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Flatland: A New Introduction

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Flatland first appeared over one hundred years ago and a dozen different editions have been published since then. Why now put out a new edition of this book in the Princeton Science Library? For each generation of readers, Edwin Abbott Abbott's classic story of encounter between beings from different dimensions has had different significance. In 1884, the social satire on the limited perspective of Victorian England was as important as the comments on the use of analogy in treating higher dimensions, and both of these elements were clarified by the introduction to the second edition, purportedly written by an editor but actually written by Abbott himself. When the book was reissued in 1926, the main stimulus for considering higher dimensions was relativity theory, and a new introduction was written by William Garnett, a physicist student of Abbott's. In 1952, when the first modern edition appeared, it was again a physicist, Banesh Hoffman, who wrote the introduction, referring to the connection between the dimensional analogy and the curvature of space. Other recent introductions have come from science-fiction and fantasy writers, a computer scientist, and a social historian.

In our own day, there are new reasons to reconsider this book and its fundamental ideas. Abbott challenged his readers to imagine trying to understand the nature of phenomena in higher dimensions if all they could see directly were lowere-dimensional slices. That is precisely the situation that radiologists face today as they analyze the slides produced by CAT scans or magnetic resonance imaging, in attempts to reconstruct the forms of objects in space by studying their planar cross-sections. But Abbott did not want his readers to stop at two-dimensional representations of the third dimension: What would happen if we were to apply the same techniques one dimension higher, and attempt to conceive objects from a spacial fourth dimension by considering three-dimensional cross-sections? For mathematicians in Abbott's day, that challenge was daunting. Their methods were adequate for visualizing surfaces in three-dimensional space, but it was only with great difficulty that these techniques could be applied to objects from higher dimensions. Frequently, researchers relied more and more on formal and abstract methods, leaving images behind.

Now, however there is a new interest in visualization, primarily because of the emergence of modern computer graphics, which can literally bring these higher dimensions into view. We can manipulate objects in four dimensions and see their three-dimensional slices tumbling on the computer screen. But how do we interpret these images? There is no better introduction to the problem of dealing with higher-dimensional slicing phenomena than this book, Flatland.

It is not necessary to read an introduction to appreciate Flatland, and at this point the reader might be encouraged to go ahead and read the book, returning to the introduction only later on to see if some of the questions that arise are answered here. in this short commentary, we shall address the satirical elements in Flatland, sketch Abbott's background and interests, provide overviews of the style and historical context of the work, and investigate Flatlands influence on contemporary approaches to new modes of visualization of higher dimensions, particularly computer graphics.

Social Satire in Flatland

When Edwin Abbott Abbott wrote his little masterpiece over one hundred years ago, he did it for several reasons. Some of these were obvious to his readers and remain obvious today. Others had to be explained by Abbott himself in the introduction to the second edition, which followed one month after the first, at the end of 1884. Still others were immediately clear to Victorian readers but need some clarification for readers in our day. Fortunately, there is enough evidence, direct and indirect, in this remarkable author's forty-five books and other writings to give us answers to many questions, though not all, since the subject itself continues to encourage us to raise new ones.

The first subject that has to be addressed is the treatment of women in Flatland. Abbott was a social reformer who criticized a great many aspects of the limitations of Victorian society. He was a firm believer in equality of educational opportunity, across social classes and in particular for women. He participated actively in the efforts to bring about changes, and the frustration he felt from the resistance of the educational establishment is mirrored in the satire of Flatland. This was the first generation in which women were permitted to attend classes at Oxford and Cambridge, but their access was still quite limited. Although there were many schools where a boy could be trained for the demanding university entrance examinations, there were few comparable opportunities for girls, and many of the young women who gained entrance to universities, like Abbott's daughter, had received much of their education at home, often from private tutors. It was partially to aid in this effort that Abbott composed his Hints for Home Teaching directed at parents who wished to help their children prepare for higher education. Abbott was also a vocal leader in the Teachers' Training Syndicate, formed and primarily supported by the major female educators of Victorian England, who extensively praised Abbott for his efforts on behalf of education reform, in particular for proposing alternate ways of qualifying for entrance into university studies. The narrow-minded attitudes that blocked these efforts show up quite clearly in Flatland, where females are presented as incapable of comprehending the education given to males. Many people ignored one whole dimension of women's existence, as symbolized by the representation of women in Flatland as straight line segments. Under the guise of protecting women, they kept them away from the means by which they might better their station. Abbott's sentiments in this matter are clear by the end of Flatland, where the narrator comes to realize that the very (female) virtues his society has been putting down are the ones that are to be most prized.

