Computer Laboratory Magnification of Idiosyncracies

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Mathematics teachers exhibit at least three recognizable personality types: numerical, algebraic, and geometric. Preferences for one or the other of these will affect the way any course is taught, and the course in which the differences often show up most clearly is calculus. The differences in approach are particularly evident when calculus courses are augmented by computer laboratories. Such laboratories do give opportunities for new insights into calculus, but along with these benefits, there is a danger that some of the valuable aspects of older courses will be discarded too swiftly. In this fairly personal essay, I would like to discuss experiences with computers and calculus stretching back over twenty years, indicating some of the changes that have taken place, describing some successful projects as well as some dead ends, and giving a caution about the kinds of calculus courses we are developing as we respond to the technology of the computer laboratory.

With the advent of pocket calculators, the numerically minded teachers had a field day, producing a great many examples that have found their way into a variety of standard texts. Almost any computer laboratory will give students access to considerable calculating power, so mathematicians with numerical tendencies have the opportunity to introduce even more examples and topics, bringing numerical analysis techniques into the earliest calculus courses. In recent years, as a number of algebraic routines have become available on reasonably small machines, the teachers who have always stressed algebraic manipulation have recognized the chance to incorporate larger numbers of algebraic and algorithmic techniques into elementary courses. And recently sophisticated computer graphics programs have appeared for producing graphs of functions or families of functions, and for manipulating images on screens in computer laboratories. Naturally these have invited instructors with a strong geometrical bent to bring graphing technology into beginning courses.

Only in rare instances will the inclusion of new techniques shorten the amount of time given to a topic. More often, in order to make room for these innovations, some topics have to be substantially reduced or even eliminated. It is important for mathematicians of all persuasions to take part in the discussions that shape a particular calculus course, especially when it involves a computer graphics laboratory. This is the only way to prevent a course from becoming too slanted in the direction of numerical or algebraic or geometric emphasis. There might be a time in the future when a new race of teachers and students are equally at home with all three aspects, but until then it seems that we are in for an even greater divergence in the appearances of the computer laboratory component of our calculus courses.

The Fall of 1990 marks the twentieth anniversary of the first computer calculus course I ever taught. During this time we have tried several different approaches in calculus and differential geometry courses, primarily at the third semester level and above, as reported in [1] and [2] . During the 1990-91 academic year, I will teach a lecture course in second semester calculus, so in this essay I will concentrate on topics that might appear in a laboratory associated with such a course, with the hope that much of what I say will apply equally well to other levels.

Frontier Experiences

Within a year after I had arrived as a young assistant professor at Brown University in 1967, the chairman of the mathematics department asked if I would be willing to participate in a two-year experiment using computers in an introductory calculus course. For the first year of the project, the lectures were to be given by Professor Philip Davis of the Division of Applied Mathematics, and the supplementary computer lectures would be given by my colleague Charles Strauss, with whom I had already begun to work on computer graphics films in differential geometry research. In the second year of the project, I would give the mathematics lectures and Charles would once again handle the computer aspect. I agreed, and I learned quite a bit. In particular I learned the difference between pure and applied mathematics.

In the first semester, students were introduced to a number of natural topics: bisection routines for finding roots, Newton's method, numerical limits of sequences and difference quotients, and various approximations of definite integrals. As he was making up the final exam, Phil Davis included a question asking the students to estimate the integral of the square of the cosine from minus pi to pi by using three techniques: left-hand endpoint Riemann sums, the trapezoid rule, and Simpson's rule, all with error estimates. I said, "And then you'll ask them to evaluate the integral and see how close their answers are to the exact value?" "No," he explained, "because that would give them the wrong idea. The integrals that they will have to deal with in the real world won't have closed-form solutions, so they shouldn't be looking for them. They should not have to worry about techniques like trigonometric identities that they will never have to use." I tried to defend classical topics in methods of integration by pointing out that one of the ways to do that particular integral was integration by parts, a technique that any applied mathematician would have to agree was a valuable one in the real world. He agreed with me in principle, but the problem was not changed, the first year anyway. The next year, when I taught the course, in addition to the numerical techniques, students were expected to solve the problem explicitly using three different techniques of integration. But I did feel bound to tell the students that there were professors who felt the such algebraic exercises in techniques of integration have been made obsolete by the computers we were just beginning to use.

