FIELDS

DAN ABRAMOVICH, JAN DENEF, AND KALLE KARU

Contents

1. Introduction

th-toroidal-reduction **Theorem 1.1.** *Let k be a field of characteristic zero. Let $f : X \to B$ be a dominant morphism* [1] *of k varieties, and let $Z \subset X$ be a proper closed subset.* [2] *Then there exists a diagram* ←1 ←2

$$\begin{array}{ccccc} U_{X'} & \subset & X' & \stackrel{m_X}{\to} & X \\ \downarrow & & \downarrow f' & & \downarrow f \\ U_{B'} & \subset & B' & \stackrel{m_B}{\to} & B \end{array}$$

where m_B and m_X are projective birational morphisms and such that

(1) the inclusions on the left are nonsingular strict toroidal embeddings.

(2) f' is a toroidal quasi-projective[3] *morphism.* ←3

(3) Let $Z' = m_X^{-1}Z$. Then Z' is a strict normal crossings divisor, $Z' \subset X' \smallsetminus U_{X'}$.

(4) The restricted morphism $U_{X'} \to m_X(U_{X'})$ is an isomorphism.

Note that when f is proper, so is f', so f' becomes projective.

Toroidal embeddings and toroidal morphisms are defined in Section 2 below. When both varieties X' and B' are nonsingular, the embeddings $U_{X'} \subset X'$, $U_{B'} \subset B'$ and the morphism f' of the theorem can be described as follows: the requirement that the embeddings are strict toroidal is equivalent to the statement that $X' \smallsetminus U_{X'}$, $B' \smallsetminus U_{B'}$ are strict normal crossings divisors. The requirement that f' is toroidal is equivalent to the following: after

Date: September 23, 2010.

Abramovich support in part by NSF grants, DMS-0901278. Karu was supported by NSERC grant RGPIN 283301.

[1](Dan) changed to dominant - need to verify all is compatible

[2](Dan) Changed to subset - need to verify all is compatible

[3](Dan) added quasi projective - verify

base change to an algebraic closure $\bar{k}$ of k, for each closed point $x \in X'_{\bar{k}}$, $b = f'(x) \in B'_{\bar{k}}$, there exist uniformizing parameters $x_1, \ldots, x_n$ for $\hat{\mathcal{O}}_{X'_{\bar{k},x}}$ and $b_1, \ldots, b_m$ for $\hat{\mathcal{O}}_{B'_{\bar{k},b}}$, such that

(1) Locally at x, the product $x_1 \cdots x_n$ defines the divisor $X'_{\bar{k}} \setminus U_{X',\bar{k}}$.
(2) Locally at b, the product $b_1 \cdots b_m$ defines the divisor $B'_{\bar{k}} \setminus U_{B',\bar{k}}$.
(3) The morphism f' gives b_i as monomials in x_j.

Here we say that $z_1, \ldots, z_n$ are uniformizing parameters for a local $\bar{k}$-algebra A if there exist constants $c_1, \ldots, c_n \in \bar{k}$, such that $z_1 - c_1, \ldots, z_n - c_n$ form a system of regular parameters for A.[4]

4→

sec-notation

2. Notations and definitions

We work over a field k of characteristic zero. A variety defined over k is an integral separated scheme of finite type over k. If X is a variety defined over k, we let $X_{\bar{k}}$ be its base extension to an algebraic closure $\bar{k}$ of k. Note that the scheme $X_{\bar{k}}$ might not be a variety.

A *modification* is a proper birational morphism of varieties. An *alteration* is a proper, surjective, generically finite morphism of varieties. An alteration $Y \to X$ is called a *Galois alteration* with Galois group G if the function field extension $K(Y)/K(X)$ is Galois with Galois group G, and if the action of G on $K(Y)$ is induced by an action on Y keeping the morphism $Y \to X$ invariant. [5]

5→

2.1. **Divisors.** Let X be a smooth variety defined over k (or more generally a smooth scheme over k), and $D \subset X$ a divisor. We say that D is a *strict normal crossings divisor* if for every point $x \in X$ there exists a regular system of parameters $z_1, \ldots, z_n$ at x, such that every irreducible component of D containing x has local equation $z_i = 0$ for some i. A divisor is a normal crossings divisor if it becomes a strict normal crossings divisor on some étale cover of X. The condition of being a (strict) normal crossings divisor is stable under base extension to algebraic closure: if $D \subset X$ is a (strict) normal crossings divisor, so is $D_{\bar{k}} \subset X_{\bar{k}}$.

Let a finite group G act on a (not necessarily smooth) variety X over k (or more generally a scheme of finite type over k), mapping a divisor $D \subset X$ into D. We say that D is *G-strict* if the union of translates of each irreducible component of D is normal. In the case where G is the trivial group 1, we say that D is a strict divisor. Thus, a strict normal crossings divisor is both strict and normal crossings divisor.

ec-toroidal-embeddings

2.2. **Toroidal embeddings.** We refer to [13, 7] for details about toric varieties. If V is a toric variety, we denote by $T_V \subset V$ the big algebraic torus in V. Toric morphisms are always assumed to be dominant.

