The 9-Vertex Complex Projective Plane

W. Kühnel and T. F. Banchoff

In the early days of topology, most of the objects of interest were defined in terms of *triangulations*, describing a topological space as a union of finitely many vertices, edges, triangles, and higher dimensional simplexes identified in certain ways along their boundaries. A triangulation with a relatively small number of simplexes, symmetrically placed, could make computations easier and suggest new properties of the object itself. Although subsequent approaches to algebraic topology have stressed other ways of defining properties of topological spaces, the discovery of a new particularly nice triangulation of an important space can once again bring out relationships that lead to new insights in different branches of mathematics. In this article we describe such a triangulation for one of the most significant objects in topology, the complex projective plane.

In a triangulation each edge has two distinct vertices and no two vertices determine more than one edge so the minimum number of vertices in a triangulation of a circle is three. In a triangulation of a surface or surface-with-boundary, each triangle has three distinct vertices and no three vertices determine more than one triangle. Moreover, at most two triangles come together at every edge. Since a cylinder has two edge curves a minimal triangulation requires at least six vertices and we can present a triangulation with precisely six vertices. On the other hand, it is possible to give a triangulation of the Möbius band with only five vertices (Figure 1, on next page) so in a sense the twist in the band makes it possible to triangulate using fewer vertices than in the case of the untwisted cylindrical band.

The boundary of the Möbius band is a pentagon and every vertex is connected to every other vertex. By adding a cone over the pentagon from a sixth point we obtain a surface without boundary called the *real projective plane* with six vertices, fifteen edges (connecting all distinct pairs of vertices) and ten triangles. We denote this special triangulation by $\mathbf{RP}^2_6$. It can be described by taking the icosahedral triangulation of the 2-sphere and then identifying opposite vertices, edges and faces.

This triangulation $\mathbf{RP}^2_6$ has numerous special properties. Although it is not possible to construct any non-self-intersecting real projective plane in ordinary Euclidean 3-space, we can construct a one-to-one mapping of $\mathbf{RP}^2_6$ into Euclidean 4-space by first building a

Tom Banchoff (left) and **Wolfgang Kühnel** (center) at work (?) at Oberwolfach, 1980.

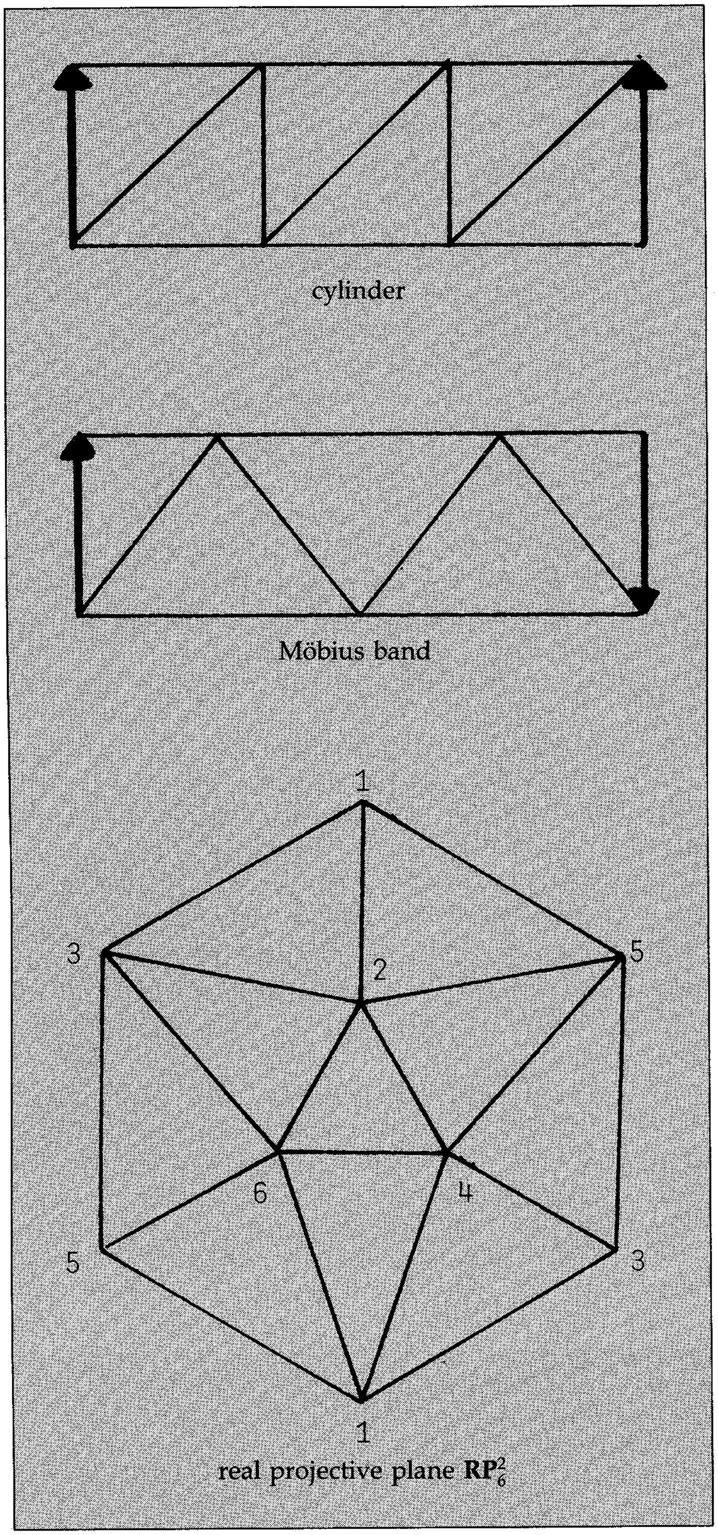

Figure 1

5-vertex Möbius band in a 3-dimensional subspace and then choosing the sixth vertex so that it does not lie in the subspace.

For any triangulation with six vertices there is a natural way to obtain an embedding into the boundary of a regular 5-simplex Δ^5 in $\mathbf{R}^5$. We number the vertices of the triangulation of $\mathbf{RP}^2_6$ and of Δ^5 and let vertices with the same indices correspond. If we extend this mapping linearly over edges and triangles, we obtain an embedding of the real projective plane into 5-space which we again call $\mathbf{RP}^2_6$.

This embedding has several important properties which will guide us in our later investigation:

1. Symmetry—Any symmetry of the icosahedron will determine a permutation of the vertices of $\mathbf{RP}^2_6$ and therefore a symmetry of $\mathbf{R}^5$ which preserves $\mathbf{RP}^2_6$.

2. Duality—Three vertices of Δ^5 determine a triangle of $\mathbf{RP}^2_6$ if and only if the remaining three vertices do not.

3. Tightness—Since $\mathbf{RP}^2_6$ contains all edges of Δ^5 any half-space of $\mathbf{R}^5$ will divide $\mathbf{RP}^2_6$ into at most two pieces. Almost every height function on $\mathbf{R}^5$ will have exactly one maximum and one minimum when restricted to $\mathbf{RP}^2_6$. A surface with this property is called *tight*.

4. Secant-Tangency—Any chord joining two points of $\mathbf{RP}^2$ lies in one of the boundary 4-simplexes of Δ^5. This follows since any two triangles of $\mathbf{RP}^2_6$ have at least a vertex in common.