In those days the graphing programs were truly rudimentary, often consisting of a series of asterisks printed out at different heights on a page to indicate the positions of function values. In a sense, that gave a better idea of what the computer was actually doing than some of the more slick routines of today that draw very smooth curves even though they are using the same finite amount of information. As a visual person, I found myself stressing the graphical aspects of calculus, just as I had before the computer was available. I was quite struck by the way that the differences between my teaching approach and that of my colleague in applied mathematics were accentuated by the way we made use of the computer aspect.

As it happens, the two-year experiment was not sufficiently successful to be continued after the seed money ran out in the late 1960s. In those days, there were not many computers available, and it was necessary to hire student consultants for a number of hours each week to help students deal with a comparatively unwieldy programming language and somewhat unreliable terminals. Some students with computer experience found the additional instruction boring; the majority found learning elementary programming difficult and not particularly helpful in understanding calculus. The time was not yet ripe.

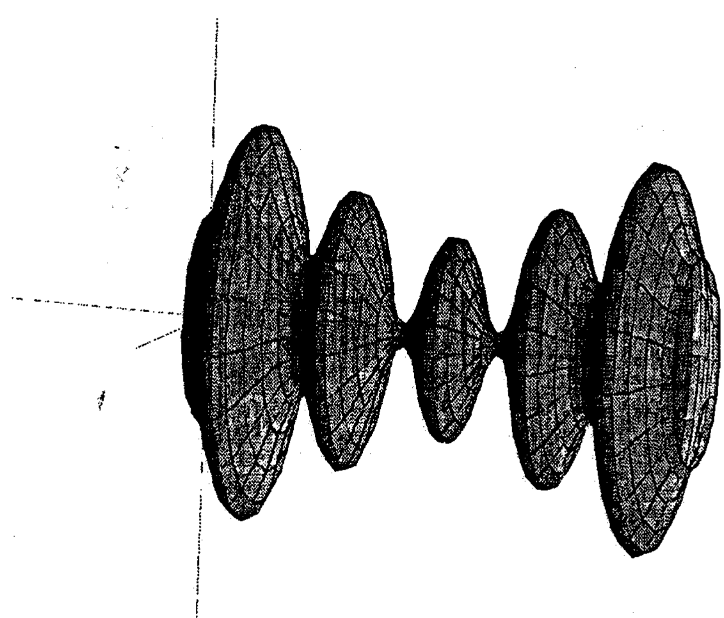
Recent Laboratory Developments

Much has changed in the ensuing twenty years. These days, nearly every student enters calculus with enough exposure to word processors to be able to handle pull-down menus and command sequences immediately. Readily available software packages make it possible for the students to investigate a large number of phenomena as soon as they know what a program does, without having to go to the trouble of learning how to write programs from scratch. The logistical details that loomed so great two decades ago are for the most part forgotten.

Speed of response is much greater that it used to be, although the increase is not so noticeable in the fairly uncomplicated problems of first-year calculus (often the same problems that we used in the pioneer days). A dramatic change is the enhanced quality and speed of the graphics displays, with high resolution and/or color adding immeasurably to the effectiveness of the images that students can now explore even with relatively unsophisticated machines. Three-dimensional display techniques are now widely available and easy to use, leading to vivid presentations of topics like volumes of revolution, one of the very first and most successful examples of visual display. As it happens, some of these earliest examples are important to remember. They should not be discarded too rapidly in our rush to take up new ideas.

Revolution, Si! Volumes of Revolution, No?