[4](Dan) a bit of rephrasing in the whole paragraph - verify
[5](Dan) added Galois alteration, verify

An open embedding of varieties $U \subset X$ defined over $\bar{k}$ (or more generally of schemes of finite type over $\bar{k}$) is called a *toroidal embedding* if for every closed point $x \in X$ there exists a toric variety V, a closed point $v \in V$, and an isomorphism of complete local $\bar{k}$-algebras:

$$\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{V,v},$$

such that the completion of the ideal of $X \smallsetminus U$ maps isomorphically to the completion of the ideal of $V \smallsetminus T_V$. The pair (V, v), together with the isomorphism, is called a local model at $x \in X$. A toroidal embedding $U \subset X$ over $\overline{k}$ is called *strict* if $D = X \smallsetminus U$ is a strict divisor.

An open embedding $U \subset X$ defined over k is called a toroidal embedding if the base extension $U_{\bar{k}} \subset X_{\bar{k}}$ is a toroidal embedding. The toroidal embedding is strict if the divisor $D = X \smallsetminus U$ is strict. Note that if the toroidal embedding $U \subset X$ is strict, then the toroidal embedding $U_{\bar{k}} \subset X_{\bar{k}}$ is also strict, but the converse may not hold.

Let $U_X \subset X$ and $U_B \subset B$ be two toroidal embeddings defined over $\bar{k}$, and let $f : X \to B$ be a dominant morphism mapping U_X to U_B. (We write such a morphism as $f : (U_X \subset X) \to (U_B \subset B)$.) Then f is called a *toroidal morphism* if for every closed point $x \in X$ there exist local models (V, v) at $x \in X$ and (W, w) at $f(x) \in B$, and a toric morphism $g : V \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,x} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{V,v} \\ {\scriptstyle \hat{f}^{\#}}\uparrow & & \uparrow{\scriptstyle \hat{g}^{\#}} \\ \hat{\mathcal{O}}_{B,f(x)} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{W,w} \end{array}$$

Here $\hat{f}^{\#}$ and $\hat{g}^{\#}$ are the ring homomorphisms coming from f and g.

A morphism $f : (U_X \subset X) \to (U_B \subset B)$ between toroidal embeddings defined over the field k is called a toroidal morphism if its base extension to $\bar{k}$ is a toroidal morphism.

The composition of two toroidal morphisms is again toroidal [2].

sec-toroidal-actions

2.3. **Toroidal actions.** An action of a finite group G on a toroidal embedding $U \subset X$ defined over $\bar{k}$ is called a *toroidal action* at a closed point $x \in X$ if there exists a local model (V, v) at $x \in X$ and a group homomorphism $G_x \to T_V$ from the stabilizer G_x of x to the big torus $T_V \subset V$, such that the action of G_x on the complete local ring

$$\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{V,v}$$

factors through the action of T_V on V via the homomorphism $G_x \to T_V$.[6] ←6 (In particular, the image of G_x must lie in the stabilizer $T_{V,v}$ of v.) The action is *toroidal* if it is toroidal at every closed point. The action of G is called *strict toroidal* if it is both strict and toroidal.

[6](Dan) Changes - shoudl rethink the compatibility statement?

Let G act on $U \subset X$ toroidally, and assume the quotinet X/G exists.[7] Then the quotient $U/G \subset X/G$ is again a toroidal embedding (the local models are given by toric varieties V/G_x). If $U \subset X$ is a strict toroidal embedding and G acts strictly toroidally, then the quotient $U/G \subset X/G$ is again a strict toroidal embedding.

Let $f : (U_X \subset X) \to (U_B \subset B)$ be a toroidal morphism that is G-equivariant under toroidal actions of G on both embeddings. We say that f is *G-equivariantly toroidal* [8] if local models can be chosen so that they are compatible with the morphism f and the actions of G on X and B. Assuming again that the quotients X/G and B/G exist, such an f induces a toroidal morphism of the quotient toroidal embeddings[9]

$$(U_X/G \subset X/G) \to (U_B/G \subset B/G).$$

For a toroidal embedding $U \subset X$ defined over k, an action of G is called toroidal if the induced action on $U_{\bar{k}} \subset X_{\bar{k}}$ is toroidal. The action is strict toroidal if it is both strict and toroidal. The quotient of a (strict) toroidal embedding by a (strict) toroidal action is again a (strict) toroidal embedding. A G-equivariant morphism $f : (U_X \subset X) \to (U_B \subset B)$ is called G-equivariantly toroidal if the base extension to $\bar{k}$ is G-equivariantly toroidal. Such a morphism induces a toroidal morphism of the quotient toroidal embeddings.

sec-semistable

2.4. **Semistable families of curves.** The reference here is [4].

A flat morphism $f : X \to B$ over the field k is a *semistable family of curves* if every geometric fiber of f is a complete reduced connected curve with at most ordinary double point singularities.

Consider a semistable family of curves $f : X \to B$ over $\bar{k}$. If $x \in X$ is in the singular locus of f, then X has a local equation at x:

$$\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{B,f(x)}[[u,v]]/(uv - h)$$

for some $h \in \hat{\mathcal{O}}_{B,f(x)}$. Note that $h = 0$ defines the image of the singular locus of f in B. It follows from this that if $U \subset B$ is a toroidal embedding and $f : X \to B$ is a semistable family of curves, smooth over U, then $f^{-1}(U) \subset X$ is also a toroidal embedding and the map f is toroidal.