Already in the last century, the geometer Veronese described an algebraic mapping of the real projective plane into $\mathbf{R}^5$ with analogous properties. We view the real projective plane as the unit sphere $S^2 = \{(x_0,x_1,x_2)|x_0^2 + x_1^2 + x_2^2 = 1\}$ in $\mathbf{R}^3$ with antipodal points identified, i.e. (x_0,x_1,x_2) is identified with $(-x_0,-x_1,-x_2)$. To find an embedding into $\mathbf{R}^6$ first of all, we choose six quadratic functions that take the same value at antipodal points of S^2 and define a mapping

$$(x_0,x_1,x_2) \rightarrow (x_0^2,x_1^2,x_2^2, \sqrt{2}x_1x_2, \sqrt{2}x_2x_0, \sqrt{2}x_0x_1)$$

It is an easy algebraic fact that two distinct points of S^2 have the same image if and only if they are antipodal so we have an embedding of $\mathbf{RP}^2$ into $\mathbf{R}^6$ which we call $\mathbf{RP}^2_v$. Even though this mapping sends the real projective plane into $\mathbf{R}^6$, the sum of the first three coordinates is 1 so the image lies in a five-dimensional affine hyperplane. Just as $\mathbf{RP}^2_6$ is situated in the boundary of a regular 5-simplex, $\mathbf{RP}^2_v$ is situated in a 4-sphere given by the intersection of the hyperplane and the unit sphere, since

$$(x_0^2)^2 + (x_1^2)^2 + (x_2^2)^2 + (\sqrt{2}x_1x_2)^2 + (\sqrt{2}x_2x_0)^2 + (\sqrt{2}x_0x_1)^2 = (x_0^2 + x_1^2 + x_2^2)^2 = 1$$

This algebraic embedding has the following properties:

1. Symmetry—Any rotation of $\mathbf{R}^3$ determines an isometry of $\mathbf{R}^5$ preserving $\mathbf{RP}^2_v$.

2. Duality—For any point of S^2, say (0,0,1), the equatorial great circle $(\cos\theta,\sin\theta,0)$ will be sent to a circle doubly covered $(\cos^2\theta, \sin^2\theta, 0, 0, 0, \sqrt{2}\cos\theta \sin\theta)$ lying in the hyperplane where the third coordinate is zero. This hyperplane supports $\mathbf{RP}^2_v$ since the third coordinate $(x_2)^2$ is never negative so all of $\mathbf{RP}^2_v$ lies to one side of the hyperplane. This circle is not the boundary of any surface-with-boundary in $\mathbf{RP}^2_v$.

3. Tightness—Any hyperplane of $\mathbf{R}^5$ separates $\mathbf{RP}^2_v$ into at most two pieces. Almost every height function on $\mathbf{R}^5$ restricted to $\mathbf{RP}^2_v$ will have exactly one maximum and one minimum.

4. Secant-Tangency—Any chord joining two points of $\mathbf{RP}^2_v$ is parallel to a tangent line of $\mathbf{RP}^2_v$. (If we project orthogonally to a 4-space and get an immersion, then it will never have double points.)

These properties are very special and indeed each of the properties 3 and 4 characterizes $\mathbf{RP}^2_v$ among all smooth embeddings of $\mathbf{RP}^2$ in $\mathbf{R}^5$ not lying in an affine hyperplane. Pohl and Kuiper have in fact shown that any *topological* embedding of the real projective plane into $\mathbf{R}^5$ not lying in a hyperplane which has the tightness property alone must either be $\mathbf{RP}^2_6$ or $\mathbf{RP}^2_v$, i.e. either this special triangulated example or the algebraic Veronese embedding. (cf.[9])

With respect to the *complex projective plane*, the situation has been less satisfactory. Although much was understood about the algebraic case, there was little information about special triangulations.

To define $\mathbf{CP}^2$ we start with the collection of triples of complex numbers (z_0,z_1,z_2) in $\mathbf{C}^3$ with $z_0\bar{z}_0 + z_1\bar{z}_1 + z_2\bar{z}_2 = 1$ and we identify two triples if one can be obtained from the other by multiplying by $e^{i\theta}$, a complex number of modulus 1. Algebraic geometers have studied the complex algebraic embedding of $\mathbf{CP}^2$ corresponding to the Veronese embedding. Of more interest here is a real algebraic embedding of this space given by Mannoury [10] in 1900, using coordinate functions analogous to those of the Veronese embedding:

$$(z_0,z_1,z_2) \rightarrow (z_0\bar{z}_0, z_1\bar{z}_1, z_2\bar{z}_2, \sqrt{2}z_1\bar{z}_2, \sqrt{2}z_2\bar{z}_0, \sqrt{2}z_0\bar{z}_1).$$

As before we can show that two triples have the same image if and only if they differ by multiplication by a complex number of modulus 1, so we get an embedding of $\mathbf{CP}^2$ which we denote by $\mathbf{CP}^2_m$.

Note that the first three coordinates are real and the last three are complex, so $\mathbf{CP}^2_m$ can be considered as an embedding into $\mathbf{R}^9$. Since the sum of the first three coordinates is 1, the image of the embedding lies an 8-dimensional affine hyperplane which we denote by $\mathbf{R}^8$. The points of $\mathbf{CP}^2_m$ all lie at distance 1 from the origin in $\mathbf{R}^9$ so $\mathbf{CP}^2_m$ actually lies in a 7-sphere in the 8-dimensional hyperplane.

As in the case of $\mathbf{RP}^2_v$ we have several special properties of this embedding.

1. Symmetry—Any unitary transformation of $\mathbf{C}^3$ determines an isometry of $\mathbf{R}^8$ which preserves $\mathbf{CP}^2_m$.

2. Duality—The set opposite a point such as (0,0,1) in $\mathbf{C}^3$ will be a collection of $(z_0,z_1,0)$ with $z_0\bar{z}_0 + z_1\bar{z}_1 = 1$ and $(z_0,z_1,0)$ identified with $(e^{i\theta}z_0, e^{i\theta}z_1, 0)$. This is mapped into the hyperplane where the third coordinate is zero, which supports $\mathbf{CP}^2_m$.

Moreover, the mapping $(z_0,z_1,0) \rightarrow (z_0\bar{z}_0, z_1\bar{z}_1, 0,0,0, \sqrt{2}z_0\bar{z}_1)$ send the 3-sphere $\{z_0\bar{z}_0 + z_1\bar{z}_1 = 1\}$ in $\mathbf{C}^2 \equiv \mathbf{R}^4$ to a 2-sphere in such a way that two vectors have the same image if and only if they differ by multiplication by a complex number of modulus 1. This map of the 3-sphere to the 2-sphere is the famous Hopf mapping described in [7]. This image 2-sphere is not the boundary of any 3-manifold-with-boundary in $\mathbf{CP}^2_m$.

3. Tightness—The intersection of $\mathbf{CP}^2_m$ with any half-space is both connected and *simply-connected.* Almost any height function when restricted to $\mathbf{CP}^2_m$ has exactly one maximum and one minimum and no critical points of Morse index 1 or 3, and this is the condition for a *tight* embedding of $\mathbf{CP}^2$.

4. Secant Tangency—Any secant line joining two points of $\mathbf{CP}^2_m$ is parallel to a tangent line of $\mathbf{CP}^2_m$.

Recently Kuiper [8] has shown that tightness characterizes $\mathbf{CP}^2_m$ among all smooth embeddings, i.e. any tight smooth embedding of $\mathbf{CP}^2$ into $\mathbf{R}^8$ not lying in an $\mathbf{R}^7$ must be the image of $\mathbf{CP}^2_m$ under an invertible projective transformation. In the same publication, Kuiper asks if there are non-differentiable embeddings of $\mathbf{CP}^2$ into $\mathbf{R}^8$ with the tightness property (problem 12).

