Algebraic geometry minus fields

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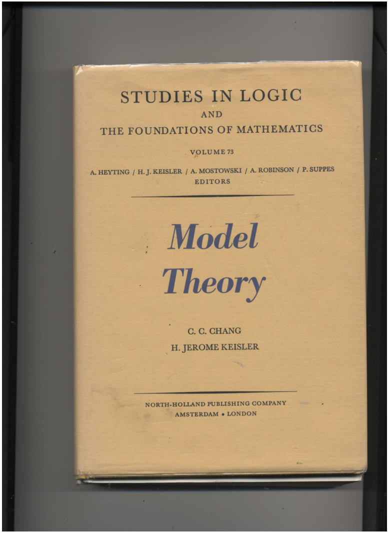
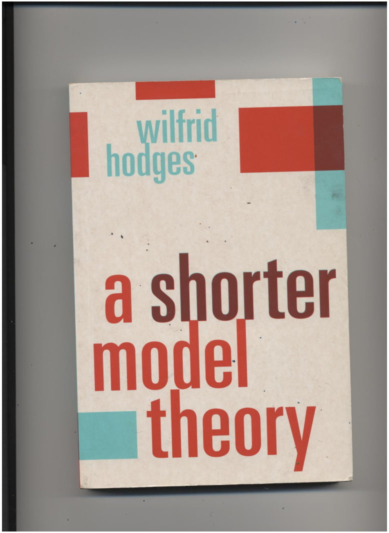
What is model theory?



1.1: What is model theory?
Model theory is the branch of mathematical logic which deals with the relation between a formal language and its interpretations, or models. . . . The line between universal algebra and model theory is sometimes fuzzy; our own usage is explained by the equation

$$\textit{universal algebra} + \textit{logic} = \textit{model theory}.$$

And what is model theory?



And what is model theory? Model theory is about the classification of mathematical structures, maps and sets by means of logical formulas. One can classify structures according to what logical sentences are true in them; in fact the term 'model' comes from the usage 'structure A is a model of sentence ϕ,' meaning that ϕ is true in A.

And what is model theory?

In 1973 C. C. Chang and Jerry Keisler characterized model theory as

universal algebra plus logic.

They meant the universal algebra to stand for structures and the logic to stand for logical formulas. This is neat, but it might suggest that model theorists and universal algebraists have closely related interests, which is debatable. Also it leaves out the fact that model theorists study the sets definable in a single structure by a logical formula. In this respect model theorists are much closer to algebraic geometers, who study the sets of points definable by equations over a field. A more up-to-date slogan might be that model theory is

algebraic geometry minus fields.

Algebraic geometry minus (?) fields

In fact some of the most striking successes of model theory have been theorems about the existence of solutions of equations over fields. Examples are the work of Ax, Kochen and Ershov on Artin's conjecture in 1965, and the proof of the Mordell-Lang conjecture for function fields by Hrushovski in 1993. -W. Hodges

Algebraically closed fields and model theory

- Steinitz's 1910 theorem showing that the isomorphism class of an algebraically closed field is determined by its characteristic and transcendence degree motivated the development of classification/stability theory.
- Tarski in 1929 gave a decision procedures and quantifier elimination theorems for the theory of the real and complex numbers.
- Algebraically closed fields are sometimes recoverable from an abstract hypothesis of "complicated geometry" (e.g. the Hrushovski-Zilber theorem on Zariski geometries).

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Theorem

Let X be an algebraic variety over an algebraically closed field K and $\Xi \subseteq X(K)$ a set of points. Suppose that for every n that if $Y \subseteq X^n$ and $Z \subseteq X^n$ are two subvarieties meeting Ξ^n in Zariski dense sets, then so does $Y \cap Z$. Then for every family $\{Y_b\}_{b \in B}$ of subvarieties of X, the Zariski closures $\{\overline{Y_b(K) \cap \Xi}\}_{b \in B}$ fit into a constructible family of varieties.

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The hypotheses of the theorem apply immediately to the special varieties of the Manin-Mumford and André-Oort conjectures, while a slight strengthening is required for the Mordell-Lang conjecture.

Manin-Mumford, again (and again)

Theorem (Manin-Mumford conjecture, Raynaud)

If A is an abelian variety over $\mathbb{C}$ and $X \subseteq A$ is a closed subvariety and we denote by $A(\mathbb{C})_{\mathrm{tor}} := \{x \in A(\mathbb{C}) : (\exists n \in \mathbb{Z}_+)[n]_A(x) = 0\}$, then $X(\mathbb{C}) \cap A(\mathbb{C})_{\mathrm{tor}}$ is a finite union of cosets of subgroups of $A(\mathbb{C})_{\mathrm{tor}}$.

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The Manin-Mumford conjecture has many proofs (including a model theoretic version due to Hrushovski, later explained more geometrically by Pink and Rössler, and of which I will say something later today). I wish to describe a new proof (due to Pila and Zannier) based on the model theory of real analytic geometry.

o-minimality

- Tarski proved that every set definable in $(\mathbb{R}, +, \cdot, <, 0, 1)$ is defined by quantifier-free formulas.
- A one-variable quantifier-free formula is equivalent to a finite Boolean combination for formulas of the form $f(x) = 0$ or $g(x) > 0$ for nonzero polynomials f and g. The sets so defined are finite and finite unions of open intervals, respectively.
- An ordered structure $(M, <, \ldots)$ is o-minimal if every definable subset of M is a finite union of points and open intervals.
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Real analytic o-minimality

- Wilkie showed in 1991 that $(\mathbb{R}, +, \cdot, <, 0, 1, \exp)$ is o-minimal.
- For us, the most important result is a 1994 theorem of Macintyre, Marker and van den Dries that the expansion of the real field by the exponential function and the restrictions of real analytic functions to boxes of the form $[-1, 1]^n$ is o-minimal.
- In particular, such functions as the Weierstrass $\wp$-function and the j-function restricted to bounded regions are definable in an o-minimal structure.

