Investigating Volumes: The Air France Cup

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Geometry is about phenomena. More than in any other branch of mathematics, geometry can be seen and felt. But it is more than the experience of geometric form that characterizes the subject – we have a natural process of abstraction and measurement that we encounter first on an intuitive level and then gradually refine throughout our mathematical education. That progressive aspect cannot be stressed too much, nor can the interconnections between geometric forms and other parts of mathematics. In my opinion, we often tend to withhold concepts from students until we feel that they are ready for them. Our attitudes are quite justified when the concept is extremely abstract, like some of the constructions in algebra or measure theory, but they can be mistaken when it comes to the presence of geometry in early education. One of my favorite illustrations of the abuse of this treatment of geometry has to do with the volume of a cone. For many students, the first time they see the formula for the cone's volume is the same day that they calculate this volume in a calculus course, as an example of a volume of revolution. At least, that is the way it always seems when I present the topic to calculus students. There should be an "aha!" from the class as a whole, when all at once there appears on the blackboard a justification for a rule that had been learned years earlier, and experienced years before that. Instead, the cone volume is submerged in a collection of frequently artificial stock problems at the end of a section on applications of integration.

Even more distressing is the fact that some students will remember a formula for the volume of a cone but have no idea about the volume of a pyramid. (After all, a pyramid is not a surface of revolution, so it only shows up in an even more optional section on applications, for volumes with known cross-section.) Worse still is the fact that many students do not seem to appreciate that the two formulas are related – the one-third that appears as a coefficient in each case might almost be viewed as a coincidence.

In my essay on *Dimensions* in the collection *On the Shoulders of Giants*, I began with the work of Friedrich Froebel, who championed this experimental approach to geometry in the early part of the nineteenth century. He concentrated on pre-school, and presented shapes to his pupils for them to play with. It was directed play to be sure, led along fairly well-determined paths by well-prepared Kindergarten teachers, but still the spirit of play was preserved. Pre-schoolers were not expected to prove theorems. But they could be expected to recognize that four square tiles fill a tray with edge length twice that of a small square, and eight cubes fit in a critical box with edge length twice that of a small cube. Moreover, they recognized that a similar relationship held for nine tiles or 27 small cubes when edge lengths tripled. Later on, they would recognize the patterns in the exponents that lead to content formulas in all dimensions, and they would say "aha!" (or something equally appreciative).

The block play of Froebel's students led to other decomposition theorems as well, so for example, they would not be surprised to see that a triangle has half the area of an associated parallelogram, or even that a pyramid has one-third the volume of an associated prism. That perhaps is a key to some of the confusion that some students experience later on – the cone is viewed as something quite separate from anything else. It doesn't seem to have the same sort of relationship with anything two-dimensional that occurs between a pyramid and a triangle. The fundamental fact is that cones don't pack – no matter how we arrange them in a box, there is space left over. That isn't the case for triangles of course – they can be repeated so that they fill the entire plane with no space remaining. The comparable figure in three-space is not a cone but a pyramid (a cone over a polygonal region). Not every pyramid can be used to fill space, but some can, and in particular it is possible to divide up a cube into three congruent pyramids. I think of that as being a fundamental insight, something that every person should know. There are not so many facts that I put on that same level.

To get from that fact to the general theorem about volumes of pyramids requires a stretching of the case for a pyramid in a cube, a variable stretching in different directions. If we double the length in a given direction, then the volume doubles. That's another fundamental insight. Such concepts should be familiar long before they become formalized in the calculus sequence.

Perhaps what I would suggest at the very least is that we incorporate some of these principles into the precalculus course. It isn't enough just to teach people how to factor polynomials, use the quadratic formula, and handle elementary trigonometric and exponential expressions, and to parse standard word problems. We should also give them some appreciation of area and volume so they won't be encountering these notions for the first time only after they are already confused by the concept of limit.

We can then reinforce these fundamental notions in our college geometry courses. An example I have used several different times in classes at quite different levels is the Air France Cup. That airline provides a plastic cup that is circular on the top and square on the bottom. There is a hole in the tray into which one can drop the cup so that it goes down half way. What is the shape of the holes and what is the volume of the cup? The answer to this, like the answer to so many mathematical questions, is, "it depends." It depends on the radius of the top rim, the length of the edges of the bottom square, and the fact that the top and bottom are in parallel planes. The volume also depends on the distance between these parallel planes, although the shape of the middle slice does not, at least if we make a further crucial assumption, namely that the cup is convex, and that it is the smallest convex set containing the top and bottom. And neither the volume nor the shape of the halfway slice depends on whether or not the center of the circle lies directly above the center of the square.