Abbot was one of the first to recognize the implications of a "two cultures" society. The men in Flatland epitomize the rational, emphasizing the importance of that which can be measured empirically and described in precise scientific language. Qualitative properties not susceptible to such quantification are relegated to the world of women, who have an absolute corner on not only intuitive aspects of knowledge but also the abstract concepts such as loyalty and love which are difficult to translate into a strictly utilitarian construct. A Square, the two-dimensional narrator of Flatland, considers himself enlightened when he propounds the view that the strain of maintaining two separate languages, one for conversation among men and one for talking with women, exacts too great a toll on young minds. There is always the danger that the language of men will be revealed to women, reminiscent of the prophibitions in earlier societies against teaching slaves to read.

As a religious man with a well-developed traditional sense of morality, Abbott clearly did not subscribe to the prevailing scientistic view of knowledge, and he more than once pointed out the dangers of letting one side of the personality completely dominate the other. Abbott was a teacher who extolled balance, and Flatland reduces to an absurdity the single-minded tendency of choosing either the totally rationalistic or the totally intuitive.

Another aspect of the satire in Flatland is the treatment of those who did not fit in. In the rigid society of Victorian England, there was little tolerance for irregularity. It was often associated with criminal tendency, and some theories blamed deviant behavior on an abnormal shape in the frame of the skull. Frequently, the unusual were segregated from the rest of society in asylums. The rest of society maintained a fascination with the freakish element, and asylums often had viewing galleries so ordinary people could observe the activities and antics of the inmates. Abbott's suggestion that irregulars be eliminated in a Swiftian exaggeration, especially when coupled with the cruel plan of keeping a number of these unfortunates available as object lessons, an expendable supply of individuals with no rights at all, to be studied by the regular Flatland pupils. Especially pointed are his remarks about the appearance of irregularities among the upper classes. His readers could certainly supply their own examples of men destined for high station who had failed to fulfill the prerequisites for completing a university education. Such people are not fit for lower employment, and so they simply cause trouble within society. Abbott's solution is worthy of Jonathan Swift's Modest Proposal: anyone who fails the final examinations of the university will be incarcerated for life or subjected to a painless death. No reader should miss the satirical intent of such a paragraph.

Abbott definitely saw education as a means for students to transcend the social class into which they were born, and he was particularly censorious of the efforts of those who used education to perpetuate class distinctions. A basic liberal education should be provided to all students, so that some of them might go on to higher education and others might go into business from a more enlightened perspective. Abbott's students at the City of London School learned practical science and art as well as theoretical subjects. Abbott resigned from the headmastership just seven years after the writing of Flatland, during a crisis over the splitting of the curriculum into separate "modern" and classical sides. In Flatland, Abbott echoes this crisis as he satirically contrasts the "feeling" of the lower classes with the more refined "seeing" of the educated part of society. By preferring the more remote way of sensing, the higher classes built a barrier between themselves and the lower strata.

Several of Abbott's students who broke societal barriers were only too happy to thank their old master. Most famous was the Prime Minister of England, H. H. Asquith, Lord Oxford, a man from humble background who gained entrance to Oxford as a result of his classical education under Abbott. Bramwell Booth, the second director general of the Salvation Army, also thanked Abbott for his encouragement and for promoting the development of his self-worth. Sir Israel Gollancz went on from the City of London School to study at the University of London at a time when educational opportunities for Jewish students were meager. He always sent his Shakespearian volumes to his old teacher, and induced him to accept membership in the British Academy, which Gollancz had helped to found. It is evident that Abbott was proud of his former pupils and that he preferred the kind of education that allowed people to rise on the basis of merit rather than of the social class into which they were born. The prevailing system is the target of his satire in Flatland.