A similar statement holds for a semistable family of curves $f : X \to B$ defined over k: Suppose $U \subset B$ is a toroidal embedding and f is smooth over U, then

$$f : (f^{-1}(U) \subset X) \to (U \subset B)$$

is a toroidal morphism of toroidal embeddings.

2.5. **Resolution of singularities.** We will use any one of the canonical resolution of singularities algorithms [12, 6, 3]. [10]

[7] (Dan) Added assumption on existence of quotient, also in next paragraph
[8] (Dan) make lesss vague?
[9] (Dan) add reference? A-DJ? (also added assumption on quotient
[10] (Dan) Why do we need canonical? also optimize this subsection

A canonical resolution of singularities algorithm applied to a variety X defined over k produces a modification $X' \to X$, with X' nonsingular and both X' and the morphism defined over k. If $U \subset X$ is a toroidal embedding, then the resolution algorithm gives a toroidal embedding $U' \subset X'$ and a toroidal modification $f : (U' \subset X') \to (U \subset X)$. Here $U' = f^{-1}(U)$.

Let $U \subset X$ be a nonsingular toroidal embedding. Applying a canonical embedded resolution of singularities to the divisor $D = X \smallsetminus U$ gives a toroidal morphism $f : (U' \subset X') \to (U \subset X)$ of toroidal embeddings, such that $U' \subset X'$ is not only nonsingular, but also strict. Indeed, the components of $X' \smallsetminus U'$ are the components of the strict transform of D and the exceptional divisors. All these components are nonsingular.

Combining these two steps, we can toroidally modify any toroidal embedding $U \subset X$ to a nonsingular strict toroidal embedding. If $f : (U \subset X) \to (U_B \subset B)$ is a toroidal morphism, then there exist toroidal modifications of both embeddings to nonsingular strict toroidal embeddings such that f induces a toroidal morphism between them. Indeed, we first apply the resolution algorithms to B, then resolve the indeterminacies of f, and finally apply the resolution algorithms to X.

3. Proof.

The purpose of this section is to prove the main theorem.

3.1. **Reduction to the projective case.** We first blow up Z on X and replace Z by its inverse image; therefore we may assume Z is the support of an effective Cartier divisor. This modification is projective in Grothendieck's sense; below we further modify X by a quasi-projective variety, so the composite modification will be also projective in Hartshorne's sense.

Let us now reduce to the case where both X and B are quasiprojective varieties. By Chow's lemma ([8], 5.6.1) there exist projective modifications $m_X : X' \to X$ and $B' \to B$ such that both X' and B' are quasiprojective varieties. Replacing X' with the closure of the graph of the rational map $X' \dashrightarrow B'$, we may assume that $X' \to B'$ is a morphism, and that m_X is still a projective morphism. Indeed the closure of the graph is contained in $X' \times_B B'$, and is hence projective over X'. Let Z'' be the union of $m_X^{-1}(Z)$ and the locus where m_X is not an isomorphism.

Now we reduce to the *projective* case. Choose projective closures $X' \subset \overline{X}$ and $B' \subset \overline{B}$, and again by the graph construction, we may assume that $\overline{X} \to \overline{B}$ is a morphism. Let $\overline{Z} = Z'' \cup (\overline{X} \smallsetminus X')$. Then the theorem for $\overline{Z} \subset \overline{X} \to \overline{B}$ implies it for $Z \subset X \to B$, because $m_X^{-1}(Z)$ is the support of an effective Cartier divisor. Thus, we may assume that X and B are projective varieties. Replacing Z by a larger subset we may assume at the same time that Z is the support of an effective Cartier divisor on X. [11] ←11

[11](Dan) Changes to accommodate X a subset, Cartier divisors etc

3.2. **Structure of induction.** We proceed by induction on the relative dimension $\dim X - \dim B$ of $f : X \to B$. In the proof we will repeatedly replace X and B with suitable projective modifications, to which f extends, until the requirements of the theorem are satisfied. This is certainly permitted if we replace Z by a proper closed subset of the modification of X, which contains the inverse image of Z and the locus where the modification is not an isomorphism. We will always (often without mentioning) replace Z in this way, taking it large enough so that it is the support of an effective Cartier divisor.

sec-relDim0

3.3. **Relative dimension 0.** Assume that the relative dimension of f is zero. The proof in this case is a reduction to the Abhyankar's lemma:

lem-abh **Lemma 3.4.** *(Abhyankar, cf.* [10]*) Let X be a normal variety and $f : X \to B$ a finite surjective morphism onto a nonsingular variety, unramified outside a divisor D of normal crossings. Then $(B \setminus D \subset B)$ and $(X \setminus f^{-1}(D) \subset X)$ are toroidal embeddings and f is a toroidal morphism. Moreover, if $f : X \to B$ is Galois with Galois group G, then G acts toroidally on X, and the stabilizer subgroups of G are abelian.* □

12→ [12] We start the reduction by constructing an alteration $\tilde{X} \to X$, such that $\tilde{X} \to B$ is a Galois alteration. Since f is surjective, it is generically finite. Let L be a normalization of the function field $K(X)$ of X over the function field $K(B)$ of B. Then $L/K(B)$ is a finite Galois extension with Galois group G. We choose a projective model $\tilde{X}$ of L such that G acts on $\tilde{X}$. If we fix an embedding $K(X) \subset L$ then for each $g \in G$ we get a rational map $\phi_g : \tilde{X} \to X$ corresponding to the embedding $gK(X) \subset L$. We let

$$\overline{\Gamma} \subset \tilde{X} \times \prod_{g \in G} X$$

be the closure of the graph of $\prod_{g \in G} \phi_g$. Then the group G acts on $\overline{\Gamma}$ and projection to one of the factors X gives us a morphism $\overline{\Gamma} \to X$. So, we may replace $\tilde{X}$ by $\overline{\Gamma}$ and assume that the rational map $\tilde{X} \to X$ is a morphism.