Early in the summer of 1990, I participated in a site visit to review the first year of operation of a carefully conceived calculus curriculum project incorporating numerical and graphical techniques with symbolic manipulation routines. Since it would obviously take a certain amount of course time to bring in all these new concepts, I asked what topics in the traditional course would go by the boards to make room. The presenter quickly offered up as a sacrifice the subject of volumes of revolution. I was told that some



of those who discuss calculus reform generally consider this topic expendable. Someone recalled a slogan from a calculus curriculum session: "Revolution, Si; Volumes of Revolution, No!" I begged to disagree. I have always considered it a high point when it was possible to use calculus to prove the correctness of a manifestly useful formula like the volume of a cone or the volume of a sphere, something which, up to that point, had merely been a memorized fact. I have never found students unappreciative of Pappus' theorem, as they learn to calculate the volume of a figure of revolution in terms of the area of its cross-section and the distance travelled by its center of gravity. I count on the fact that students already appreciate this theorem when I generalize it in the study of tube domains in differential geometry. Anyone who works in CAD-CAM recognizes the importance of figures of revolution as primitive objects for geometric modeling. I do not appreciate the suggestion that volumes of revolution be removed to make room for symbolic manipulation packages for algebraic techniques of integration. If there are those who want to replace one of my favorite applications with something more "relevant", I am prepared to challenge them.

This strong defense of volumes of revolution exposes my prejudices if not my idiosyncrasies. I know what topics I liked when I took calculus, and I know which ones have continued to show up in my research and teaching at higher levels. These are the calculus topics I present with excitement, one of the chief characteristics of any course likely to engage students. Each professor has his or her own list of such topics. Having a computer with strong graphics capabilities encourages me to develop geometric topics even further. Having symbolic manipulation software leads some of my algebra colleagues even further away from where I think the course should go. The same can be said for the number manipulation facilities of machines, leading to more problems featuring numerical approximation. The machines will accentuate our differences before they bring us all together. There isn't time in the curriculum to take advantage of all the enhancements that could occur if we exploited fully the symbolic, the numerical, and the graphical aspects of computers in the teaching of elementary calculus. Until there is general agreement about which topics should be left aside to accommodate even an abbreviated version of these three basic thrusts, we will experience some frustrations, as will our students. We can be assured that twenty years from now the students and teachers of the future will consider our dilemmas completely irrelevant, and they will be dealing with problems undreamed of today. The one thing that is safe to predict is that there will still be problems and differences of opinion.

Various Laboratory Models

All this being said, there are a number of comments that can be made based on our experiences over the past twenty years. The major difference between the first time I taught calculus with computers and the present situation is the availability of laboratories filled with machines. In the old days, computer assignments were in reality homework problems, done individually by students who had to find the best times to avail themselves of scarce resources. Any cooperation between students was purely incidental. All that changes in the modern laboratory setting.

Over the years, we have tried a number of different models for laboratory interaction, keeping some aspects and rejecting others. A list of various approaches might be useful to others who are developing their own strategies.

1) The Science Lab--Based on traditional biology or physics laboratories, this model features setup demonstrations, followed by carefully laid out experiments to be recorded on standardized "lab report" forms. Students work singly or in pairs during a set period, and are graded on the accuracy and completeness of their reports. In this model, everyone is supposed to come up with essentially the same results, and sharing of experiences is of minor significance. Assignments are like standard homeworks, with fixed answers that the grader already knows. Success in this model depends on the quality of the materials used. The exploratory aspect is minimized, leading to greater efficiency at the expense of spontaneity.

2) The Language Lab--Here students are given interactive drill materials, with feedback to indicate how well they are mastering pronunciation and reading and listening comprehension. Students work individually, at their own pace, through standardized materials, possibly with the assistance or intervention of instructors who can listen in on what is happening. Once again, this model requires careful preparation and selection of materials. It may be most effective in remedial situations, especially when students in a class have widely differing backgrounds.