Abbott's Background and Interests

What sort of person was the writer of Flatland? Clearly Abbott was impressed by the power of language, and Flatland is about language at several levels. Throughout the book there are overt and hidden references to Abbott's favorite author, Shakespeare. It is unlikely that his readers would recognize many of the allusions, since although Victorians did attend Shakespearian plays, they did not study them in the schools, except at the City of London School. There a tradition of prize examinations based on Shakespeare led the bright and highly competitive students to memorize large portions of plays, and to apply to Elizabethan English some of the linguistic analysis that was common with scholarship in classical languages. Abbott wrote his first major book, *A Shakespearean Grammar*, in 1870, and several new editions appeared in his lifetime, even though he lamented as late as the turn of the century that headmasters throughout England were unwilling to turn over to the study of English literature even a small portion of the time allotted to Latin and Greek. Abbott himself writes that in preparing a new edition of his work on Shakespeare, he reread all of the plays for a third time. It was a close reading, too. Abbott had been trained by his father, Edwin Abbott, headmaster of the Philological School in Marylebone, London, and compiler of the concordance of the works of Pope that is till the major reference for that subject. Abbott aided his father in this enterprise and wrote the introduction to his father's life's work.

Abbott was somewhat of an intellectual radical, going so far as to suggest that the techniques of literary criticism developed by classical scholars should be used with the same vigor to examine the Scriptures. In his controversial article on the Gospels for the ninth edition of the *Encyclopaedia Britannica*, Abbott suggests linguistic tests for the dating and the authorship of the Gospels and Epistles. He refers to the "method of curves" as a way of analyzing style and structure, and he laments that the work involved is too much for a single person, perhaps anticipating our present day when computer searches routinely carry out the kinds of investigation Abbott could only dream of. Such suggestions were naturally threatening to those who wished to base their faith on less scientific grounds.

Part of the rigidity of the educational system was due to the central place played by examinations, first for gaining entrance to the universities, and secondly for determining the sort of degree that would be awarded. Abbott himself did quite well under this system. By the time he had graduated from the City of London School, he had won prizes in just about all of the subjects in the curriculum. He had delivered prize declamations in English, French, German, Latin, and Greek, and he had even won a prize in mathematics. At Cambridge he was first in classics and Chancellor's Medallist, the same year that the top mathematics prizes went to another graduate of the City of London School, Thomas Steadman Aldis, who was First Wrangler and Smith's Prizeman. It was unusual for all four top prizes to go to boys from the same school, and unheard of that that school should be a day school, not one of the English public schools that were the traditional leaders in preparatory education.

The Style of Flatland

The narrative style of *Flatland* is somewhat different from some of the more familiar reports of visits to exotic lands, since the story is told not by the visitor but by the person visited. It is as if the story of Gulliver were told by the mayor of Lilliput or the adventures of Alice by the White Rabbit. It is only in the latter half of the book that the narrator A Square can relate the changes that took place in his own perceptions as a result of the remarkable events caused by the visit of a being from a higher dimension. The reader has to stay with the story all the way through in order to appreciate the change that takes place in the storyteller. A similar thing happens with the narrator of another book written in exactly the same year, Huckleberry Finn. In the beginning of that book, the narrator shares the prejudices of his society with respect to slavery, and by the end of his journey down the river, his experiences have caused him to reject those views. Even the relatively enlightened ideas that the author expresses at the end of the book may seem inadequate in our present age, and they might not represent accurately the views of the author. Still, the novel itself is valuable for the tale it tells about the development of sensibility in a strange environment. That is what we have in *Flatland* as well. We have not erased racial prejudice from our world, nor do we treat women equally or provide for the needs of the handicapped. The social lessons of *Flatland* are still being learned.

The Historical Context of Flatland

Much of what Abbott wrote in *Flatland* came about in response to the mathematical and philosophical concerns of his day, when geometers attempted to visualize phenomena in the fourth and higher dimensions. They drew pictures and made models, and even attempted to use stereoscopic images in order to see what projections and slices of four-dimensional objects look like. They were often frustrated by their inability to see more than one incomplete image at a time. All that has changed with the advent of computer technology. We cannot answer all the questions raised by the investigators in the nineteenth century, but we can reconsider them in new ways. Abbott's two-dimensional narrator led his readers through a good part of the exercises any citizen of Flatland would go through in perception training, refining the sense of touch and sight so as to come to terms with all reasonable shapes. Similarly, we in three-space learn to interpret the sense data that come to use, gradually coming to the point where we can resolve ambiguities and draw correct inferences from the views presented to us. We can refine our own abilities to model and draw so as to construct the most effective images to communicate accurately about them. As the concepts become more and more complicated, our limitations become tested, and ultimately we experience frustration. This is especially true when an object or a configuration resists representation on the visual plane that is our most trusted means of receiving information. The problem may have to do with the intrinsic complexity of something in our own space, or it might be due to an entirely different reason, because some phenomenon comes from a space beyond our own, from a higher dimension. That is the analogue of the problem faced by A Square in *Flatland*.