Consider the quotient variety $\tilde{X}/G$. Since G fixes the the field $K(B)$, we have a birational morphism $p : \tilde{X}/G \to B$, hence a rational map $p^{-1} \circ f : X \to \tilde{X}/G$. Let $X_0 \subset X \times_B \tilde{X}/G$ be the closure of the graph of this rational map, and note that the projection $\tilde{X} \to \tilde{X}/G$ factors through X_0:

$$\begin{array}{ccc} & & \tilde{X} \\ & \swarrow & \downarrow \\ X_0 & \to & X \\ f_0 \downarrow & & \downarrow f \\ \tilde{X}/G & \to & B \end{array}$$

[12] (Jan) rewrite using flattening?

In the diagram above the horizontal maps are modifications. Since $\tilde{X} \to \tilde{X}/G$ is finite, so is $f_0 : X_0 \to \tilde{X}/G$. Thus, we have modified $f : X \to B$ to a finite morphism $f_0 : X_0 \to \tilde{X}/G$.

Let $D \in \tilde{X}/G$ be the branch locus of f_0. We let $B' \to \tilde{X}/G$ be a resolution of singularities such that the inverse image of $D \cup f(Z)$ is a strict divisor of normal crossings, and we let X' be the normalization of $X_0 \times_{\tilde{X}/G} B'$. Then the projection $f' : X' \to B'$ is a finite morphism that ramifies over a divisor of normal crossings. By Abhyankar's lemma such a morphism is toroidal.

Note that, by construction, $U_{B'} \subset B'$ is a nonsingular strict toroidal embedding. Applying resolution of singularities to X' and its divisor $X' \smallsetminus U_{X'}$, we may assume that $U_{X'} \subset X'$ is also a nonsingular strict toroidal embedding. The morphism $f' : (U_{X'} \subset X') \to (U_{B'} \subset B')$ and $Z' \subset X'$ satisfy the statements of the theorem. This finishes the proof of the theorem in case $\operatorname{rel.dim} f = 0$.

Assume now that we have proved the theorem for morphisms of relative dimension $n-1$, and consider the case that f has relative dimension n, with X and Y projective varieties, f surjective, and Z the support of an effective Cartier divisor.

3.5. **Preliminary reduction steps.** The idea of the proof in case of relative dimension n is to factor the morphism $f : X \to B$ as a composition $X \to P \to B$, where $X \to P$ has relative dimension 1 and $P \to B$ has relative dimension $n - 1$. We then apply the induction assumption to the morphism $P \to B$, after having replaced $X \to P$ by a semistable family of curves (section 2.4), using semistable reduction. In order to apply the semistable reduction theorem [5], we need the the map $X \to P$ to have geometrically irreducible generic fiber. Let us construct such a factorization.

3.5.1. *Normalizing.* First, we may replace X with its normalization, therefore we can assume X is normal, replacing Z as explained above. Let $\eta \in B$ be the generic point of B.

3.5.2. *Using Bertini's theorem.* By the projectivity assumption we have $X \subset \mathbf{P}^N_B$ for some N. Let $L \subset \mathbf{P}^N_\eta$ be a general enough $(N-n)$-plane, so that $L \cap X$ is finite and contained in the nonsingular locus of X, and such that no line in L is tangent to X. The set of such L contains a nonempty open subset U_1 of the Grassmannian $\mathbb{G}(N-n, \mathbf{P}^N_\eta)$ of $(N-n)$-planes in $\mathbf{P}^N_\eta$. Let $\mathbf{P}^N_\eta \dashrightarrow \mathbf{P}^{n-1}_\eta$ be the projection from $L \subset \mathbf{P}^N_\eta$. This gives a rational map $X_\eta \dashrightarrow \mathbf{P}^{n-1}_\eta$ that is not defined at the finite set of points $L \cap X$. Blowing up these points gives a projective morphism $\tilde{X}_\eta \to \mathbf{P}^{n-1}_\eta$ with fibres $M \cap X$ for all $(N-n+1)$-planes $L \subset M$. Indeed the blowup $\tilde{X}_\eta$ is the closure of the graph of $X_\eta \dashrightarrow \mathbf{P}^{n-1}_\eta$. Note that $\tilde{X}_\eta$ is normal because X is normal and the center of the blowup is a finite set of nonsingular points. Since X is normal, a general enough $(N-n+1)$-plane in $\mathbf{P}^N_\eta$ is disjoint from the singular locus of X. Thus by Bertini's Theorem (see e.g. [11], Chapter