3) The Mathematical Video Arcade--The video game approach seems well suited to a certain kind of mathematical competition, solving problems very quickly. This approach treats mathematics rather like chess, so students study certain kinds of problems the way they would pour over books of openings or gambits or endgames. High performance in such specialized competitions may be related to mathematical talent, but not necessarily. Eye-hand coordination may or may not be an essential element

of the experience. Logical puzzles may be particularly well adapted for this kind of presentation, although it does not seem especially well suited to communicating the ideas of calculus.

4) The Studio Art Class-- After an initial presentation by the laboratory leader (possibly the instructor but more likely a student assistant), students singly or in pairs investigate examples of a particular phenomenon, formulating and testing conjectures. Near the end of the session, students are invited to present their best insights to the class as a whole. If the group is small enough, members of the class can circulate and see the images or animations prepared by each of them. A leader prepares a "critique", responding to each of the pieces, stressing the positive aspects and making suggestions for improvement. Such a laboratory works well when coordinated with standard lecture presentations, especially when the conjectures that arise from the lab investigations can be treated in a formal way in the class. Selection of topics is quite important, providing a sufficiently rich set of phenomena while at the same time providing some guidance so the class is not overwhelmed. The looser structure of exploration in a laboratory based on such a model is harder to lead, but the results can be quite dramatic when it succeeds.

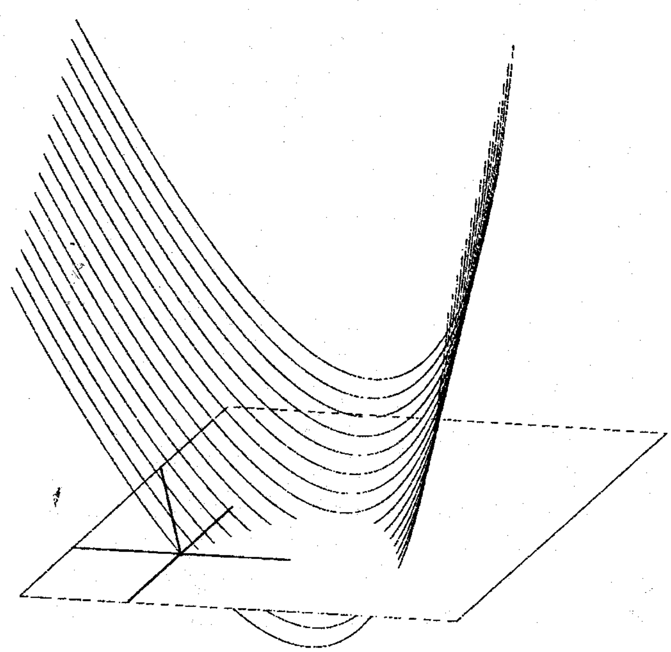
Communicating Results--Saving and Hard Copying

Effective implementation of laboratory experiences demands some way for students to save their work, for transmission to the instructor, and, to a certain extent, to the rest of the class. Ideally students should be able to save not only numerical results and static images but also image sequences, even animations, showing how a given example can be modified to yield families of examples. Students should be able to attach notes to files, indicating how to interpret the images which it generates. In some cases, a printed image is sufficient to illustrate a given discovery or to motivate a conjecture. For these, there should always be some facility for hard copies of material on graphics screens, although there should be some restraints on the number of images saved in printed form. Even better is a procedure for saving an entire process, so that the instructor or another student can enter the program and continue investigating it. In this way, the explorations developed by members of the class can serve as the basis of later laboratory topics or theoretical discussions in the classroom.

Follow-Up Opportunities

Students should be polled more than once during a term about the ease of using existing programs and about "wish lists" of features desired in future versions of the programs. This is important feedback for those responsible for the maintenance and improvement of the software, and it often suggests new directions for subsequent courses. Two years ago, after I had requested such a list at the end of a third semester honors calculus course, four of the best students petitioned to continue in a semester-long group independent study so that they could implement a number of suggestions that they and other students had formulated at the end of the course. That independent study led to substantial improvements in the program we were using, and paved the way for the current version, which is coupled with another newly developed senior honors program for the study of curves and surfaces in introductory differential geometry.