Convexity should be a familiar concept throughout the curriculum, beginning with pre-school tiles and appearing in every course. Still it may take some work for students to figure out the shapes of intermediate slices just based on the three-dimensional convexity assumption. They can be guided, of course, by some limiting cases – if the square has zero side length, then we have a conical Dixie cup, with known volume and cross-sections, and if the circular rim has zero radius, we have a square-based pyramid, again well-understood.

Since it is the curved sides of the figure that may be a source of confusion, it may be helpful to consider a more polyhedral example, with a square on top and another square on the bottom (dubbed the "Square France Cup" by one student). Now there is a simplification on one hand but a complication on the other. If the squares are of the same size and with edges parallel, then the convex set is a prism if the centers lie over each other, and a square-based parallelepiped with the same volume as that of the prism, if the centers are not so situated.

Things are not too much more difficult if the squares have unequal edge lengths. In this case, the convex set is a frustrum of a square-based pyramid and the cross-sections are squares, with edge length changing linearly as we go from the bottom to the top. This problem was solved already nearly four thousand years ago by the Egyptians (for whom the calculation of the volume of an incomplete pyramid was of practical importance as well as theoretical interest). Completing the frustrum for a pyramid leads to a solution of the volume problem, using a bit of elementary algebra (**Figure 1**). This solution also indicates the presence of a mixed term, relating the top and bottom edge lengths. It is not as well known as it should be, despite the efforts of papyrus scribes.

Note that

$\frac{x}{r} = \frac{x+h}{s}$ by similarity. Thus,

$$V = \frac{1}{3}(s^2(x+h)) - \frac{1}{3}(r^2x)$$

$$= \frac{1}{3}(s^2(\frac{sx}{r}) - r^2x)$$

$$= \frac{1}{3}(s^3(\frac{x}{r}) - r^3(\frac{x}{r}))$$

But $sx = rx + hr$, so $x = \frac{rh}{s-r}, \frac{x}{r} = \frac{h}{s-r}$

so $V = \frac{1}{3}(h)(\frac{s^3 - r^3}{s-r}) = \frac{h}{3}(s^2 + sr + r^2)$.

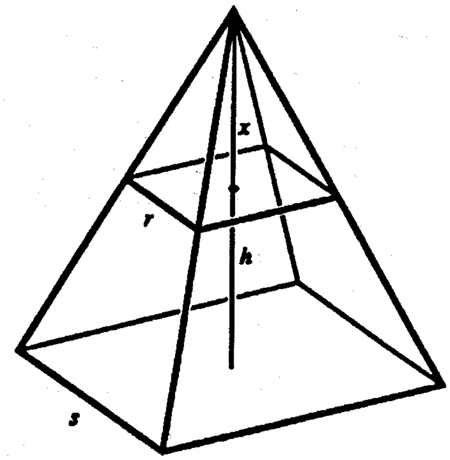

Figure 1.

We recognize the parts that reduce to the volume of a pyramid if $r = 0$ or if $s = 0$, and we have this intermediate part as well. Note that the halfway slice is a square with side $(1/2)(r+s)$, so with area $(1/4)s^2+rs+(1/4)r^2$. This expression contains the term rs which appears in the volume formula for the frustrum of the pyramid and we might expect a relationship between the area of this middle slice and the full volume of the solid.

We can easily use this approach to find the volume of a cup with rectangles of the same shape as its rims, just by applying the stretching principles. Another modification of the formula is necessary, however, if the top rectangle is rotated by a quarter turn. The convex hull is not so hard to determine in this case – it has a boundary consisting of four trapezoids – and the middle slice is a square (**Figure 2**).

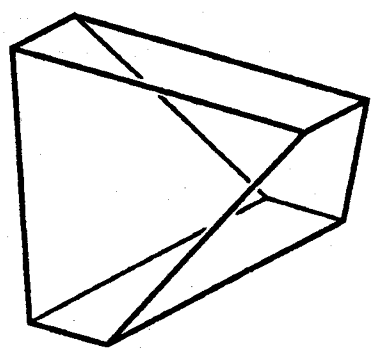

Figure 2.