III, Corollary 10.9 and Remark 10.9.2) there exists a nonempty open subset U_2 of the Grassmannian $\mathbb{G}(N-n+1, \mathbf{P}^N_\eta)$ of $(N-n+1)$-planes in $\mathbf{P}^N_\eta$, such that the scheme-theoretic intersection $M \cap X$ is nonsingular for each $M \in U_2$. Let Γ be the closed subset of $\mathbb{G}(N-n, \mathbf{P}^N_\eta) \times \mathbb{G}(N-n+1, \mathbf{P}^N_\eta)$ consisting of the pairs (α, β) with $\alpha \subset \beta$. Note that Γ is irreducible, since it is an image of an open subset of an affine space. Hence, the image of the projection $(U_1 \times U_2) \cap \Gamma \to U_1$ is dense in U_1, because the projections of Γ on the two Grassmannians are surjective. We conclude that there exists a nonempty open set $U_1' \subset U_1$ of planes in the Grassmannian $\mathbb{G}(N-n, \mathbf{P}^N_\eta)$, such that the generic fibre of the morphism $\tilde{X}_\eta \to \mathbf{P}^{n-1}_\eta$ is smooth, whenever $L \in U_1'$. Because the field k is infinite, the k-valued points are dense in the Grassmannian, hence we may choose the plane L to be defined over k.

3.5.3. *Using Stein factorization.* The rational map $X_\eta \dashrightarrow \mathbf{P}^{n-1}_\eta$ gives a rational map $X \dashrightarrow \mathbf{P}^{n-1}_B$, defined over k. Let us replace X with the normalization of the closure of the graph of this map, so we may assume we have a morphism $X \to \mathbf{P}^{n-1}_B$, with X normal. The generic fibre of this morphism is smooth (it is the same as the generic fibre of $\tilde{X}_\eta \to \mathbf{P}^{n-1}_\eta$). Let $X \overset{g}{\to} P \to \mathbf{P}^{n-1}_B$ be the Stein factorization, where $g : X \to P$ is a projective morphism of relative dimension 1 with geometrically connected fibers, and the second morphism is finite (see [9], 4.3.1 and 4.3.4). Then the generic fibre of g is geometrically irreducible.

We are now ready to apply the semistable reduction theorem to the morphism g.

Definition 3.6. Let $\alpha : X_1 \to X$ be an alteration, and $Z \subset X$ an irreducible divisor. The *strict altered transform* $Z_1 \subset X_1$ of Z is the closure of $\alpha^{-1}(\eta)$ in X_1, where η is the generic point of Z. The strict altered transform of a reducible divisor is the union of the strict altered transforms of its components.

3.7. **Semistable reduction of a family of curves.** By [5], Theorem 2.4, items (i)-(iv) and (vii)(b), there exists a commutative diagram of morphisms of projective varieties

$$\begin{array}{ccc} X_1 & \overset{\alpha}{\to} & X \\ \downarrow g_1 & & \downarrow g \\ P_1 & \overset{a}{\to} & P \\ & & \downarrow \\ & & B \end{array}$$

and a finite group $G \subset \mathrm{Aut}_P P_1$, with the following properties:

1. (1) The morphism $a : P_1 \to P$ is a Galois alteration with Galois group G (i.e. $P_1/G \to P$ is birational).
2. (2) The action of G lifts to $\mathrm{Aut}_X X_1$, and $\alpha : X_1 \to X$ is a Galois alteration with Galois group G.

(3) There are disjoint sections $\sigma_i : P_1 \to X_1$, $i = 1, \dots, \kappa$, such that the strict altered transform $Z_1 \subset X_1$ of Z is the union of their images and G permutes the sections σ_i.
(4) The morphism $g_1 : X_1 \to P_1$ is a semistable family of curves with smooth generic fibre, and $\sigma_i(P_1)$ is disjoint from $\operatorname{Sing} g_1$ for each i.

Note that the image of $\alpha^{-1}(Z) \setminus Z_1$ in X has codimension at least two (by definition of the altered transform), hence it lies over a proper closed subset of P. The same holds for the locus in X_1/G where the modification $X_1/G \to X$, induced by α, is not an isomorphism, hence its image in X lies over a proper subset of P. Indeed X is normal, hence any rational map from X to a complete variety is regular outside a subset of codimension ≥ 2. Thus we can find an effective Cartier divisor on X_1/G whose support Z' contains $\alpha^{-1}(Z)/G$, such that $X_1/G \to X$ is an isomorphism outside Z', and $Z' \setminus (Z_1/G)$ lies over a proper closed subset of P/G.

We may replace X, P, Z by X_1/G, P_1/G, and Z', cf. the discussion just above section 3.3. Then $X_1/G = X$, $P_1/G = P$, and $\alpha^{-1}(Z) \setminus Z_1$ lies over a proper closed subset of P_1. Note however that X might not be normal anymore. Finally, observe that the singular locus of $g_1 : X_1 \to P_1$ lies over a proper closed subset of P_1, since g_1 is flat (because semistable) with smooth generic fibre (by (4)).