Abbott was not the first person to challenge his readers with the enigmas of perceptions in different dimensions. Plato had already used a powerful image that crossed dimensional boundaries when he posited a race of men whose only sense impressions came from the shadows they could see on the wall of a cave. How little of the solid reality of the third dimension would such limited creatures be able to appreciate, and how little of the true nature of transcendent reality do we comprehend when all we can see are the solid shadows of ideal objects in the mind of God. In the nineteenth century in Germany and England, several writers seized on the idea of imagining the experience of beings confined to two dimensions. Gustav Fechner made up the story of shadow man cast on a wall by a slide projector. Karl Friedrich Gauss and Hermann von Helmholz both investigated the kind of geometry that would be discovered by an intelligent flatworm sliding about on the surface of a solid object. Helmholz's ideas were well represented in British intellectual journals, which ran debates about the nature of higher dimensions and our ability to comprehend them. Charles Howard Hinton examined the physical properties of the life of flat creatures on a two-dimensional surface in the early 1880's, and it is likely that Abbott could have seen one or more of his articles before he wrote *Flatland*. Abbott's best friend, Howard Candler, the "H.C" to whom *Flatland* is dedicated, was mathematics master at the Uppingham School, were Hinton spent several years as science master. Hinton wrote that his purpose was to examine scientific principles, while the purpose of Abbott's book was not science but rather "his philosophy and his lessons."

Abbott was one of the first writers to realize the full power of the dimensional analogy for investigating contact among beings from different dimensions. In his later theological works, he developed this idea further as an analogy for the all-seeing power of God, who makes revelations to a lower world and interacts with creatures there. Abbott was also one of the first to exploit the notion of slicing as a means of gaining information about solid objects by studying planar cross-sections. He was familiar with the process of calculating the volume of a solid ball by keeping track of the areas of circular slices, and he probed the psychological process of trying to appreciate a from by keeping track of its different cross-sections. It is difficult for A Square to comprehend the nature of a sphere from the third dimension. It would be even more difficult for him to comprehend the surface of a doughnut.

The challenge for use is apparent. Just as a sphere penetrating Flatland is viewed by A Square as a circle growing and then shrinking in time, so also if we were visited by a hypersphere from a space of four dimensions, we might see a sphere growing and then shrinking in time. The ability to treat such a sequence of impressions as the gradual revelation of an entity from a higher dimension is the first exercise for anyone who wishes to accept the challenge of *Flatland*. More difficult is the exercise implied in the book concerning the analogue of a square in three and higher dimensions. In the text, the number of corners of analogues of cubes in different dimensions can be suggested by arguments based on the characteristics of various formulas. It is not so easy to imagine the ways in which such cubes and hypercubes appear when they are put through a slicing place or hyperplane at different angles.

Flatland and New Modes of Visualization

Today the major reason for our interest in *Flatland* is that for the first time we can achieve some of the dreams of our ancestors a century ago and obtain direct visual experience of phenomena in a dimension higher than our own. The breakthrough that brings this about is modern computer graphics. Images moving on a computer screen can give us a view of complicated two-dimensional phenomena like networks and patters, and they can also enable us to investigate complex objects in three dimensions, like the plans of a building or a machine part. As we watch, the turn of a dial makes the image spin around on the screen, revealing new aspects of the structure and new relationships as different parts of the object come into view.

The building and the machine part are three-dimensional, with every point specified by three coordinates. Taking those coordinates two at a time gives us the different elevations, front, side, and top views. Computers can calculate quickly enough to display mixed views from any desired angle. By showing a sequence of views, thirty views every second, each slightly different from the previous one, the computer can create an animated film that reveals the structure of a three-dimensional object much better than any single picture.