This example makes it clear that the volume of the convex hull (the smallest convex set containing them) depends on more than just the areas of the top and bottom rims. An extreme case of some importance occurs when the rectangles shrink to segments. If the segments are <u>parallel</u>, we get a rectangle as the convex hull, with zero volume. If the segments are in perpendicular directions, then the convex hull is a tetrahedron. We can divide this into two triangle-based pyramids,

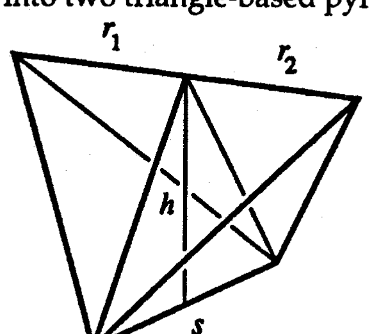

Figure 3.

so we can calculate the volume by knowing the area, $(hs/2)$ so we have $V = 1/3(hs/2)\,(r_1+r_2) = (1/6)\,hsr$. Note that this formula works if r and s are different, just as long as the lines which contain the two segments are perpendicular (**Figure 3**).

We can see this another way by working with subtraction rather than addition. We complete the figure to a prism with parallelogram base (**Figure 4**).

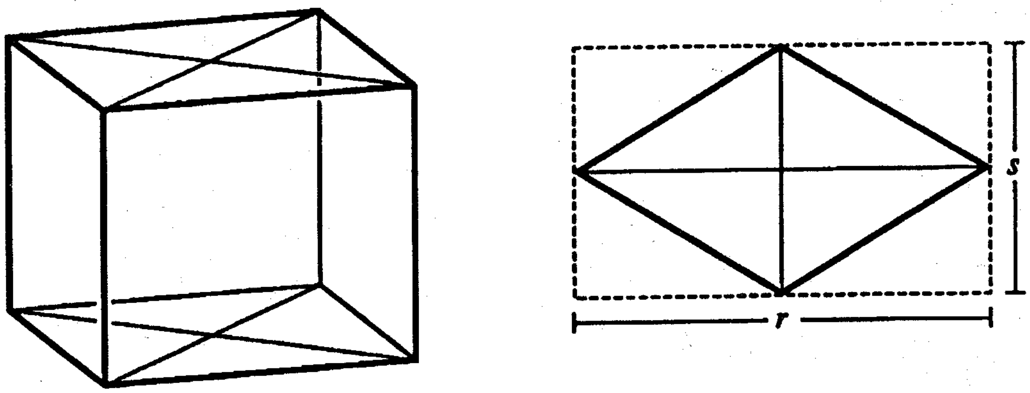

Figure 4.

The area of the parallelogram will be $(rs)/2$, and the tetrahedron remains in the middle after we slice off four corners of the prism.

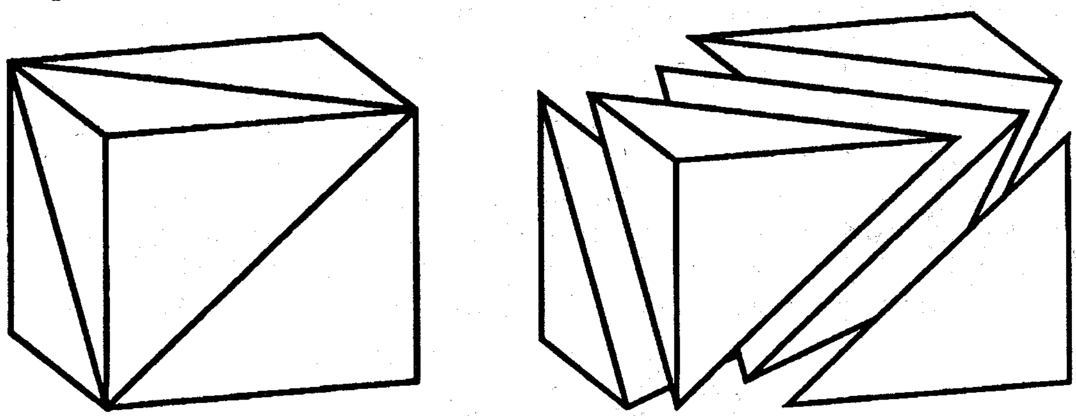

Figure 5.

Each corner slice has volume $(1/3)h((1/4)\,rs)$ so the volume of the inside tetrahedron is $1/2(rs)\,h - 4((1/3)h((1/4)rs)) = (1/6)\,rsh$ (**Figure 5**).