3.8. **Using the inductive hypothesis.** Let $\Delta \subset P$ be the union of the loci [13] over which P_1 or X_1 are not smooth, and the closure of the image of $\alpha^{-1}(Z) \setminus Z_1$ in P. Note that Δ is a proper closed subset of P. We apply the inductive assumption to $\Delta \subset P \to B$, and obtain a diagram ←13

$$\begin{array}{ccccc} U_{P'} & \hookrightarrow & P' & \stackrel{m}{\to} & P \\ \downarrow & & \downarrow & & \downarrow \\ U_{B'} & \hookrightarrow & B' & \to & B \end{array}$$

such that $P' \to P$ and $B' \to B$ are projective modifications, the left square is a toroidal morphism of nonsingular strict toroidal embeddings, $m^{-1}\Delta$ is a divisor of strict normal crossings contained in $P' \setminus U_{P'}$, and m is an isomorphism on $U_{P'}$.

We may again replace P, B by P', B', writing U_P, U_B instead of $U_{P'}, U_{B'}$, and further we may replace X, X_1, P_1, σ_i by their pullbacks to P', and Z by the union of its inverse image and the inverse image of $P' \setminus U_{P'}$. With the *pullback* to P' of a variety over P, we mean here the irreducible component of the base change to P' that dominates the given variety. After these replacements the properties (1), (2), (3), (4) and the equalities $X_1/G = X$, $P_1/G = P$, are still true. Moreover, both $\alpha^{-1}(Z) \setminus Z_1$ and the singular locus of g_1 lie over $P \setminus U_P$. Now, P and B are nonsingular and $P \setminus U_P$ is a strict normal crossings divisor on P. Moreover $P \to B$ is toroidal, and $P_1 \to P$ is unramified over U_P.

[13] (Jan) changed the loci - verify

Finally, we replace P_1 by its normalization and X_1, σ_i by their pullbacks to the normalization. By Lemma 3.4, since P_1 is normal, it inherits a toroidal structure given by $U_{P_1} = a^{-1}(U_P)$ as well, so that $P_1 \to P$ is a finite toroidal morphism. Moreover, $P_1/G = P$ because P is normal.

To summarize, in addition to properties (1)-(4) above, and the equality $P_1/G = P$, we also have that the morphisms $a : (U_{P_1} \subset P_1) \to (U_P \subset P)$ and $(U_P \subset P) \to (U_B \subset B)$ are toroidal. The embedding $U_{X_1} \subset X_1$, where $U_{X_1} = g^{-1}U_{P_1} \setminus \cup_i \sigma_i(P_1)$, is a toroidal embedding and the morphism $g_1 : (U_{X_1} \subset X_1) \to (U_{P_1} \subset P_1)$ is a G-equivariant toroidal morphism, by section 2.4 and (4). Note also that the divisor $\alpha^{-1}(Z)$ lies in $X_1 \setminus U_{X_1}$.

sec-torification

3.9. **Torifying the group action.** By Abhyankar's Lemma 3.4, G acts toroidally on $(U_{P_1} \subset P_1)$ and its stabilizers are abelian. If G acts toroidally on $(U_{X_1} \subset X_1)$ and if the morphism $g_1 : X_1 \to P_1$ is G-equivariantly toroidal, then the induced morphism $X_1/G \to P_1/G = P$ is toroidal, cf. section 2.3. Moreover, by resolution of singularities we find a nonsingular strict toroidal embedding $U_{X'} \subset X'$ and a toroidal modification $X' \to X_1/G$. [14] We then 14→ obtain a toroidal morphism $X' \to X_1/G \to P \to B =: B'$, as required by Theorem 1.1 (because the morphism $X_1/G \to X$ induced by α is a modification). However, in general, G does not act toroidally on X_1.

We follow section 1.4 of [1] to construct a suitable modification of X_1 on which G acts toroidally. In [1] this modification is obtained by two blowups, each followed by normalization. However, there one works over an algebraically closed field $\bar{k}$, thus we need to verify that the ideals blown up are actually defined over k, so that the modification is also defined over k. We recall the construction of the ideals to be blown up and explain why they are defined over k. For the convenience of the reader we also recall why these constructions yield a toroidal action, although this is all done in [1].

3.9.1. *Blowing up the singular locus.* A first situation where G does not act toroidally on X_1 happens when an element of G_x exchanges two components of a fiber $g_{1,\bar{k}} : X_{1,\bar{k}} \to P_{1,\bar{k}}$ passing through a point $x \in X_{1,\bar{k}}$. This problem is solved in [1] by blowing up the singular scheme S of the morphism $g_{1,\bar{k}}$, hence separating all nodes. Note that S is the subscheme of $X_{1,\bar{k}}$ defined by the first Fitting ideal sheaf of $g_{1,\bar{k}}$. This ideal sheaf is obtained from the first Fitting ideal sheaf of g_1 by base change. Thus S is defined over k. Let X_2 be the blowup of X_1 along S, and X_2^{nor} the normalization of X_2. The action of G on X_1 lifts to an action of G on X_2 and X_2^{nor}. Let U_{X_2} be the inverse image of U_{X_1} in X_2. Note that U_{X_2} is nonsingular because the morphism g_1 is smooth on U_{X_1}. We identify the inverse image of U_{X_1} in X_2^{nor} with U_{X_2}.