What about a triangulation of $\mathbf{CP}^2$ that would have similar properties as $\mathbf{CP}^2_m$, much as $\mathbf{RP}^2_6$ and $\mathbf{RP}^2_v$ share corresponding properties? Until recently there was no such good triangulation of the complex projective plane. We now tell how such a triangulation was found and how its properties have been determined.

In 1980 the first named author set out to try to find a triangulation of $\mathbf{CP}^2$ which would sit in the boundary of the 8-simplex Δ^8 in $\mathbf{R}^8$ in much the same way that $\mathbf{CP}^2_m$ sits in a 7-sphere in $\mathbf{R}^8$. This in particular meant that the number of vertices of such a triangulation should be $\alpha_0 = 9$. Since the triangulation would have the tightness property, all $\binom{9}{2} = 36$ possible edges should be used so α_1, the number of 1-simplexes, should be $\alpha_1 = 36$. Moreover, since $\mathbf{CP}^2$ is simply-connected, with no non-contractible 1-dimensional curves, the desired tightness condition indicates that the intersection of $\mathbf{CP}^2$ with any half-space of $\mathbf{R}^8$ should be simply-connected as well, and this implies that all $\binom{9}{3} = 84$ 2-simplexes of Δ^8 should be included in the triangulation so $\alpha_2 = 84$.

However, at this point one must take account of a sphere. This will exhibit both M_9^4 and $\mathbf{CP}_m^2$ as realizations of the mapping cone of the Hopf mapping from S^3 to S^2 and this leads to a correspondence between the points of these two 4-manifolds. This process becomes clearer if we first look at the analogous situation for $\mathbf{RP}_6^2$ and $\mathbf{RP}_v^2$. It turns out in both the real and the complex cases there are technical difficulties involved in carrying over arguments from the smooth situation to the case of triangulated manifolds and simplical mappings. We will not be able to go into detail here as we present the general description of this proof.

Thus far we have been considering $\mathbf{RP}^2$ as the 2-sphere S^2 with opposite points identified. Another useful way to describe this space is to start with $\mathbf{R}^2$ and add a "line at infinity" consisting of one abstract point for each family of parallel lines in $\mathbf{R}^2$. This is topologically the same as taking an open unit 2-disc D^2 and identifying to a point the intersection of any line through the origin with the boundary of the disc. This identification is just the antipodal identification on the disc, and we may consider $\mathbf{RP}^2$ as a realization of the *mapping cone* of the mapping $h : S^1 \to \mathbf{R} \cup \{\infty\}$ which sends each point (x,y) of S^1 to x/y if $y \neq 0$ and which sends $(1,0)$ and $(-1,0)$ to the point at infinity.

The triangulation $\mathbf{RP}_6^2$ has a duality property that opposite to any 2-simplex spanned by three vertices (e.g. 123) there is a non-boundary 1-cycle given by the other three vertices and the corresponding edges, (e.g. $\partial(456)$). Opposite to each Δ^2 in $\mathbf{RP}_6^2$ there is the boundary of another triangle in Δ^5 not lying in $\mathbf{RP}_6^2$. Any point of $\mathbf{RP}_6^2$ not lying either in $\Delta^2 = 123$ or in the boundary of $\overline{\Delta}^2 = 456$ can be expressed uniquely as t times a point of $\partial\Delta^2$ plus $(1 - t)$ times a point of $\partial\overline{\Delta}^2$. Any level set corresponding to a value t between 0 and 1 will be a 1-dimensional sphere expressed as a union of convex cells (in fact a nonagon). As t approaches 1, this level set goes to $\partial\Delta^2$. As t approaches 0, this level set approaches a decomposition of the 1-sphere into three pairs of segments which are mapped to the three edges of $\partial\overline{\Delta}^2$ by the Hopf mapping which identifies each point x of S^1 with its antipodal point $(-1)x$ (see figure 3A).

The corresponding decomposition of $\mathbf{RP}^2$ given in coordinates uses the height function $\phi(x_0,x_1,x_2) = (x_0)^2$. Consider the 2-ball $D^2 = \{[x_0,x_1,x_2] \| x_0^2 \geqslant 1/2\}$ with boundary $S^1 = \{[1/\sqrt{2},x_1,x_2] \| (x_1)^2 + (x_2)^2 = 1/2\}$, and the 2-sphere $\overline{S}^1 = \{[0,x_1,x_2] \| x_1^2 + x_2^2 = 1\}$. Any point not on either D^2 or $\overline{S}^1$ can be written uniquely as $[r,x_1,x_2]$ with $0 < r < 1/\sqrt{2}$ and therefore it uniquely determines points $[1/\sqrt{2}, x_1,x_2]$ on S^1 and $[0,x_1,x_2]$ on $\overline{S}^1$. All level sets of ϕ between ½ and 0 are 1-spheres and as the level approaches 1/2, this 1-sphere approaches the boundary of the ball D^2. As the level approaches 0, this 1-sphere approaches a double covering of the non-bounding 1-sphere $\overline{S}^1$.

We now carry out the same analysis in the complex case. To obtain a different description of $\mathbf{CP}^2$ we start with the complex plane $\mathbf{C}^2$ and add a "line at infinity" consisting of one (abstract) point for each equivalence class of parallel complex lines in $\mathbf{C}^2$. This is topologically the same as taking the open unit 4-ball $B^4 \subsetneq \mathbf{R}^4 \cong \mathbf{C}^2$ and for each complex line ℓ through the origin, identifying the intersection $\ell \cap \partial B$ to a point. This is just the identification given by the *Hopf map* $h : \partial B^4 = S^3 \to S^2$ which in coordinates reads as $h(z,w) = z/w \in C \cup \{\infty\} \cong S^2$ for $(z,w) \in \mathbf{C}^2$ satisfying $z\bar{z} + w\bar{w} = 1$. (One can express this fact by saying that $\mathbf{CP}^2$ represents the *mapping cone* of $h : S^3 \to S^2$).

Now as already mentioned above, our M_9^4 has the *duality property* that opposite to any 4-simplex spanned by five vertices (e.g.: 56789) there is a non-bounding 2-cycle given by the boundary of the 3-simplex spanned by the remaining four vertices (e.g.: $\partial(1234)$). Opposite to each Δ^4 there is the boundary of a Δ^3. Any point of M_9^4 not lying either in the 4-simplex $\Delta^4 = 56789$ or in the boundary of $\Delta^3 = 1234$ can be expressed as t times a point of $\partial\Delta^4$ plus $(1 - t)$ times a point of $\partial\Delta^3$. Any level set corresponding to a value t between 0 and 1 will be a 3-dimensional sphere expressed as a union of convex cells. As t approaches 1, this level set goes to $\partial\Delta^4$; and as t approaches 0, this level set approaches a decomposition of the 3-sphere into four solid tori $S^1 \times \Delta^2$ which are mapped to the four triangles of $\partial\Delta^3$ by the Hopf map (see figure 3B). (A more precise argument would use the technique of "collapsing", a standard method in PL topology).