We can use this result to solve the Square France Cup problem when the top square is rotated 45° with respect to the bottom. In this case the convex hull is a square antiprism (**Figure 6**).

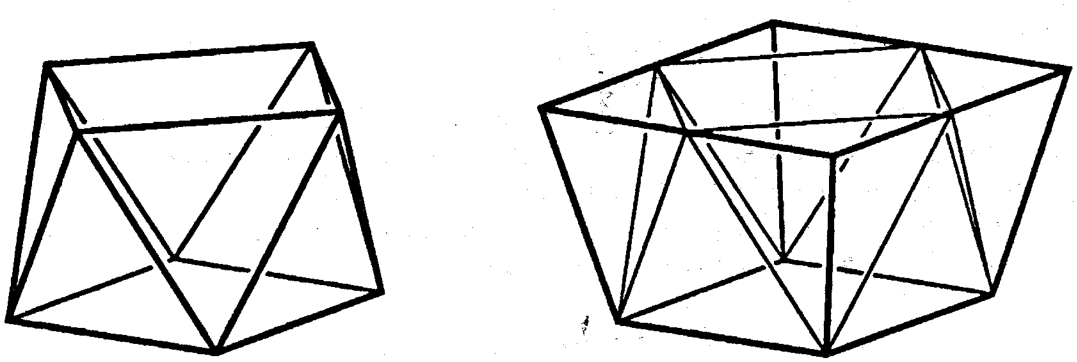

Figure 6.

We may obtain such a figure by removing four triangle-based pyramids from a frustrum of a pyramid.

We can also decompose this figure another way which leads to the solution of the original problem (**Figure 7**).

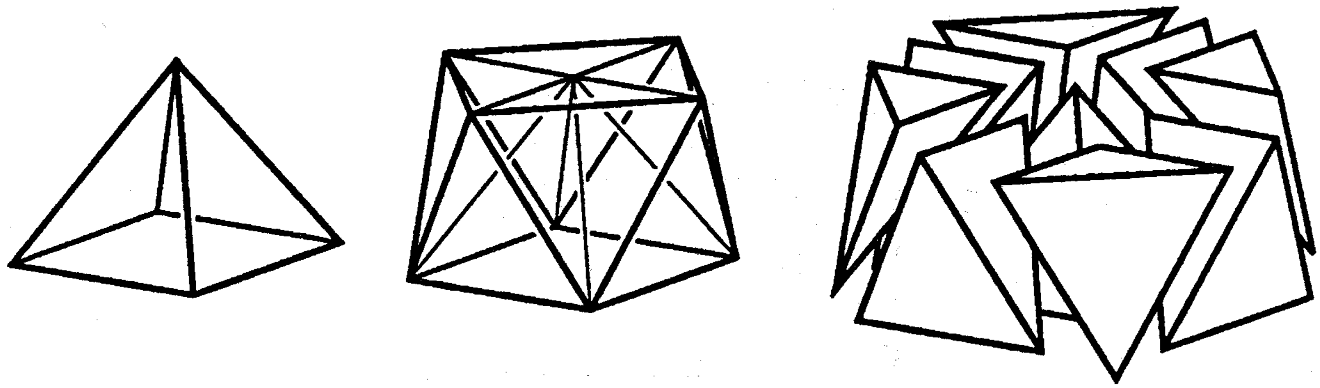

Figure 7.

If we remove from the figure the square-based pyramid with the center of the top square as its apex, we can decompose the remaining figure into the four triangle-based pyramids with their apices at the vertices of the lower square, and four tetrahedra with one edge in each of the two parallel planes, lying in perpendicular directions. The entire volume is then,

$$\frac{1}{3}s^2h + 4\cdot\left(\frac{1}{3}h\right)\left(\frac{r^2}{4}\right) + 4\cdot\frac{1}{6}\cdot h\cdot s\cdot\left(\frac{r}{\sqrt{2}}\right)$$

$$=\frac{1}{3}(h)(s^2+r^2+\sqrt{2}\,rs).$$

Practically the same decomposition works for the Air France Cup. We note that the planes through the sides of the square base which just touch the circular top will contain triangles lying in the boundary of the convex hull.