If such animations are valuable for studying three-dimensional objects, they are ever more crucial in giving us insight into phenomena in four or more dimensions. The computer techniques from architecture and machine design help us to approach collections where each point is specified by four or more coordinates. Again, we show the coordinates two or three at a time, and use our imaginations to gain an appreciateion of the overall relationships so we can more effectively predict what will come next.

Dramatic evidence of the power of such techniques can be found in the field of radiology, where CAT scans and magnetic resonance imagine has produced data sets that can monitor the development of a tumor over the course of weeks, with all the information available to be investigated on a computer graphics screen. Precisely the same kinds of techniques enable geologists to study the data on global warming trends over tens of thousands of years, or the economics of a large urban area, with each ward contributing numerical observations in a dozen different categories. The use of computer graphics in statistics has changed that subject in dramatic ways, especially in the area of exploratory data analysis.

Time and the Fourth Dimension

"The fourth dimension? That's time, isn't it?" It isn't hard to imagine A Square's rejoinder to such a question. He would turn the question back on the poser and remind him that in Flatland, the comparable response to the introduction to the concept of the third dimension is, "The third dimension? That's time, isn't it?" For beings whose universe has a spatial fourth dimension, the fifth dimension would be the one assigned to time, and in general we can expect that time will come in right after the dimensions that are reserved for space. For some modern physicists, there are ten dimensions which act like space, and the eleventh is time. The trouble is that when we treat time as a dimension, we do not expect it to act like the homogeneous interchangeable dimensions of space. There is a perfectly fine mathematics of space-time, but it is not the one that

Abbott is pointing to in *Flatland.* There we are asked to contemplate a fourth spatial dimension equivalent in all ways to the first three. As it happens, in order to introduce the geometry of space-time, almost all physics writers lower the dimension to that of one or two dimensions of space to go with one dimension of time, thereby reducing the total number of dimensions to two or three, allowing easier visualization. Relativity in Flatland helps us to understand relativistic effects in our own universe.

Flatland was written a full generation before Albert Einstein developed his theories of relativity, and most of the people who thought about dimensions in Abbott's day were concerned with the nature of space rather than space-time. (A good reference for the place of geometry in Victorian England is the recent book *Mathematical Visions* by Joan Richards.) There were several writers who recognized that time could be treated as a dimension similar to the other three, but with certain differences. Hermann Minkowski had already worked out some of the consequences of measuring space in a way that treated one direction differently from the others, in the same way that time measurement is different from the measurement of space. Edmond Laguerre in France had developed a geometry of spheres that was four-dimensional in that each sphere could be specified by three dimensions for the center and one more dimension for the radius. But to tell whether or not two spheres intersected, it was necessary to use formulas in which the radius coordinate behaved like time rather than space. Another important geometry involving four variables was the geometry of lines in space. We can specify all non-horizontal lines by indicating the points where the line intersects a pair of horizontal planes, describing each of these points by two coordinates. Such four-dimensional geometries were well understood by mathematicians as abstract constructs, but resistance remained to the concept of a physical fourth dimension equivalent in every way to the accepted three dimensions. It is the challenge of visualizing such a homogeneous four-dimensional space that formed the basis of *Flatland.*

Higher Dimensions

It is important to note that Flatland is not specifically about the fourth dimension. One of the problems of thinking of the fourth dimension as time is that it tends to stop things at that stage. One is not likely to go on to a fifth dimension in order to find a fourth dimesion of space (although that is precisely what happens in Madeleine L'Engle's book A Wrinkle in Time, where the author takes space-time for granted and has here powerful beings taking shortcuts through a five-dimensional void). In Flatland, there is to be no limit on the number of dimensions, and indeed the point of the story is that the Sphere, at first an omniscient observer, is revealed as subject to his own dimensional prejudices, unable to conceive of a dimension beyond the third, as a four-dimensional being might begin by doubting the reality of anything of dimension five or higher. All of us are slaves to the prejudices of our own dimension. Even the frontispiece of the book invites readers to contemplate words of up to ten dimensions in the mammoth cloud around the perspective drawing of the name of the book.

