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Hypergraphics

Visualizing Complex Relationships in Art, Science and Technology

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Real-Time Computer Graphics Analysis of Figures in Four-Space

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Real-time interactive computer graphics provides the opportunity for a research mathematician to investigate directly the geometric properties of curves and surfaces as they undergo transformations in 3- and 4-dimensional space. In this paper we will describe two films containing examples which indicate some of the power of this method and which illustrate the ways in which the graphics facility aids in the visualization of complex geometrical relationships. At the end of the paper, we give a brief technical description of the equipment that makes this sort of investigation possible and accessible.

The first film is entitled The Hypercube: Projections and Slicing. The second is Complex Function Graphs: Squaring and Exponential Functions. The first film examines one of the most basic and best understood figures in geometry, the square and its higher dimensional analogues. The four coordinates (-1,-1), (-1,1), (1,1), (1,-1) describe the square in a 2-dimensional coordinate system and the eight coordinates (±1,±1,±1) give the vertices of its 3-dimensional analogue, the cube. Although we can represent the square all at once on the plane surface of a computer graphics terminal, we can only present pictures of a cube by projecting to a chosen plane either by straight on or orthogonal projection or by central projection from a given point. No single view gives the entire picture, but by rotating the configuration, or, what is equivalent, by continually changing the position of the plane to which we project, we obtain a sequence of pictures which we can then

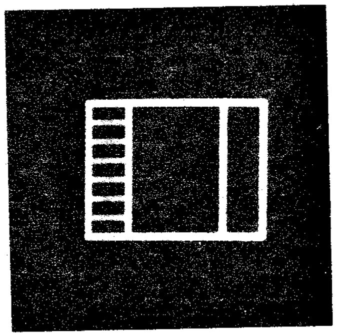


Figure 1

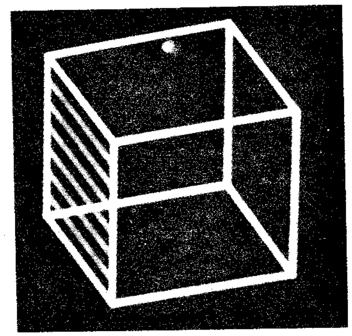


Figure 2

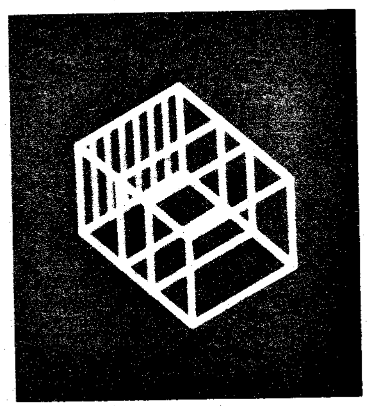


Figure 3

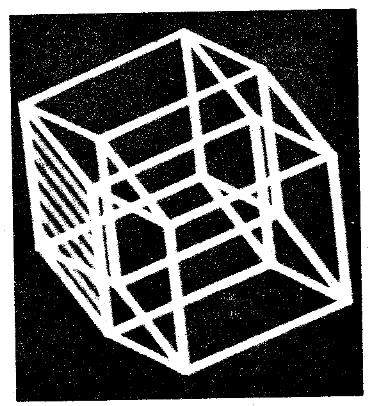


Figure 4

present as a motion picture film. With a machine that is fast enough to compute many pictures per second, it is possible to turn a dial or manipulate a "joystick" airplane-type control and have the figure respond by rotating about various axes in 3-space, thus producing a collection of images on the screen which we readily perceive as the shadows of a revolving cubical framework in 3-dimensional space.

Certain rotations produce more information than others, and some rotations are special enough to produce a high degree of visual ambiguity which is resolved or clarified only by a subsequent rotation. For example, the straight-on view of a cube looks like a square and then, as it begins to rotate about an axis parallel to one edge, like a collection of squares or rectangles sliding through one another, an effect well-described as the "revolving door illusion". (Figure 1) A rotation of 120 degrees about the vertical followed by 60 degrees about the horizontal produces an image which contains six parallelograms but which we readily interpret as the images of the six identical square faces of a cube. (Figure 2)