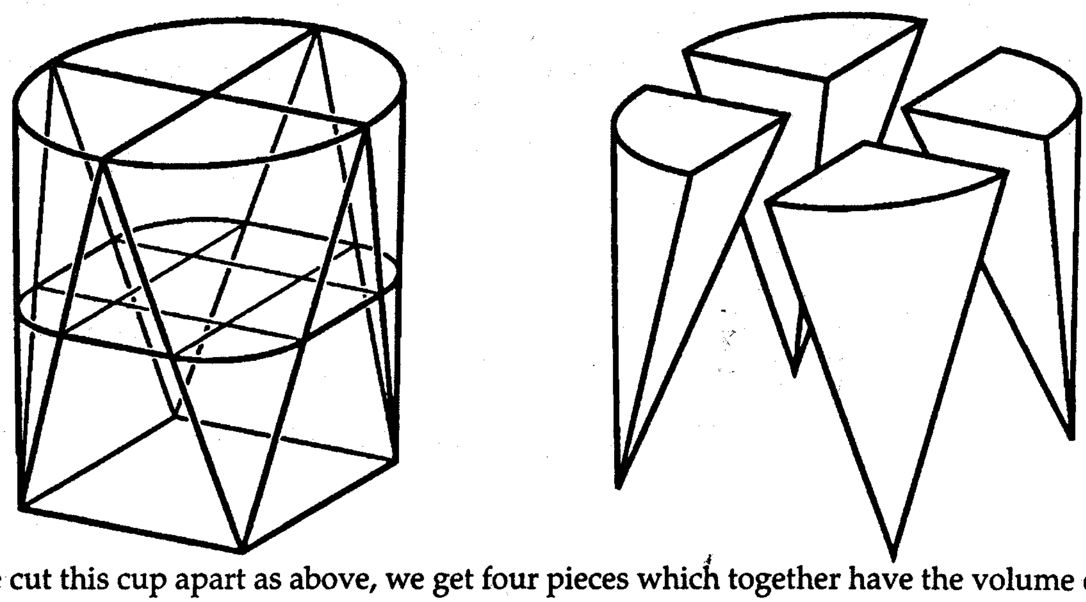

Figure 8.

In fact, if we cut this cup apart as above, we get four pieces which together have the volume of a circular cone, and the remaining pieces are the same in the Square France Cup. Thus, the volume is $(1/3)s^2h + (\pi/3)t^2h + 4(1/6)(tsh) = (1/3)h(s^2 + \pi t^2 + 2ts)$ (**Figure 8**).

In the previous diagrams we can also see the answer to the shape of the halfway slice.

We have four quarter-circles each with radius $t/2$, contributing a total area $\pi(t/2)^2$, together with a square of side length $s/2$ and four rectangles of side lengths $s/2$, and $t/2$. Thus, the area of the middle slice is $\pi t^2/4 + s^2/4 + st$ (**Figure 9**).

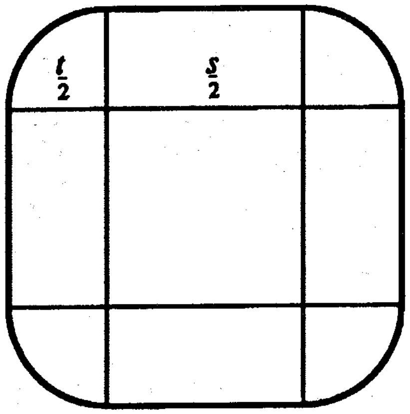

Figure 9.

So, after a long process, we have solved one particular geometric problem. In that process, we have used any number of important geometric techniques, and it is possible to continue these ideas in several different directions, to study formulae for different shapes of the top and bottom for example, or to generalize to cases where the planes of the two figures are not parallel. It is also possible to consider generalizations of this result to higher dimensions! The methods used up to this point have been almost entirely synthetic, and indeed they could be used already in secondary school or lower. A small bit of trigonometry makes it possible to extend the result for a tetrahedron to the case where the opposite edges lie in lines with directions making an angle θ (leading to the formula $1/6\, hrs \sin\theta$). We can introduce coordinates, and begin to write volume formulae in terms of coordinates, again with generalizations for higher-dimensional space.

The Air France Cup is a good example of an easily stated problem that leads to investigate a great many different geometric topics. The final formula is not so important as the process by which students learn to recognize various aspects of the problem, formulating related problems and seeing how the solution of a special case can lead to a general method. The process of generalization appears to be quite natural. Students can grow to appreciate the way mathematicians approach geometric phenomena, and that is one of the greatest insights we can hope to convey.

The illustrations for this article were rendered by Davide Cervone, using the program Aldus Freehand on a Macintosh computer.

References

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