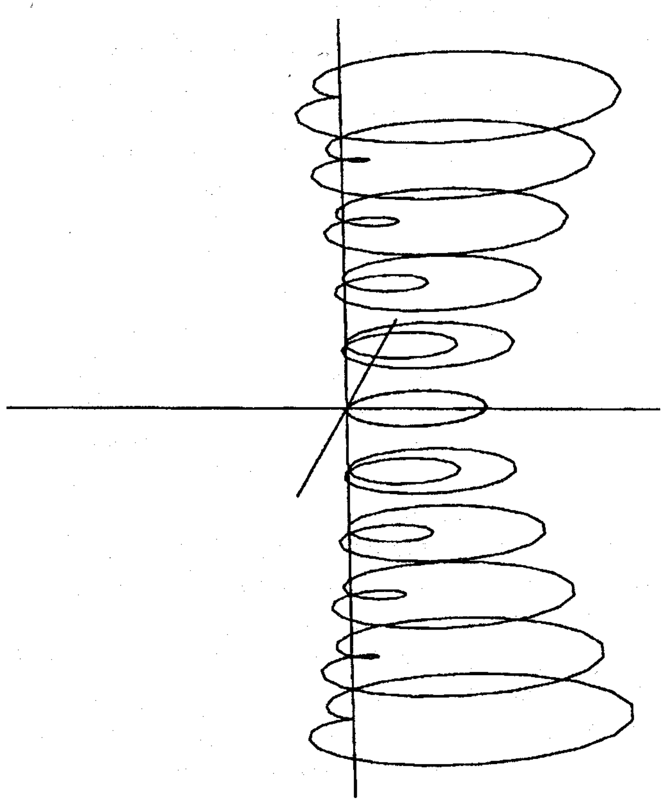
It is important to give students an opportunity to follow up their accomplishments in one course with the chance to make further contributions in the future. This is one of the big advantages of computer graphics as a subject, with its encouragement of group activities and the relative accessibility of funds for the support of summer research work with computer research teams. As mathematicians, we are probably best advised to try not to woo students away from computer science, but rather to encourage them to continue to take mathematics courses along with their computer studies. There is enough evidence around that mathematical experience enhances the ability to be accepted at top-ranked programs in computer science in the United States and in other parts of the world. We do well to urge students to continue mathematical studies no matter what may be their eventual concentrations.



Investigating Families of Examples

Over the years, we have found that laboratory experiences are especially effective when they treat not collections of unrelated examples but rather families of examples. How many zeroes does the function $f(x) = x^2 + 2x + c$ have? The answer depends on c, and as c varies, we get an entire collection of answers. A closely related question is to find the domain of $1/(x^2 + 2x + c)$, and to sketch the graphs of the different functions in this family for different values of c. A graphics computer can be especially useful in indicating this dependence. We can also analyze the integral of f(x) from -1 to 1 and consider the dependence of the answer on c. We can then go on to investigate the one-parameter family of functions $f(x) = x^3 + cx$, or the two-parameter family $f(x) = x^4 + cx^2 + bx$. Students can decide what are the crucial properties to look for, and then determine what are the relationships between c and b at the exceptional positions at which the qualitative behavior changes.

We can apply the same approach to study other types of families, for example families of polar coordinate function graphs. The equations $r(\theta) = \cos(\theta) + c$ determine very different curves depending on the nature of c. More gener-



ally in the study of parametric equations, we can investigate what happens as we vary the value of c in the parametric curve $x(t) = t^2$, $y(t) = t^3 - ct$ for various values of c.