The corresponding decomposition of $\mathbf{CP}^2$ given in coordinates uses the height function $\phi(z_0,z_1,z_2) := z_0\bar{z}_0$: consider the 4-ball $B^4 = \{[z_0,z_1,z_2] \| z_0\bar{z}_0 \geqslant 1/2\}$ with boundary $S^3 = \{[1/\sqrt{2},z_1,z_2] \| z_1\bar{z}_1 + z_2\bar{z}_2 = 1/2\}$ and the 2-sphere $S^2 = \{[0,z_1,z_2] \| z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}$. Any point not in either this 4-ball or in this 2-sphere can be written uniquely as $[r,z_1,z_2]$ where $0 < r < 1/\sqrt{2}$, and therefore it uniquely determines points $[1/\sqrt{2},z_1,z_1] \in S^3$ and $[0,z_1,z_2] \in S^2$. All ϕ-level sets between 1/2 and 0 are 3-spheres, and as the level approaches 1/2, this 3-sphere approaches the boundary of the 4-ball B^4. As the level approaches 0, this 3-sphere approaches the non-bounding 2-sphere $S^2 = \{[0,z_1,z_2]\}$, giving a representation of the mapping cone of the Hopf map.

This decomposition gives an explicit correspondence between the $\mathbf{CP}^2$ defined in terms of coordinates and this M_9^4 which we can now write as $\mathbf{CP}_9^2$, the *9-vertex complex projective plane.* It is a well known fact that from the Hopf map $h : S^3 \to S^2$, we obtain a nontrivial S^1-bundle over S^2, called the *Hopf bundle*, so in some way $\mathbf{CP}^2$ contains this bundle. More precisely the natural retraction $\mathbf{CP}^2 \backslash B^4 \to S^2$ gives a nontrivial 2-disc-bundle over S^2 the boundary 1-sphere-bundle of which is just the Hopf bundle.

To find a polyhedral analogue in our model $\mathbf{CP}_9^2$ we take again the 4-simplex $\Delta^4 = 56789$ and the opposite non-bounding 2-sphere $\partial\Delta^3 = \partial(1234)$. For each vertex or edge or triangle in $\partial\Delta^3$ we can take its "preimage",

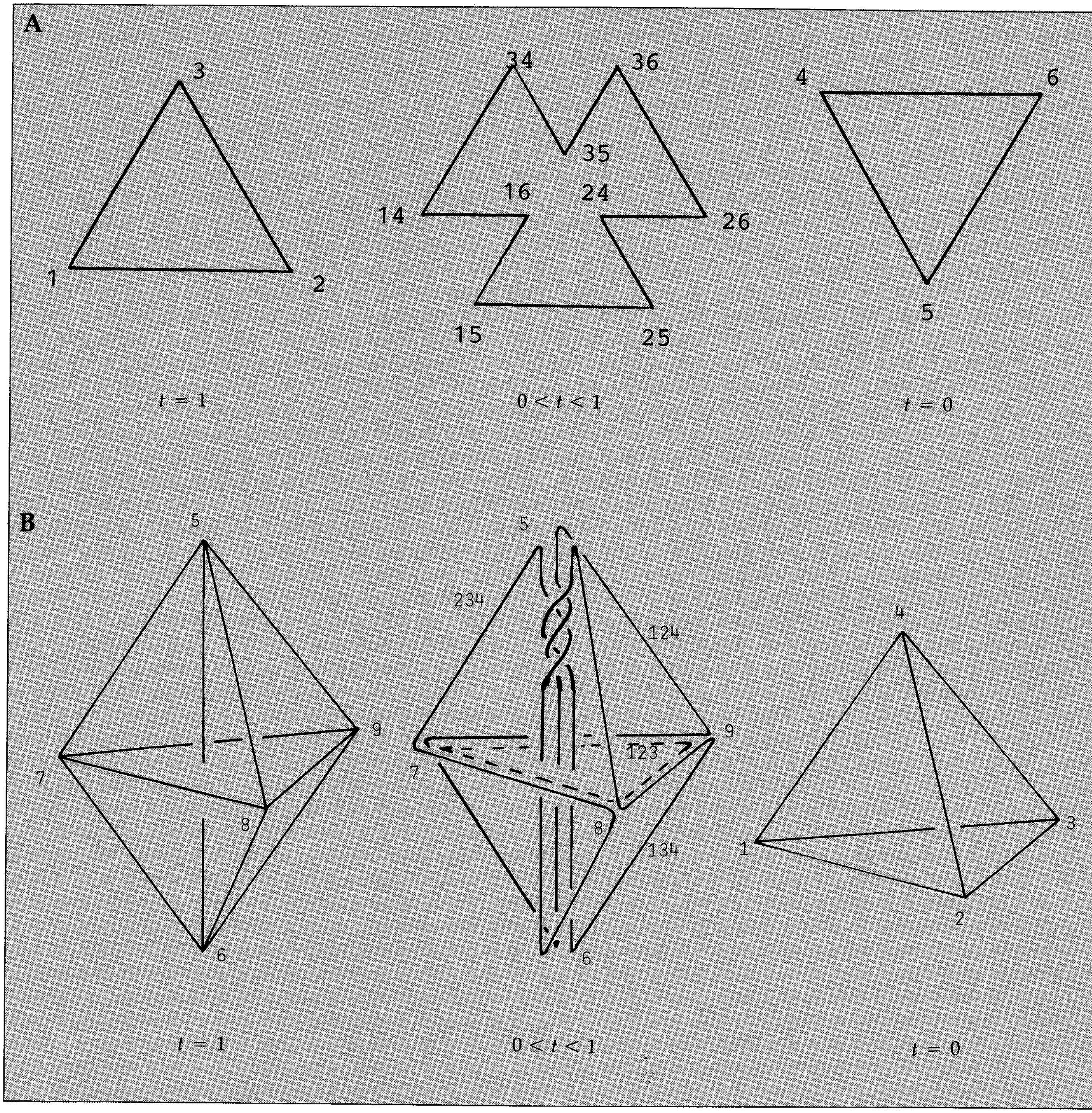

Figure 3

i.e. the collection of all those simplexes in 56789 which are directly joined with the given vertex or edge or triangle in $\partial\Delta^3$. For example, the preimage of 123 in this sense will be the boundary of the triangle 789 which is topologically a 1-sphere. The following figure 4 (on the next page) shows these various preimages, each of which is a homotopy 1-sphere. The picture describes the natural retraction $\partial(\mathbf{CP}^2_9\backslash\Delta^4) \to \partial\Delta^3$ and on closer examination it can be seen that this retraction is, up to homotopy, a nontrivial 1-sphere bundle over S^2 : the fibre "turns around" just once going around the picture. (Again, a more complete argument will use techniques from the theory of block bundles. Note that each of the fibres in figure 4 collapses to a polygon.)

So far we have given several arguments why our simplicial complex is a triangulation of $\mathbf{CP}^2$. In the following paragraphs we wish to indicate some more relationships between this model and some constructions in algebraic geometry. Let us first go back to $\mathbf{RP}^2_v$ and $\mathbf{RP}^2_6$. The Veronese embedding $V : \mathbf{RP}^2 \to \mathbf{R}^5$ splits in a natural way to