All of these ideas extend to four dimensions. Instead of using pairs or triples of real numbers, we specify the coordinates of a point by giving four numbers. The corners of the 4-cube are then given by $(\pm1,\pm1,\pm1,\pm1)$ so we get 16 vertices. We have to describe which pairs of vertices are to be connected by edges but this is accomplished by using the same rule that works for the square and the 3-cube, namely, we join two corners by an edge if and only if the coordinates are different in exactly one of the components. Thus we join $(-1,-1)$ to $(-1,1)$ but not to $(1,1)$, we join $(-1,1,1)$ to $(-1,1,-1)$ but not to $(1,-1,1)$, and we join $(-1,1,-1,-1)$ to $(-1,1,1,-1)$ and $(1,1,-1,-1)$ but not to $(-1,1,1,1)$ or $(1,1,1,1)$. Just as two edges come from each vertex of the square, we get three edges from every vertex of the 3-cube and four from every vertex of the 4-cube. We can use this fact to count the numbers of vertices in each case, since for example we have 8 vertices for the 3-cube, each with 3 edges, yielding 24. But this counts each edge twice, so we end up with 12 edges for the 3-cube. Similarly we get $(16 \times 4)/2 = 32$

edges for the 4-cube.

Naturally it is possible to analyze these combinatorial configurations in great detail and to proceed to generalize them to arbitrary dimensions, and indeed it is this manipulation of analytic geometry concepts which most often characterizes higher dimensional geometry. For four dimensions however, it is still possible to gain a considerable amount of geometric intuition visually, by interpreting projections of the vertices and edges of the 4-cube into 3-dimensional space. For over a century geometers have drawn pictures and constructed models of such projections, with varying degrees of success in communicating insights into the geometry of an object which cannot be fully described by any model in 3-space. The power of the computer graphics approach is that we can take any projection into 3-space and manipulate it by rotations and projections as if it were a wire-frame model in some room which we were viewing through the window provided by the television screen of the graphics terminal. But in addition to these rotations about an axis in 3-space, we can also turn a dial that effects a rotation in 4-dimensional space and we can watch the effect on the images of the projections into 3-space and then down to the 2-dimensional screen. Occasionally we will project two slightly displaced images which we then observe using stereoscopic viewing apparatus, but in general we achieve enough of a 3-dimensional effect by having the figure rotate slowly and letting the motion clues provide the illusion of depth.

For the 4-cube again there are certain rotations which are difficult to understand fully at first because the figure is not sufficiently unfolded in the 3-space into which it is projected. The first rotation produces the 4-dimensional analogue of the revolving door illusion for the 3-cube, a set of sliding cubes which stop after a 120 degree rotation to resemble the framework of a box kite. (Figure 3) In the film one cube is colored red and the opposite cube is green, so it is easier to fix attention on a particular part of the rotating 4-cube. This is especially useful as we proceed to rotate in a different direction to get a figure in which it is possible to see all 16 vertices that we expect on the 4-cube and all 32

edges. (We have 12 edges each on the red 3-cube and the green 3-cube, and eight more edges joining their corresponding vertices.). (Figure 4)

This figure, rotated in two different ways in 4-space, is still flat in one direction, which we can see if we rotate so that the image of one of the 3-cubes in the boundary is seen to lie in a 2-dimensional plane. A further rotation in 4-space then gives a projection which has a truly general position, with four edges at each vertex no three of which lie in a plane. (Figure 5)

It is possible to view the same sequence of rotations using central projection rather than orthographic, so that we get a perspective effect, with the back face smaller than the face in the foreground. (Figure 6) Rotating one of these central projections of the 4-cube produces some of the best effects in this Hypercube film. (Figure 7)

Another techniqe which is extremely effective in analyzing figures of high complexity is the technique of <u>slicing</u>. We choose a collection of parallel planes and we show what the successive slices are as the planes move across the figure.

In 2-dimensions, we slice a square by a line and we get a family of parallel segments, all of the same size if the slicing line is parallel to an edge and growing and then decreasing segments in the case where the slices are perpendicular to a diagonal. (Figure 8)

In 3-dimensions if we slice a cube square first we get a family of parallel squares, and edge first gives rectangles growing and then decreasing back to an edge.

Most interesting is the set of slices corner first, perpendicular to a long diagonal. An exercise in spatial visualization which confuses many students is to ask what the slice is which occurs half way through. The answer is provided by the picture. (Figure 9) Halfway through we cut each of the six squares on the boundary of the 4-cube exactly the same way and we get a regular hexagon.