3.9.2. *Local description.* First we recall why $U_{X_2} \subset X_2^{\text{nor}}$ is a toroidal embedding. Let x be a closed point of $X_{2,\bar{k}}^{\text{nor}}$. Choose a local model (V, v) of $P_{1,\bar{k}}$ at the image of x in $P_{1,\bar{k}}$, compatible with the G-action as in section

[14](Dan) not necessary to say what has to happen with Z since this is what we woudl have done..

2.3, and let A be the completion of the local ring of the toric variety V at v. As can be seen from the local description (section 2.4) of the semistable family $X_{1,\bar{k}}$ over $P_{1,\bar{k}}$, there are only 3 possible cases for the completion of $X_{2,\bar{k}}$ at the image of x in $X_{2,\bar{k}}$, namely one of the following formal spectra:

$$\operatorname{Spf} A[[y,z]]/(zy^2 - h),\ \operatorname{Spf} A[[y,z]]/(y^2 - h), \text{ or } \operatorname{Spf} A[[z]],$$

where in the first two cases $h \in A$ is a monomial (i.e. a character of the big torus of V), and $h(x) = 0$. The third case holds if and only if the image of x in $X_{1,\bar{k}}$ is not in the singular locus of $g_{1,\bar{k}}$. Only in this case it is possible that x belongs to the inverse image Γ of $\cup_i \sigma_i(P_1)$ in $X_{2,\bar{k}}$, and then we may assume that $z = 0$ is a local equation for Γ at x. The first formal spectrum and the third one are completions of appropriate (not necessarily normal) equivariant torus embeddings. The same [15] holds for each component of the ←15 formal spectrum of the normalization of $A[[y]]/(y^2 - h)$. Hence $U_{X_2} \subset X_2^{\text{nor}}$ is a toroidal embedding. Note also that the divisor $X_2^{\text{nor}} \smallsetminus U_{X_2}$ is $G-$strict.

3.9.3. *Analyzing the group action.* In the first case, the ideal generated by y, as well as the ideal generated by z, is invariant under the action of G_x. Hence multiplying y and z by suitable units with residue 1, we may assume that G_x acts on y and z by characters of G_x. Indeed, replace y by $|G_x|^{-1} \sum_{\sigma \in G_x} (y/\sigma(y))(x)\, \sigma(y)$. Thus the action of G on $X_{2,\bar{k}}^{\text{nor}}$ is [16] toroidal ←16 at x.

In the third case, if $x \in \Gamma$ then a similar argument as in the first case shows that the action of G on $X_{2,\bar{k}}^{\text{nor}}$ is toroidal at x.

However, in the second and third case, if $x \notin \Gamma$ then the action on $U_{X_{2,\bar{k}}} \subset X_{2,\bar{k}}^{\text{nor}}$ is in general not toroidal at x, but because G_x is abelian, we can choose the local formal parameter z such that G_x acts on it by a character (indeed consider the representation of G_x on the vector space over $\bar{k}$ generated by the G-orbit of z). When this character is nontrivial, the action is not toroidal at x. Indeed, the divisor locally defined at x by $z = 0$ is not contained in the toroidal divisor $X_{2,\bar{k}}^{\text{nor}} \smallsetminus U_{X_{2,\bar{k}}}$. Moreover these locally defined divisors, as x varies, might not come from a globally defined divisor (because these are not defined in a canonical way).

3.9.4. *Pre-toroidal actions.* At any rate, the action of G on X_2^{nor} is pre-toroidal in the sense of Definition 1.4 of [1]. A faithful action of a finite group G on a toroidal embedding $U \subset X$ over $\bar{k}$ is called *pre-toroidal* if the divisor $X \smallsetminus U$ is G-strict and if for any[17] point x on X where the action is not ←17 toroidal we have the following. There exists an isomorphism ϵ, compatible with the G_x-action and the toroidal structure, from the completion of X at x to the completion of $X_0 \times \operatorname{Spec} \bar{k}[z]$ at $(x_0, 0)$, where $U_0 \subset X_0$ is a toroidal embedding with a toroidal G_x-action, x_0 is a point of X_0 fixed by G_x, the

[15] (Jan) will verify again

[16] (Jan) introduce this local notion in section 2.3

[17] (Dan) Closed?

toroidal structure on $X_0 \times \operatorname{Spec} \bar{k}[z]$ is given by $U_0 \times \operatorname{Spec} \bar{k}[z] \subset X_0 \times \operatorname{Spec} \bar{k}[z]$ and the action of G_x on $X_0 \times \operatorname{Spec} \bar{k}[z]$ comes from its action on X_0 and its action on z by a nontrivial character ψ_x of G_x. Note that the character ψ_x only depends on x and not on ϵ. Assume now that the G-action on $U \subset X$ is pre-toroidal.

3.9.5. *Blowing up the torific ideal.* In [1] (Theorem 1.7 and the proof of Proposition 1.8) it is proved that there exists a canonically defined G-equivariant ideal sheaf $\mathcal{I}$ on X, called the *torific ideal sheaf*, having the following properties. The support of $\mathcal{I}$ is contained in the closed subset of points of X where the G-action is not toroidal. For each closed point x of X, where the G-action is not toroidal, the completion of $\mathcal{I}$ at x is generated by the elements of $\widehat{\mathcal{O}}_{X,x}$ on which G_x acts by the character ψ_x. And finally, if we denote by $\widetilde{X}$ the normalization of the blowup of X along $\mathcal{I}$, and by $\widetilde{U}$ the inverse image of U in $\widetilde{X}$, then $\widetilde{U} \subset \widetilde{X}$ is a toroidal embedding on which G acts toroidally.