$$V_1 : (x_0,x_1,x_2) \to (x_0^2,x_1^2,x_2^2),$$

$$V_2 : (x_0,x_1,x_2) \to (x_0x_1,x_1x_2,x_2x_0).$$

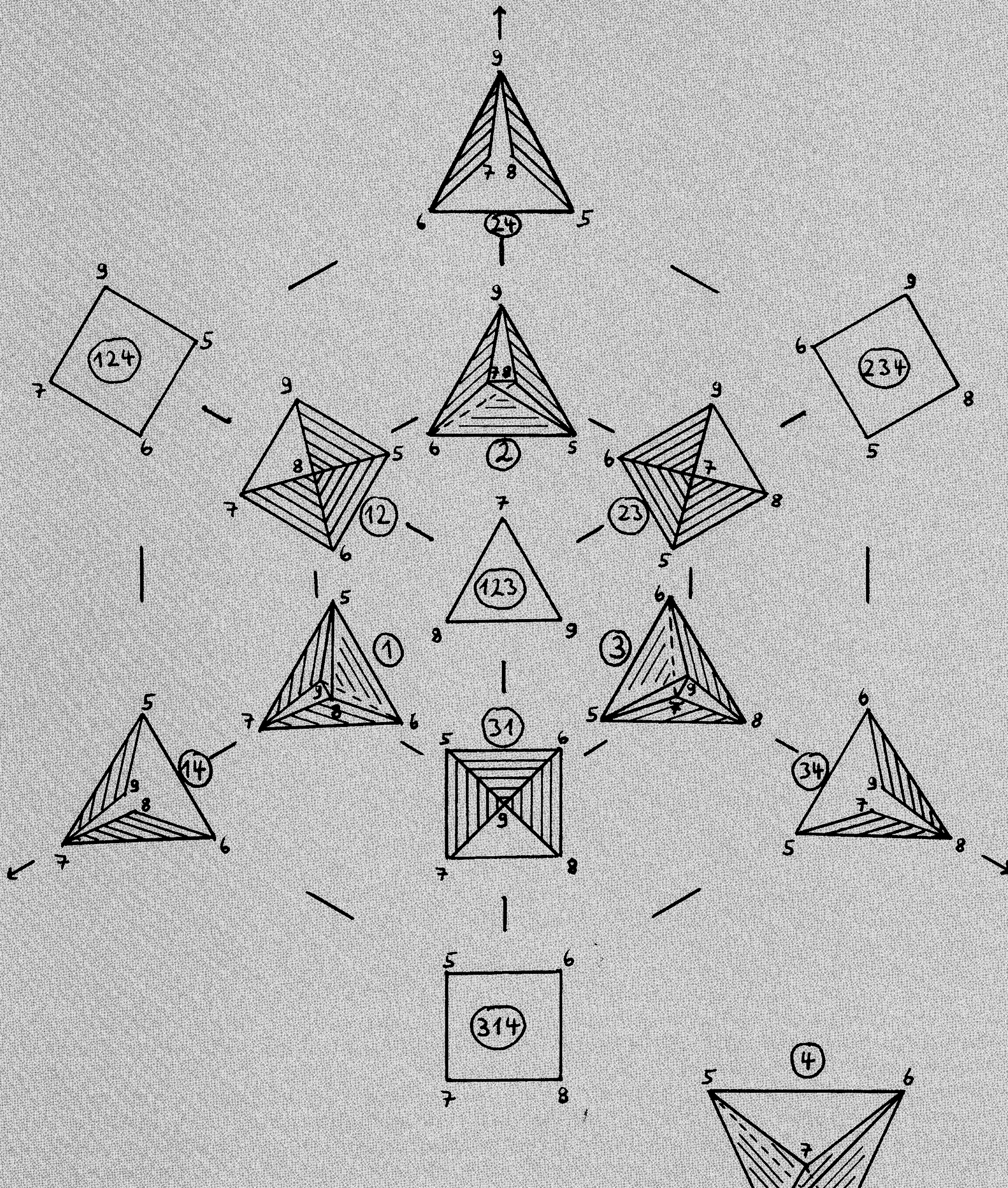

Figure 4

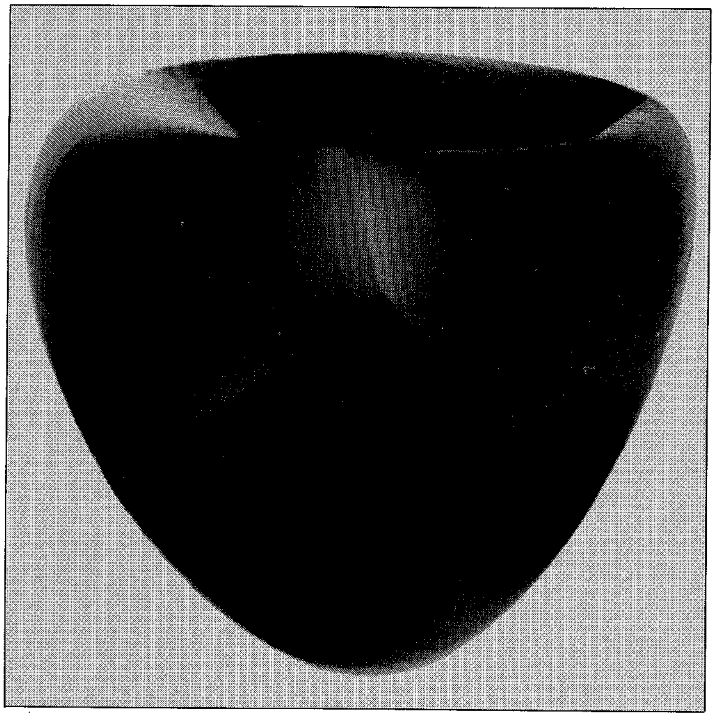

Figure 5

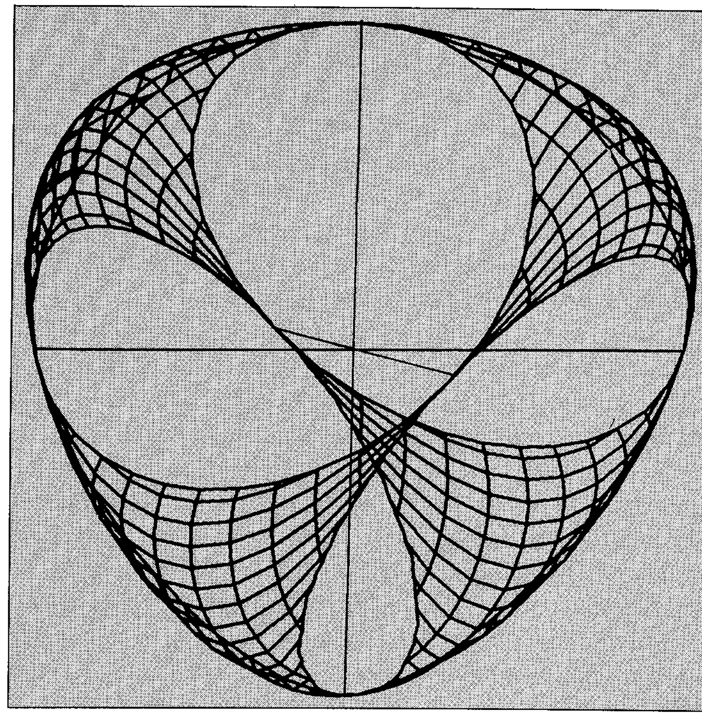

Figure 6

The image of V_1 is the plane equilateral triangle spanned by (1,0,0), (0,1,0) and (0,0,1) where the preimage of each interior point is a "torus" $S^0 \times S^0$ in $\mathbf{RP}^2$. The image of V_2 is well known as *Steiner's Roman surface* (see figure 5). The locus of self-intersection of this surface is the set of intervals $(-1/2,1/2)$ on the three coordinate axes meeting at the origin which is a triple point. It is interesting to observe that the four circles on the boundary of the convex hull (inscribed in the faces of a regular tetrahedron), together with these three straight line segments along the axes, form Fano's projective plane of order 2 with seven points and seven lines (see figure 6).