We can also use parameters in the exponent, as for example when we look at the integral of the function $f(x) = x^{(-1+a)}$ from 1 to 2. We can take the limit of this function as a approaches 0, to find an expression for the logarithm of 2. Naturally the dependence on the constant might not always lead to a good limit, as we can see by considering a family of functions which is linear from O to a and from a to 2a, with $f(x) = 0$ if $x \geq 2a$, $f(0) = 0$, and $f(a) = 1/a$. We have $\lim f(x) = 0$ for all x as a approaches 0, but the integral of f from O to 1 approaches1, since this integral equals 1 if $a > 0$.

Kinesthetic Response

Students who have grown up in the era of video games are quite aware that it is possible to make shapes move around on a screen in response to analogue input devices. In our programs, we use either numerical values entered from a keyboard, or selected from some preset options from pulldown menus, using a mouse or light pen, or by manipulating slider bars with buttons or dials. With a fast enough machine, we get close to real-time response, and the student is able to feel the effect of the parameter changes. Such display does have great advantages over chalkboards, or even overhead projectors. If we are studying a one-parameter family of functions, for example, we want to be able to see graphs of several members on the same coordinate system. If the graph moves from one position to another as the operator changes the parameter, all the better. We should always prefer the kinds of problems that can be done best by a given program and a given machine. For laboratory use, a program should give a response in not more than thirty seconds. This is not to say that certain projects might not take considerably longer than that, but such projects are better relegated to homework problems so they do not disrupt the flow of the class activity. In an actual research laboratory, students may sit for several minutes or longer in a nearly trancelike waiting state, and although such patience is a requirement for successful computer science, it does not seem desirable as part of a laboratory for an elementary calculus course.

Aims of the Calculus Course

Every once in a while we have to ask ourselves what we hope to accomplish by handing to the students tools that will enhance their abilities to do algebraic manipulations, numerical calculations, and graphs. It is all well and good to provide these means of augmenting what the students already have, but the results will not be very good unless the students already have a rudimentary idea of what is going on. We all know that it is deficiencies in algebra that often cause students the most difficulty in coming to grips with calculus. Any student can learn how to differentiate a polynomial, but in order to solve an elementary maximum-minimum problem, it is necessary to find the zeros of the polynomial. We know that a number of students have difficulty with such problems even when the original function is cubic, so the technique for finding the roots of the derivative is a simple application of the quadratic formula. Giving a student a symbolic manipulation routine for solving such a problem is the equivalent to providing a calculator for the multiplication of two two-digit numbers or the sum of two simple fractions. It is true that students who can handle cubics are sometimes stumped when a quartic function leads to a cubic derivative, and the ability to recognize the one obvious root carefully arranged by the poser of the question is often the faculty that distinguishes the honor student from the rest of the pack. We might ask what will distinguish our students in the new dispensation, when we provide each of them with a computer that automatically finds roots of polynomials. One might hope that a student could take the information given by such root-finding programs and use it to solve word problems or graphing problems. But if the student is incapable of doing that for cubic functions, it seems unlikely that he or she will have much better luck with more complicated examples. It is true that the presence of a computer takes some of the pressure off the persons designing the examinations, since it is no longer necessary to make sure that the answers "come out even". In fact, it might be considered better when they don't, thus making them closer to that real world out there. But I for one would hate to see the old problems disappear. I'm fond of the examples that I have put together over the years, with first and second derivatives that are relatively easy to factor so that the maxima and minima are nice numbers and the graph fits nicely on a standard grid. Even with the luxury of a computer, we instructors will still have to exercise a measure of care in choosing the coefficients in our polynomials and our rational functions, to make sure that the answers are reasonable, i.e. consistent with some sort of a priori analysis of the problem without which students can be led into the most extreme sort of errors.