The proof of this last property follows directly from the following local description at x of the blowup, where we may assume that X_0 is an affine toric variety, U_0 is the big torus of X_0, G_x is a subgroup of U_0, G_x acts on X_0 through U_0, $X = X_0 \times \operatorname{Spec} \bar{k}[z]$, $U = U_0 \times \operatorname{Spec} \bar{k}[z]$, and ϵ is the identity. Locally at x the ideal sheaf $\mathcal{I}$ is generated by z and monomials $t_1, \cdots, t_m$ in the coordinate ring R of X_0. Indeed, at least one monomial of R is contained in $\mathcal{I}_x$, because G_x is a subgroup of the big torus of X_0. Above a small neighborhood of x, the blowup of X along $\mathcal{I}$ is covered by the charts

$$\operatorname{Spec} R[z, t_1/z, \cdots, t_m/z], \ \operatorname{Spec} R[t_i, t_1/t_i, \cdots, t_m/t_i][z/t_i],$$

for $i = 1, \cdots, m$. These are torus embeddings of $U_0 \times \operatorname{Spec} \bar{k}[z, z^{-1}]$, hence their normalizations are toric. Moreover the embedding $\widetilde{U} \subset \widetilde{X}$ is toroidal, and G_x acts toroidally on it, at any point of $\widetilde{X}$ above x, because on the first chart the locus of $z = 0$ is contained in the inverse image of $X \setminus U$, and on the other charts the action of G_x on z/t_i is trivial. The above argument also shows that the support of $\mathcal{I}$ is disjoint from U, because $\mathcal{I}_x$ contains a monomial in R. Thus the blowup is an isomorphism above U.

If the toroidal embedding $U \subset X$ and the G-action are defined over k, then the torific ideal sheaf $\mathcal{I}$ is also defined over k, because it is stable under the action of the Galois group of $\bar{k}$ over k. Indeed this is a direct consequence of the above description of the completions of $\mathcal{I}$. Hence $\widetilde{X}$ is also defined over k.

3.9.6. *Conclusion of the proof.* We now apply this to X_2^{nor}. Let X_3 be the blowup of X_2^{nor} along the torific ideal sheaf, X_3^{nor} the normalization of X_3, and U_{X_3} the inverse image of U_{X_2} in X_3. The blowup morphism $X_3 \to X_2^{\mathrm{nor}}$ is an isomorphism above U_{X_2}. Hence U_{X_3} is nonsingular, and we identify it with the inverse image of U_{X_2} in X_3^{nor}. Note that G acts toroidally on the toroidal embedding $U_{X_3} \subset X_3^{\mathrm{nor}}$. By the argument in the beginning of section 3.9, we now see that in order to prove Theorem 1.1 it suffices to show that the composition of the morphisms $X_3^{\mathrm{nor}} \to X_2^{\mathrm{nor}} \to X_1 \to P_1$ is G-equivariantly toroidal. But this is a straightforward consequence of the above given local descriptions of $X_{2,\bar{k}}$ and the blowup of X along $\mathcal{I}$, with $X = X_{2,\bar{k}}^{\mathrm{nor}}$. This finishes the proof of Theorem 1.1. □

References

AJ [1] D. Abramovich and A. J. de Jong. Smoothness, semistability, and toroidal geometry. *J. Algebraic Geom.*, 6(4):789–801, 1997.

Abramovich-Karu [2] D. Abramovich and K. Karu. Weak semistable reduction in characteristic 0. *Invent. Math.*, 139(2):241–273, 2000.

BM [3] E. Bierstone and P. D. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.*, 128(2):207–302, 1997.

deJong [4] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, (83):51–93, 1996.

deJong-families [5] A. J. de Jong. Families of curves and alterations. *Ann. Inst. Fourier (Grenoble)*, 47(2):599–621, 1997.

EV [6] S. Encinas and O. Villamayor. Good points and constructive resolution of singularities. *Acta Math.*, 181(1):109–158, 1998.

Fulton [7] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

EGA-II [8] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.

EGA-III [9] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

GM [10] A. Grothendieck and J. P. Murre. *The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme*. Lecture Notes in Mathematics, Vol. 208. Springer-Verlag, Berlin, 1971.

HartshorneAG [11] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

Hironaka [12] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2) 79 (1964), 109–203; ibid. (2)*, 79:205–326, 1964.

KKMS [13] G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin, 1973.

(Abramovich) Department of Mathematics, Brown University, Box 1917, Providence, RI 02912, U.S.A.
E-mail address: abrmovic@math.brown.edu

(Denef) University of Leuven, Department of Mathematics, Celestijnenlaan 200 B, B-3001 Leuven (Heverlee), Belgium.
E-mail address: jan.denef@wis.kuleuven.ac.be

(Karu) Dept. of Mathematics University of British Columbia 1984 Mathematics Rd Vancouver, B.C. V6T 1Z2
E-mail address: karu@math.ubc.ca