Similar observations can be made for $\mathbf{RP}^2_6$: choose the six vertices to be the points $1/\sqrt{2}\ (1,\pm 1,0)$, $1/\sqrt{2}\ (0,1,\pm 1)$, $1/\sqrt{2}\ (\pm 1,0,1)$ in S^2. These are mapped by the Veronese embedding to the vertices of a (nonregular) 5-simplex. The first part of the Veronese mapping sends these points two by two to the three midpoints of the edges of the equilateral triangle and the preimage of each interior point of that smaller triangle is four points, a "torus" $S^0 \times S^0$ in $\mathbf{RP}^2_6$. The second part sends these six points to the vertices of a regular octahedron, and the induced simplexwise linear mapping gives a polyhedral versin of Steiner's surface containing four of the eight triangles of this octahedron— no two of which are adjacent (see figure 7). This example is classically known as the heptahedron model of $\mathbf{RP}^2$. All these mappings defined on $\mathbf{RP}^2$ are tight because they are linear projections of tight embeddings.

Similarly for $\mathbf{CP}^2$ the Mannoury embedding splits into

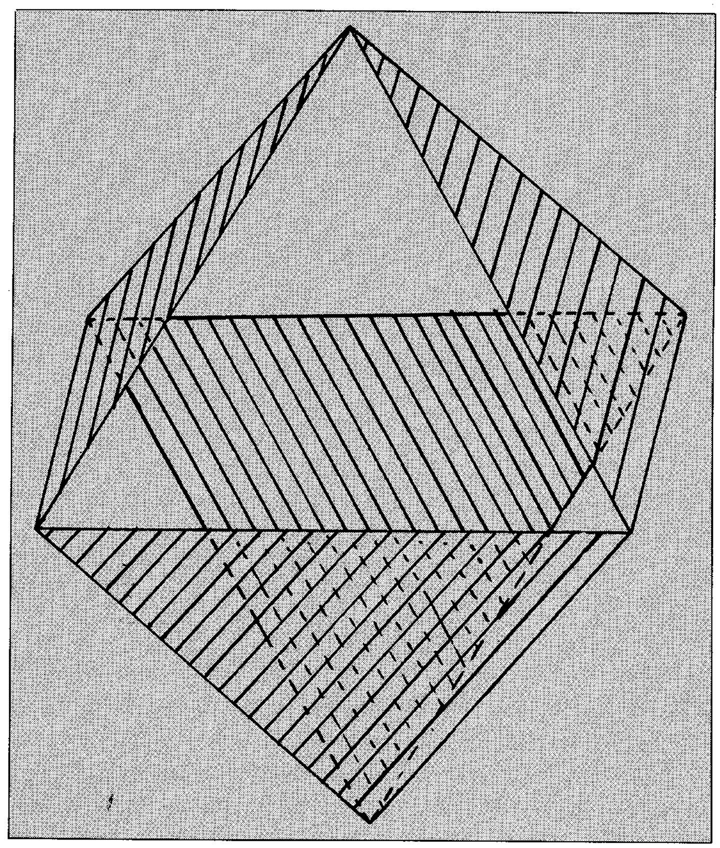

Figure 7

$$V_1 : [z_0,z_1,z_2] \to (z_0\bar{z}_0, z_1\bar{z}_1, z_2\bar{z}_2) \in \mathbf{R}^3$$

and

$$V_2 : [z_0,z_1,z_2] \to (z_0\bar{z}_1, z_1\bar{z}_2, z_2\bar{z}_0) \in \mathbf{C}^3.$$

The image of V_1 is the same equilateral triangle as in the real case but now the preimage of each interior point of the triangle is a torus $S^1 \times S^1$ in $\mathbf{CP}^2$. The image of V_2 looks more complicated. Its self-intersection behavior is analogous to the real case: the self-intersections are three unit discs in the three complex coordinate axes with singularities along their boundaries and with the origin as a triple point.

In order to get polyhedral analogues of V_1 and V_2, we first have to find a good way to choose the nine vertices as actual points in $\mathbf{CP}^2$. One approach is to use the fact from algebraic geometry that any nonsingular cubic curve in $\mathbf{CP}^2$ has exactly nine inflection points. Any such curve (sometimes called an "elliptic curve") is homeomorphic to a torus $S^1 \times S^1$ and it carries the structure of an abelian group defined by correspondences which mean roughly that the sum of any three collinear points is zero. The nine inflection points are exactly the order 3 elements in this group. Now which of the various cubic curves should we choose? The first idea might be to take the curve $z_0^3 + z_1^3 + z_2^3 = 0$ with the inflection points $[1, -\omega^v, 0]$. $[0, 1, -\omega^v]$, $[-\omega^v, 0, 1]$ where $v = 0,1,2$ and $\omega = e^{2\pi i/3} = 1/2(-1 + \sqrt{-3})$ is a cube root of unity. However it turns out that this configuration has too many symmetries to fit our triangulation in the best possible way. Instead of this we prefer to use the three cubics

$$z_0^3 + z_1^3 - z_2^3 = 0,\ z_0^3 - z_1^3 + z_2^3 = 0,\ -z_0^3 + z_1^3 + z_2^3 = 0.$$

The inflection points of these three cubics in $\mathbf{CP}^2$ are the following 18 points:

$$[1, \pm\omega^\nu, 0],\ [0, 1, \pm\omega^\nu]\ [\pm\omega^\nu, 0, 1],\ \nu = 0,1,2.$$

Now choose the 9 vertices of $\mathbf{CP}_9^2$ to be the following points

1	4	7	=	$[1,\omega,0]$	$[0,1,\omega]$	$[\omega,0,1]$
2	5	8	=	$[1,\omega^2,0]$	$[0,1,\omega^2]$	$[\omega^2,0,1]$
3	6	9	=	$[1,1,0]$	$[0,1,1]$	$[1,0,1]$

and choose the 9 barycenters of the edges 12,23,31,45, 56,64,78,89,97 to be the points

$\widehat{23}$	$\widehat{56}$	$\widehat{89}$	=	$[1,-\omega,0]$	$[0,1,-\omega]$	$[-\omega,0,1]$
$\widehat{31}$	$\widehat{64}$	$\widehat{97}$	=	$[1,-\omega^2,0]$	$[0,1,-\omega^2]$	$[-\omega^2,0,1]$
$\widehat{12}$	$\widehat{45}$	$\widehat{78}$	=	$[1,-1,0]$	$[0,1,-1]$	$[-1,0,1]$

We can now give an outline to show why this choice of points fits our triangulation.

First of all it is remarkable that the Veronese mapping sends any of these two sets of 9 points to the vertices of a regular 8-simplex. The partial mapping V_1 sends them three by three to the midpoints of the edges of the equilateral triangle (as in the real case), and V_2 sends them to three equilateral triangles in the three complex coordinate axes in $\mathbf{C}^3$. This, of course, induces piecewise linear versions of V_1 and V_2. The image of V_1 will be the same triangle as in the real case and it is interesting to observe that the interior of that triangle is covered by 27 of the 36 4-simplexes of $\mathbf{CP}_9^2$,