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Counting rational points on definable sets

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Theorem (Pila-Wilkie, 2006)

If $X \subseteq \mathbb{R}^n$ is definable in some o-minimal expansion of $\mathbb{R}$ and $\epsilon > 0$ is any positive real number, then there is a constant $C = C(X, \epsilon)$ such that for all $t \geq 1$,

$$\#\{(q_1, \ldots, q_n) \in \mathbb{Q}^n \cap (X \smallsetminus X^{alg}) : \max\{H(q_i)\} \leq t\} \leq Ct^\epsilon$$

(where we write $H(a/b) := \max\{|a|, |b|\}$ for a and b coprime integers).

Pila-Zannier proof of Manin-Mumford

- Let A be a complex abelian variety and $\pi : \mathbb{C}^g \to A(\mathbb{C})$ the covering map expressing $A(\mathbb{C})$ as a complex torus. Suitably restricted, π is definable in an o-minimal expansion of $\mathbb{R}$.
- $A(\mathbb{C})_{\mathrm{tor}}$ is the image of π on rational points.
- If $X \subseteq A$ is an algebraic subvariety, then $\pi^{-1}X$ is definable and its algebraic part comes from groups.
- By the Pila-Wilkie theorem, there are few rational points on $\pi^{-1}X$ of height $\leq t$ (and, hence, few torsion points on X of order $\leq t$), but by a Galois theoretic result of Zannier, the size of the set of conjugates over the field of definition of X of a torsion point of high order must be large.

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Difference fields

Hrushovski's earlier model theoretic proof of the Manin-Mumford conjecture (as well as some other work on p-adic versions of Pink's strengthening of the André-Oort conjecture) used the model theory of difference fields. In more recent work, the model theory of difference fields has been applied to algebraic dynamics.

- A **difference field** is a field K given together with a distinguished endomorphism $\sigma : K \to K$.
- By an algebraic dynamical system over the field K we mean a pair (X, f) where $f : X \to X$ is a map of algebraic varieties over K. Usually, f is assumed to be dominant. By a σ-variety over the difference field (K, σ) we mean a map $f : X \to X^\sigma$ over K where X^σ is the σ-transform of the variety X.
- Assuming that σ is trivial on the algebraic closure of the field of definition of $f : X \to X$, we may regard an AD as a σ-variety.

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Difference closed fields

Every difference field may be embedded into a difference closed field, an algebraically closed field for which every consistent difference equation in any finite number of variables has a solution. In particular, of $f : X \to X^\sigma$ is a σ-variety with f dominant over the difference closed field (K, σ), then the set $(X, f)^\sharp := \{a \in X(K) : f(a) = \sigma(a)\}$ is Zariski dense in X. Deep structural results have been proven about sets definable in difference closed fields many of which yield information about ADs via the (X, f) to $(X, f)^\sharp$ correspondence.

Some theorems on algebraic dynamics: periodic points over finite fields

Theorem (Poonen (exposed by Fakhruddin); using Hrushovski's theorem on the limit theory of the Frobenius)

If $f : X \to X$ is a dominant self-map of an algebraic variety defined over $\mathbb{F}_p^{alg}$, then the set of f-periodic points in $X(\mathbb{F}_p^{alg})$ is Zariski dense in X.

Some theorems on algebraic dynamics: descent, a conjecture of Szpiro

Theorem (M. Baker; model theoretic proof (and strong generalization) by Chatzidakis and Hrushovski)

If $f : \mathbb{P}^1_{\mathbb{C}(t)} \to \mathbb{P}^1_{\mathbb{C}(t)}$ is a rational function of degree at least two defined over $\mathbb{C}(t)$ for which there is some non-preperiodic point $P \in \mathbb{P}^1(\mathbb{C}(t))$ with $0 = \widehat{h}_f(P) := \lim h(f^{\circ n}(P))/\deg(f)^n$, then, f is conjugate to a rational function over $\mathbb{C}$.

Some theorems on algebraic dynamics: dynamical Manin-Mumford

Theorem (A. Medvedev and S.; consequence of a much more general classification of invariant varieties for polynomial actions)

If p is a prime number, $g(x) \in \mathbb{Z}[x]$ is a polynomial of degree at most p, and $f(x) := x^p + pg(x)$ is not linearly conjugate to a monomial or a Chebyshev polynomial, then every irreducible variety $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ which contains a Zariski dense set of tuples of f-periodic points is defined by equations of the form $f^{\circ m}(x_i) = x_j$ and $x_k = \xi$ where ξ is an f-periodic point.

Beyond?

- Transfer and quantifier elimination theorems for valued fields ground the theory of motivic integration and are behind many rationality results for Poincaré series. The most impressive recent development is the Cluckers-Hales-Loeser proof of an asymptotic transfer theorem for the fundamental lemma.
- The deep structural theorems on differential algebraic groups used in Hrushovski's proof of the function field Mordell-Lang theorem have been extended to p-adic differential analytic groups and have been used to give uniform versions of the Manin-Mumford conjecture for p-adic families of abelian varieties.

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