Curiously enough, it is a universe of ten spatial dimensions which has recently gained quite a bit of popularity in quantum physics. There is little attempt to visualize so many dimensions at once, but the framework of higher dimensions still provides a way of keeping track of a large number of variables, some so small that they can pass unnoticed when we concentrate on the usual three dimensions, even more beyond our powers of visualization. Even though the eventual aim of the physicists is to deal with spaces of these high dimensions, any popular exposition of the subject begins by describing analogues in two, three, and four dimensions, more often than not referring to the analogies presented in *Flatland*.

Another area in which higher dimensions arise in physics is in the curvature of space. In almost any treatment of this concept, either *Flatland* is cited directly, or the writer or speaker will refer to *Sphereland*, the sequel written in 1965 by the Dutch physics teacher Dionys Burger. In that book, descendants of A Square set out on long journeys and discover that their world is not flat after all, but curved into the surface of a large sphere. In the same way we can image space explorers travelling so far that they determine that our own supposedly flat three-dimensional universe is in reality finite like the surface of a sphere. At such a moment we may be sure that the Flatland analogy will be invoked to help people come to terms with this new worldview. Although strictly speaking the concept of curvature of space does not necessarily involve a higher dimensional space that our space is curved into, this is undoubtedly the most natural way to begin thinking about this idea. A more thorough treatment of these concepts is found in *The Shape of Space*, by Jeff Weeks, which uses Flatland as a starting point and describes two-dimensional worlds of varying topological complexity, shaped like doughnuts or twisted Möbius bands or Klein bottles. Understanding such two-dimensional examples is crucial for appreciating the current developments in the shape of three- and four-dimensional space. For further reading on these and other topics using the dimensional analogy, see *The Fourth Dimension* and *Geometry, Relativity, and the Fourth Dimension* by Rudolf Rucker, *The Planiverse* by A. K. Dewdney and my own volume, *Beyond the Third Dimension.*

Other scientists who have profited by recent developments in visualization of data sets in high dimensions are the statisticians, especially those engaged in the field of exploratory data analysis. Researchers quite often have to analyze a mass of data from a sociological survey or a set of physical or biological experiments. The numbers of parameters in such investigations can be very large indeed, with a single observation leading to a string of a dozen or more numbers. Searching for patterns in such data collections can reveal hidden relationships, allowing scientists to predict the size of certain variables once others are known, and suggesting trends that can lead to predictions of short- or long-range behavior. A recent book on data visualization by Edward Tufte, *Envisioning Information*, has as its first chapter "Escaping from Flatland," and at the centenary conference on *Flatland* at Brown University in 1984, John and Paul Tukey introduced a full range of techniques in exploratory data analysis by starting with the world of data visualization accessible to A Square. Software for computers of a moderate size now enables researchers to manipulate multidimensional data sets with great freedom, projecting point clouds into three or four dimensions and then analyzing the patters on a graphics screen as the collection of points is rotated in space, or as the data set is sliced to reveal the points lying in a thin slab. As such a test slab proceeds through the data set, we gain important information about the distribution of points in the whole collection. The slicing technique from *Flatland* still remains one of the most powerful tools for dealing with aggregates in higher dimensions.

The technique of slicing is also at work when a cartographer produces a contour map of a mountainous region, shading in the points that correspond to the region between two different levels. This technique does not have to stop with the analysis of three-dimensional data using two-dimensional slices since it works as well in higher dimensions. If each point in space is given a fourth coordinate indicating its temperature, then by indicating points with temperature between two given values, we obtain an analogue of a countour map in more than three dimensions. These same approaches work to help mathematicians analyze the structure of abstract functions of three or more variables, which in turn can serve as models for the analysis of more and more complicated physical phenomena. Although such methods have been theoretically possible for many years, it is only in recent times that computer have become powerful enough to carry out the procedures effectively. now even relatively small computers produce contour maps, and it will not be long before it will be easy to apply the same approach to the direct study of objects in higher dimensions. The dream of mathematicians of the last century, used so effectively by Abbott in his treatment of the ways we comprehend and the ways we communicate, will finally be within our grasp. We do not know all of the ways that this new way of seeing will change our approach to geometry and its applications, but one thing is sure–anyone who wants to make good use of these developments in visualization will being by reading the book you now hold, *Flatland*.