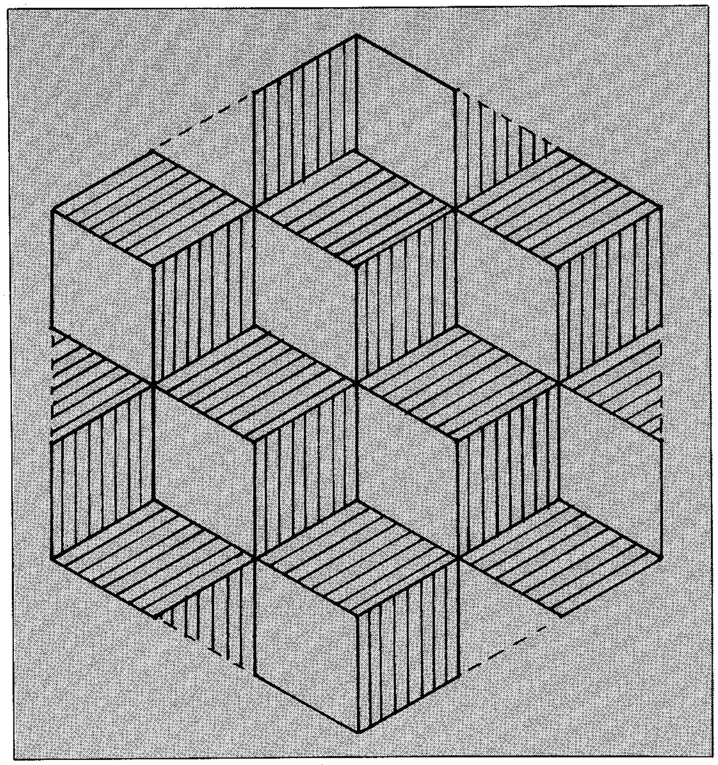

Figure 8

namely the orbit of 12459 under the symmetry group. In fact the preimage of each interior point of that triangle is a torus (in a subdivision of $\mathbf{CP}_9^2$) decomposed by 27 vertices and 27 quadrilaterals which is shown in figure 8 (we owe B. Morin this symmetric hexagonal arrangement).

Secondly the cubic $z_0^3 + z_1^3 - z_2^3 = 0$ forms a torus in $\mathbf{CP}^2$ going through the 9 points $\widehat{12},\widehat{23},\widehat{31},4,5,6,7,8,9$. The following figure 9 shows a natural polyhedral analogue in the first barycentric subdivision of $\mathbf{CP}_9^2$.

The same is true for the two other cubics by symmetry. It is very important to note the fact that the combinatorial symmetry group of $\mathbf{CP}_9^2$ generated by α, β, τ admits a representation in the group of projective (or antiprojective) transformation of $\mathbf{CP}^2$ leaving the two sets of 9 points invariant. Each such projective transformation can be considered to be a unitary transformation of $\mathbf{C}^3$ represented by a complex (3×3)-matrix. Now the correspondence is the following:

	Represented by:	
$\alpha = (147)(258)(369)$	multiplication by in $\mathbf{C}^3$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\beta = (123)(654)$	multiplication by in $\mathbf{C}^3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\gamma = (123)(456)(789)$	multiplication by in $\mathbf{C}^3$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$
$\tau = (12)(45)(78)$	complex conjugation in $\mathbf{C}^3$, $\sigma(z_0,z_1,z_2) = (\bar{z}_0,\bar{z}_1,\bar{z}_2)$.	σ

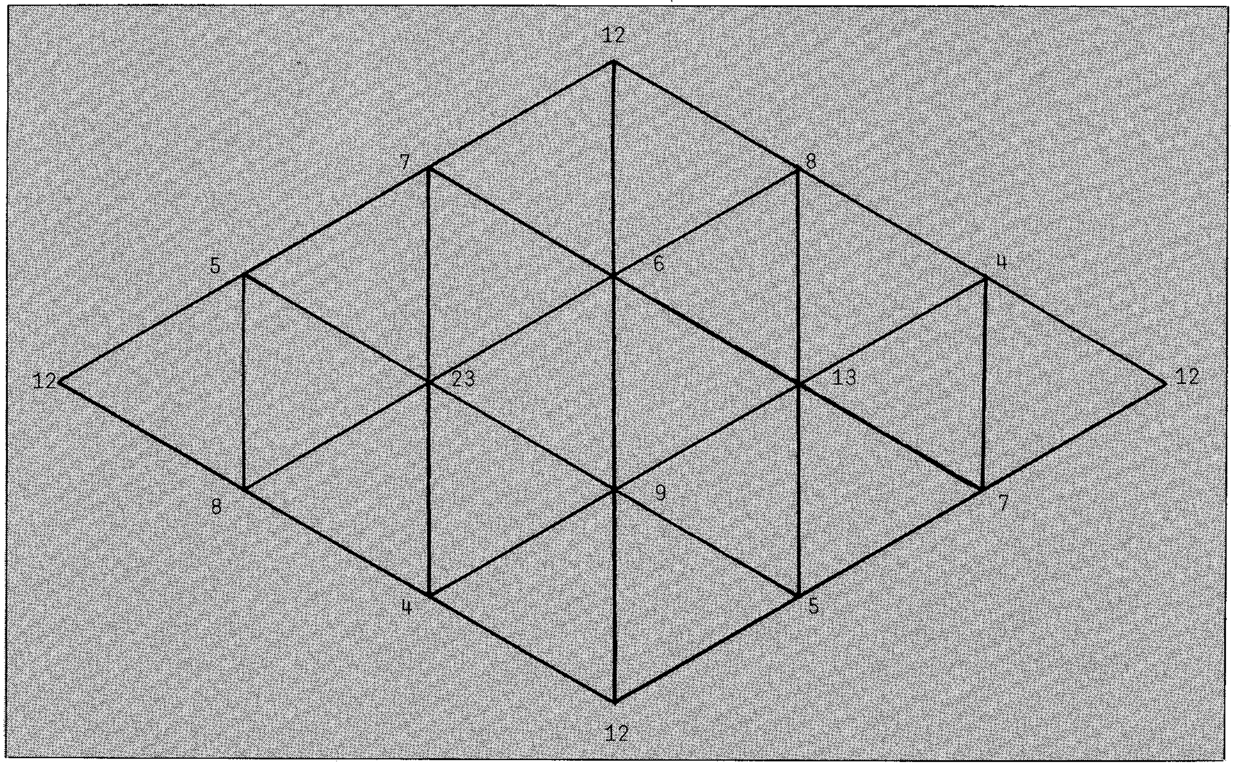

Figure 9

But note the fact that the symmetry group of the configuration of the two sets of 9 points is larger: the additional matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

represents also such a symmetry but the corresponding permutation of the vertices (15)(24)(36)(78) is *not* a symmetry of $\mathbf{CP}^2_9$. From this it follows that $\mathbf{CP}^2_9$ is not simply a quotient of well known configurations in $\mathbf{C}^3$ like the Hessian polyhedron or Gosset's polytope because the symmetry group of these quotients contains the full Hessian group of order 216 (see [2]). This makes our construction of the 54 element group different from related constructions by Coxeter. In order to get $\mathbf{CP}^2_9$ from the configuration of the two sets of 9 points we necessarily have to break the natural symmetry of this configuration.

A recent communication from M. Yoshida indicates a close connection between this example and complex crystallographic groups and this direction may lead to direct descriptions of the triangulation in $\mathbf{CP}^2$.

Furthermore it follows that the transposition τ acts like complex conjugation σ on $\mathbf{CP}^2_9$. Remember the fact that the quotient space $\mathbf{CP}^2/\sigma$ of $\mathbf{CP}^2$ mod conjugation is the 4-sphere S^4. This has been shown independently by W. S. Massey and N. H. Kuiper (see [11],[7]). Therefore we have the sequence

$$\mathbf{RP}^2 = \mathrm{Fix}(\sigma) \to \mathbf{CP}^2 \to \mathbf{CP}^2/\sigma = S^4,$$

and in a perfect analogy we have for our triangulation

$$\mathbf{RP}^2_6 = \mathrm{Fix}(\tau) \to \mathbf{CP}^2_9 \to \mathbf{CP}^2_9/\tau = \partial\Delta^5$$

where the quotient space $\mathbf{CP}^2_9/\tau$ turns out to be the boundary of the simplex spanned by the vertices $3,6,9,\hat{12},\hat{45},\hat{78}$, and where the fixed point set of τ spanned by the vertices $3,6,9,\hat{12},\hat{45},\hat{78}$ in the first barycentric subdivision is combinatorially an $\mathbf{RP}^2_6$ as shown in figure 10 (on the following page). This figure takes place in the real plane $(x_0,x_1,1)$ with the line $x_2 = 0$ at infinity. Note that the whole triangulation is given by seven straight lines in $\mathbf{RP}^2$. Now we are back to the point we started from: the 6-vertex real projective plane.