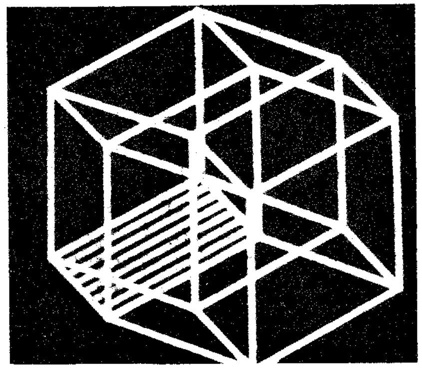
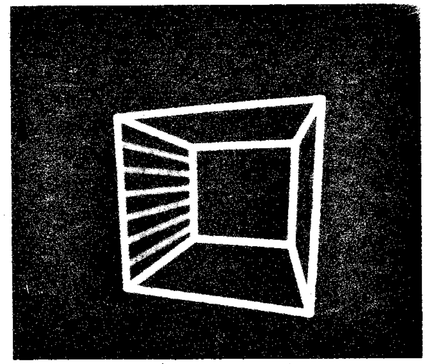


Figure 5



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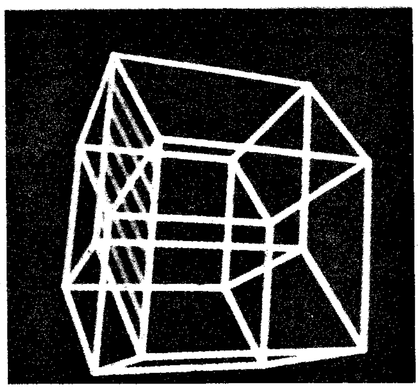

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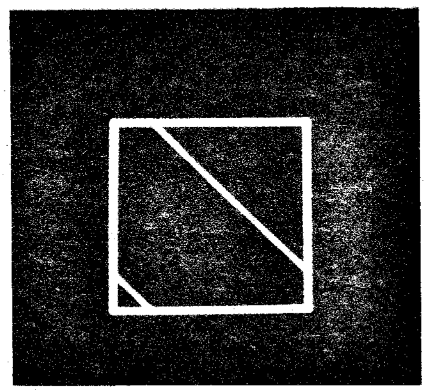

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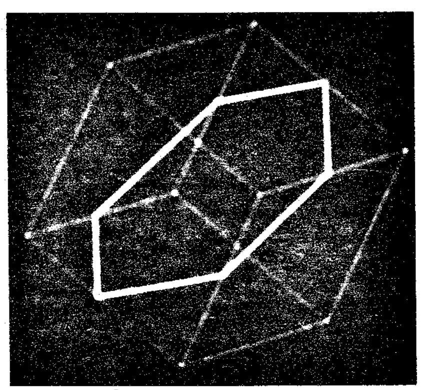