It is also true that some students have a great deal of difficulty learning to graph functions. It would be a serious mistake to lead them to think that now that graphic computers are around, it is no longer necessary to be able to graph simple functions by hand. Any student who goes to a computer to graph a quadratic function is the equivalent to a clerk who need the calculator to divide a number in half. I don't mind it when someone uses a calculator to find out the dollar equivalent of 37.20 in Dutch guilders at $.5489 per guilder, but it does bother me when I see someone using a calculator to solve that problem when the rate is $.50. And I do want to see my students come up with good looking parabolas on appropriately chosen grids, with straight tangent lines at given points. Only after that, have they earned the right to use the computer to do the same problem for a quintic.

So what are the algebra problems that I expect my students to do in calculus? I often give a simple quiz, without asking students to put their names on the paper so there is no individual pressure. I ask them to expand $(x + a)^3$ and to factor $x^3 - a^3$. A discouragingly large number of students miss at least one of those problems. Telling them that it is all right not to know how to do such problems because now there is a computer to do them is a cruel hoax. Once again, it seems a good idea to require a basic mathematical facility with simple calculations before giving a student a calculator, with simple algebraic techniques before providing a symbolic manipulator, and with simple graphing facility before providing a graphics routine.

This being said, I think that the graphing programs can be invaluable in topics like Taylor series, where the student can find the first two terms easily and then can set the machine to work to graph a number of polynomial approximations, seeing how they converge where they do and how they can fail to converge near endpoints of the interval of convergence. I don't know of anyone who draws enough approximating sums to handle that problem without a graphics computer.

What about integration? It seems strange that some teachers move wholeheartedly into using symbolic integration packages who previously would never let their students use a table of integrals, even the abbreviated one on the inside cover of the calculus book. Somehow the idea of looking for an integral in the table book was something engineers did, not mathematicians. But there is an art to using those tables, since you have to recognize what form the integral is in, and often it is necessary to manipulate the integrand a bit before it resembles one of the forms listed in the tables. That at least requires a bit of finesse. It takes less than that to invoke a command on a computer to integrate some rational function in closed form.

In a recent consultation, I was surprised to see as one of the premier examples of the use of symbolic manipulation a unit on partial fractions, one of the techniques of integration which I had always liked and which I had feared doomed to the waste can when techniques were deemed expendable. It is true that the main problem with such examples was algebraic--if you set the problem up correctly, you often ended up with two equations in two unknowns, and you had to know how to solve such things. In rare instances, it is possible to use the partial fractions technique on a rational function with a cubic or quartic denominator, but the number of difficulties is great. First of all, only the better students are able to do the factorization, and even if the denominator is presented in factored form, it requires some expertise to set up and solve a three-by-three or four-by-four system of equations, after which the results would have to be reinterpreted to express the integral as a sum of rational functions, logarithms, and inverse tangents. The whole purpose of such exercises was to enable the student to exhibit some finesse. The problems themselves were not especially interesting in terms of applications (except for the sole example of the epidemiological model), and few students ever seemed to appreciate the theorem that all rational functions could be integrated in this way, modulo the ability of the integrator to factor and solve linear systems. Now the computer provides us with expert algebraic assistance, and it will be interesting to see if students once again show an interest in integrating complicated rational functions. But I would still like to believe that anyone who uses such a machine could integrate "by hand" any rational function with a quadratic denominator.

The same holds true for graphing rational functions. You earn the right to use a sophisticated graphing routine for complicated rational functions by first showing that you can graph basic examples by hand. It's a simple message, but one that bears repetition.

It seems that a great variety of things that can be done with a computer are actually merely higher-exponent versions of things that many students never learn well enough in the elementary cases. It is not at all clear that the ability to confront quintics without flinching is going to make students better at understanding the ideas of calculus, which after all should be one of the primary long-standing goals of our teaching.

All that I have written above assumes that the purpose of the calculus course is to teach calculus, as opposed to algebraic manipulation, numerical analysis, or, for that matter, mechanical drawing or computer science. Computer laboratories offer us great opportunities for improving the quality of calculus courses, and by bringing the best parts of our mathematical personalities to this challenge, we can look forward to very stimulating experiences in the future, for ourselves and for our students.

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