We hope that the reader has enjoyed learning some aspects of the 9-vertex complex projective plane, as we have over the last two years. The first discussion of this example took place at the Geometrie-Tagung in Oberwolfach in September 1980. That began our collaboration in the exploration of the properties of this remarkable 4-manifold and its relation to different parts of mathematics. We have been encouraged by the responses of many mathematicians who have suggested new questions and pointed out additional relationships to other fields. In particular, we would like to thank M. Atiyah, P. Cartier, H. S. M. Coxeter, H. Clemens, D. Ferus, F. Hirzebruch, H. Karcher, N. H. Kuiper, C. McCrory, R. MacPherson, B. Morin, W. F. Pohl and Chr. Zeeman. We also acknowledge gratefully the hospitality of the I.H.E.S. in Bures-sur-Yvette and of the Sonderforschungsbereich "Theoretische Mathematik" at the University of Bonn which made it

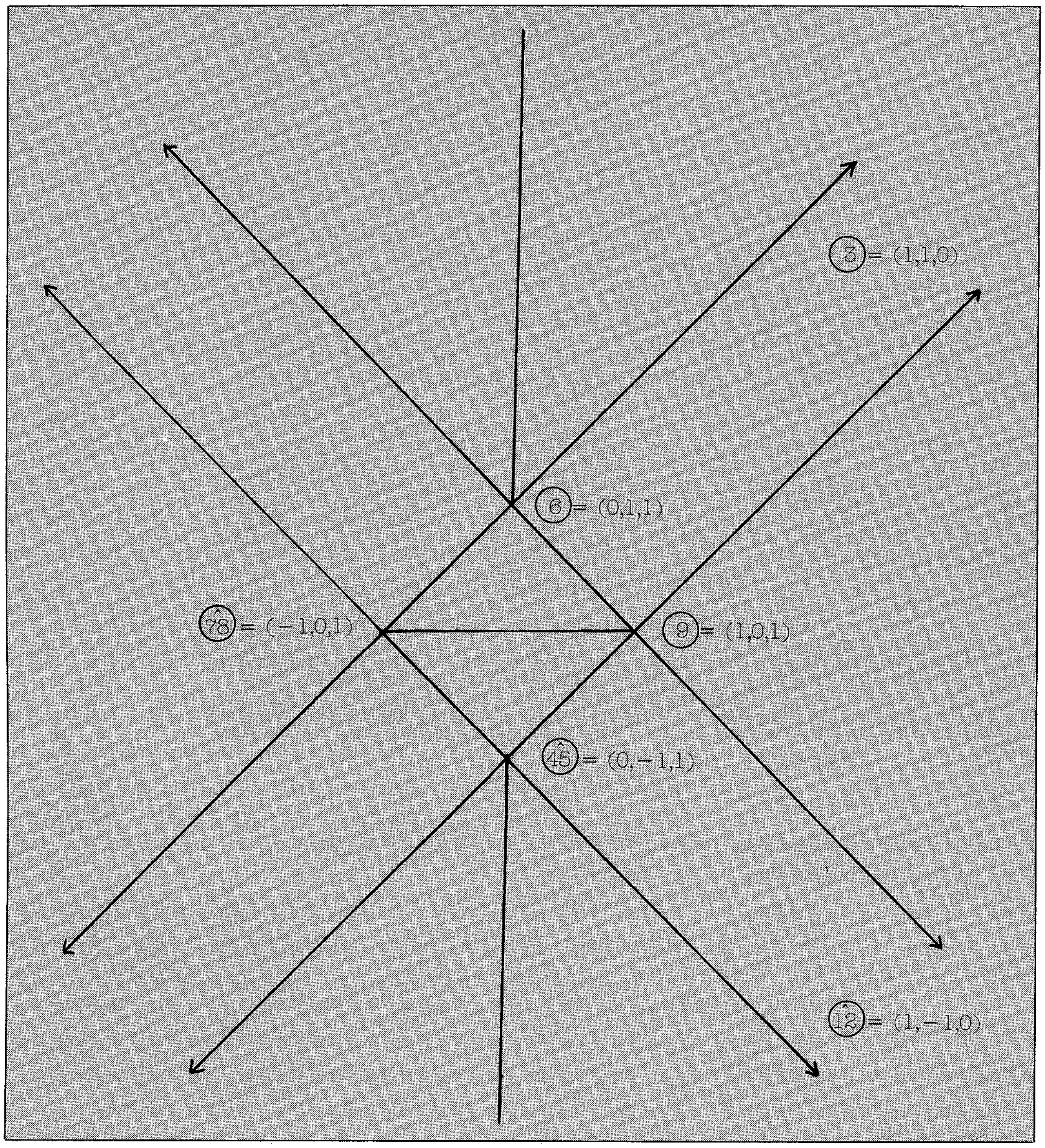

Figure 10

possible for the authors to work together on this project. The work continues and we look forward to the responses of many more mathematicians as we pursue the topics brought forward by the 9-vertex complex projective plane.

References

1. E. Bertini (1907) Geometria proiettiva degli iperspazi, Pisa, (German translation (1924) Einführung in die projektive Geometrie mehrdimensionaler Räume, Wien)
2. H. S. M. Coxeter (1974) The Equianharmonic Surface and the Hessian Polyhedron, *Annali di Mat. 98*:77–92
3. H. S. M. Coxeter and W. O. J. Moser (1980) Generators and Relations for Discrete Groups, 4th ed., Springer-Verlag (Ergebnisse der Mathematik und ihrer Grenzgebiete 14)
4. B. Grünbaum and V. P. Sreedharan (1967) An Enumeration of Simplicial 4-Polytopes with 8 Vertices, *J. Comb. Theory 2*:437–465
5. W. Kühnel and G. Laβmann (to appear) The unique 3-neighborly 4-manifold with few vertices, *J. Comb. Th.* (*A*)
6. N. H. Kuiper (1962) On convex maps, *Nieuw Archief voor Wisk. 10*:147–164
7. N. H. Kuiper (1974) The quotient space of CP(2) by complex conjugation is the 4-sphere, *Math. Ann. 208*:175–177
8. N. H. Kuiper (1980) Tight Embeddings and Maps. Submanifolds of Geometrical Class Three in E^N, Proc. Chern. Symp. Berkeley 1979, Springer-Verlag, pp. 97–145
9. N. H. Kuiper and W. F. Pohl (1977) Tight topological embeddings of the real projective plane in E^5, *Invent. Math. 42*:177–199
10. G. Mannoury (1900) Surfaces-images, *Nieuw Archief voor Wiskunde 4*:112–129.
11. W. S. Massey (1973) The quotient space of the complex projective plane under conjugation is a 4-sphere, *Geometriae Dedicata 2*:371–374

Wolfgang Kühnel
Fachbereich Mathematik
Technische Universität
Straβe des 17. Juni 135
1000 Berlin (West) 12
Germany

Thomas F. Banchoff
Mathematics Department
Brown University
Providence,
Rhode Island 02912
U.S.A.