Figure 9

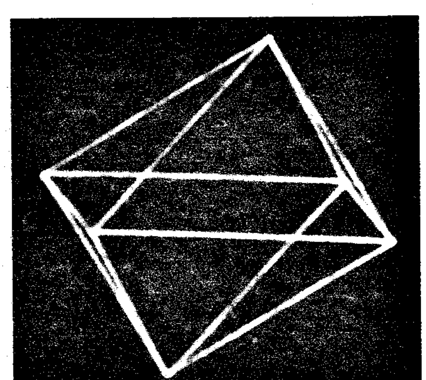

Figure 10

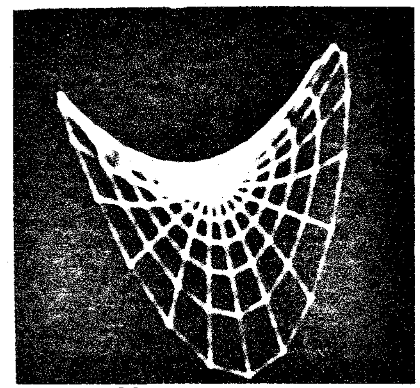

Figure 11

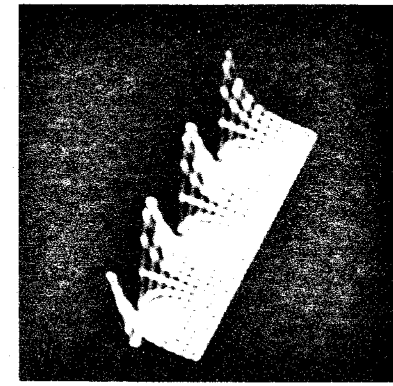

Figure 12

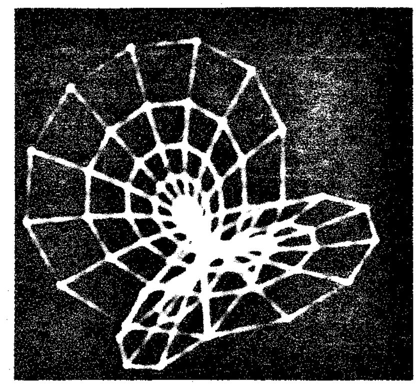

Figure 13

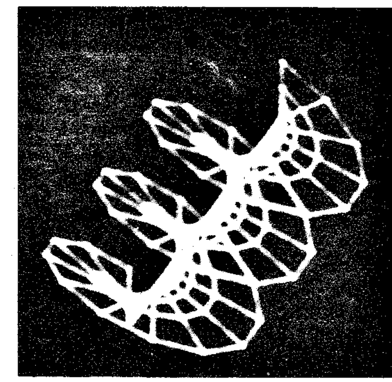

Figure 14

Exactly the same sort of program enables us to analyze the 3-dimensional slices that are produced by a family of 3-dimensional hyperplanes as they move across a 4-cube in 4-dimensional space. We obtain a family of cubes when the slicing hyperplanes are parallel to one of the 3-cube faces of the 4-cube, and a set of square prisms beginning and ending at a square if the slicing hyperplanes are taken parallel to one of the square faces. If we slice edge first, we get a set of triangular prisms, then hexagonal prisms, then triangular prisms, ending at a segment. Again, most interesting are the slices we get corner first. (Figure 10)

The slice begins with a point then moves to a small tetrahedral pyramid which grows until its corners become cut off thus producing a truncated tetrahedron. Three-eighths of the way through the cube we obtain an Archimedean semi-regular polyhedron with four equilateral triangles and four regular hexagons. Half way through we meet each of the eight bounding 3-cubes exactly the same way and the 3-dimensional slice is a regular octahedron.

The techniques of projections and slicing can be used especially effectively in the study of complex function graphs. We write the complex function $w = f(z)$ in terms of its real and imaginary parts, $w = u + iv$, $z = x + iy$, so $u = u(x,y)$, $v = v(x,y)$. We then enter the graph as a parametrized surface in Euclidean 4-space, $(x,y, u(x,y), v(x,y))$. We can project into the 3-dimensional subspace $v = 0$ to display the real part of the complex function, then rotate in 4-space to transform the image into the imaginary part, in the subspace $u = 0$. Rotating another way in 4-space, to the subspace $y = 0$, we transform the graph of the real part of the function into the graph of the real part of the inverse relation, which we can then rotate to display the transformation from the real to the imaginary part of the inverse relation.

In the film, we display the function graphs of $w = z^2$, or $u = x^2 - y^2$, $v = 2xy$ (Figure 11) and of $w = e^z$ or $u = e^x \cos y$,

$v = e^x \sin y$. (Figure 12) In each case the projections provide a visualization of the Riemann surface of the function, as well as the Riemann surface of the inverse relations, the square root (Figure 13) and the logarithm. (Figure 14) This film of the exponential function has also been described in [1].

The intersections and singular points of these projected surfaces provide some of the most significant information related to the geometry and topology of surfaces in 4-space. In particular, the origin of the graph of the square root relation is an example of a "Whitney umbrella", studied in the theory of normal singularities and normal characteristic classes. Such applications to research in geometry and topology are the subject of an address to be given at the International Congress of Mathematicians in Helsinki, August 1978, and details will appear in the Proceedings of that Congress.

Production Notes--These films were produced on a Vector General Scope attached to a Digital Scientific Meta 4 computer, augmented by a parallel processor, the SIMALE, developed at Brown University. Copies of the film for sale or rental are available from Banchoff/Strauss Productions, Brown University, Providence, Rhode Island.

References

[1] Banchoff, T. and Strauss, C., Real Time Computer Graphics Techniques in Geometry, Proceedings of Symposia in Applied Mathematics, Vol. 20, American Mathematical Society (1974), 105-111.