Osculating Tubes and Self-Linking for Curves on the Three-Sphere

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Abstract. For a smooth closed curve, the osculating tube consists of the union of the osculating circles at points of the curve. For a closed curve on the three-sphere, the self-linking number of the image of the curve under stereographic projection from a point to the opposite hyperplane depends on the point, and the difference between the numbers for two points is related to the algebraic number of times that a curve from one point to the other meets the osculating tube of the curve transversely. The self-linking number of an \((n, m)\)-torus knot on the three-sphere is shown to be \(n m\), answering a question of N. Kuiper.

Introduction

Alfred Gray made many contributions to the geometry of tubes, quite often using computer graphics both for research and for illustrations of his books and papers. In this note, we will deal with a tube that he did not study extensively, connected with the osculating circles of a parametrized curve. For a curve in the plane, the geometry of the osculating tube gives an alternate proof of the well-known fact that the osculating circles of a spiral, with monotonic curvature, are nested, with no two intersecting (This result for plane curves is developed by Alfred Gray in [6], pages 111–112).

For a space curve, the osculating tube is connected with the behavior of self-linking numbers under conformal transformations, as studied earlier by the author and James White in [1]. In this article, we develop the geometry of this osculating tube for plane curves, then curves on the two-sphere, then curves in three-space, finally we concentrate on the osculating tubes of curves on the three-sphere in four-space. The main result relates the self-linking numbers of stereographic projections of a curve on the three-sphere from two points to the positions of these points relative to the osculating tube of the curve.

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The problem of determining the self-linking numbers of torus knots on the 3-sphere was first raised by Nicolaas Kuiper in his paper on hyperbolic 4-manifolds [8], page 75, following the work on the same subject by M. Gromov, H. B. Lawson, and W. Thurston [7]. In 1991, Kuiper discovered that there was an error in his argument, and communicated that information to the author. The proof of the self-linking formula presented here is basically the one constructed by the author at that time but not published previously. The relationship with osculating tubes is new.

Definition of the Osculating Tube

Let \( X(t) \) be a twice-differentiable curve in \( \mathbb{R}^n \) defined over the interval \( a \leq t \leq b \). Assume that the velocity vector \( X'(t) \) is never zero, with length \( s'(t) \). We define the unit tangent vector \( T(t) \) to be the normalized velocity vector, so \( X'(t) = s'(t)T(t) \).

Then \( X''(t) = s''(t)T(t) + s'(t)T'(t) \) where \( T'(t) \) is perpendicular to \( T(t) \). If \( T'(t) \) is not equal to zero, we define the Principal Normal \( P(t) \) and the curvature \( \kappa(t) \) by the condition \( T'(t) = s'(t)\kappa(t)P(t) \). If \( T'(t) = 0 \), we define the curvature \( \kappa(t) \) to be 0.

Consider now a curve \( X(t) \) for which the curvature \( \kappa(t) \) is never zero. In this case, we define the radius of curvature to be \( r(t) = 1/\kappa(t) \). For each \( t \), we define the osculating circle to be the circle in the plane of \( T(t) \) and \( P(t) \) with radius \( r(t) \) centered at \( X(t) + r(t)P(t) \). The collection of all of these osculating circles forms the osculating tube

\[
O(t, v) = X(t) + r(t)P(t) + r(t)(\cos(v)T(t) + \sin(v)P(t)).
\]

A straightforward computation shows that the partial derivative vectors \( O_t(t, v) \) and \( O_v(t, v) \) are linearly dependent when \( \sin(v) = -1 \) (where \( O(t, v) = X(t) \)) and also when \( r'(t) = 0 \) and \( P'(t) \) is a multiple of \( T(t) \).

Osculating Tubes for Plane Curves

In the case where \( X(t) \) is a plane curve, \( P'(t) \) will always be a multiple of \( T(t) \) so the singularities of the osculating tube occur on the curve itself, where \( \sin(v) = -1 \) and along the entire osculating circle at a point where \( r'(t) = 0 \). Points with this last property are called vertices of the curve.

Corollary: If \( r'(t) \neq 0 \) over an interval, then the osculating tube is an immersion of the cylinder into the plane, and the osculating circles are therefore nested. A smaller osculating circle is completely contained in any larger one.

Figure 1 illustrates this phenomenon for an arc of an Archimedean spiral.

For a curve with a finite number of vertices such that the sign of \( r'(t) \) changes for each one, the osculating tube can be expressed as the union of embedded cylinders bounded by osculating circles at vertices. Figure 2 shows one of the four pieces that fit together to make the osculating tube of a non-circular ellipse.

A limiting case of this phenomenon is given by the class of piecewise circular plane curves, called PC Curves in the papers of the author and Peter Giblin [3], [4]. Such a curve is given by a sequence of circular arcs, with the tangent line of each arc at its endpoint coinciding with the tangent line of the succeeding arc at its beginning point. The osculating circle at an interior point of an arc is the circle containing the arc, while the osculating circles at a node where two arcs meet are
the circles tangent to the tangent line at the node lying between the circles of the
two arcs at the node. In this case the osculating tube is a union of pieces, one for
each node, consisting of the region between the two circles at the node. The author
and Peter Giblin will treat PC space curves in a forthcoming article.

Osculating Tubes for Curves in Three-Space

For a smooth curve $X(t)$ in three-space, we may define the Binormal $B(t) =
T(t) \times P(t)$. Then a standard calculation shows that $P'(t) = -s'(t)\kappa(t)T(t) +
s'(t)\tau(t)B(t)$, where $\tau(t)$ is the torsion of the curve at $X(t)$.

The previous calculations then show that the osculating tube of a space curve
is singular precisely along the curve itself and when simultaneously $r'(t) = 0$ and
$\tau(t) = 0$. The latter condition will not occur generically for a single curve, even for
a one-parameter family of curves.

Figure 3 below shows a portion of the osculating tube of a circular helix.

Osculating Tubes for Curves on the Two-Sphere

One particular class of interesting curves in three-space is the spherical curves
$X(t)$ with $|X(t)| = 1$, on the two-sphere of radius 1 about the origin. It is a straightforward calculation to show that the osculating circles of a spherical curve lie on
the same sphere. This can be seen as well by projecting the curve stereographically
from a point $N$ not on the curve $X(t)$ to a plane curve $\pi_N X(t)$. The osculating
circles to this plane curve lie on the plane, and they are defined as the circles that
have order of contact at least three with the curve. The images of these circles

Figure 1.
under inverse stereographic projection go to circles on the sphere with the same order of contact with the original curve \(X(t)\).

The points of the spherical curve \(X(t)\) that have torsion equal to zero are precisely the vertices of the plane curve \(\pi_N X(t)\), where the osculating circle has contact higher than three. The osculating tube of \(X(t)\) will then consist of a union of annular regions on the sphere bounded by osculating circles at torsion zeros of the curve, where each such region is the image under inverse stereographic projection of the analogous regions in the plane.

Note that if an osculating circle of \(X(t)\) passes through the point \(N\), then the image of that circle is a straight line with higher order of contact with the curve \(\pi_N X(t)\), yielding an inflection point of the projected curve.

**Self-Linking Numbers for Curves in Three-Space**

For a space curve \(X(t)\) with curvature never zero, the *self-linking number* \(\text{SL}(X)\) of \(X(t)\) is defined to be the linking number of \(X(t)\) with the normal push-off \(X(t) + \epsilon P(t)\), where \(\epsilon\) is chosen sufficiently small that the linking number is stable, not changing for any smaller positive value. This notion was first developed by Calgareanu [5] in 1959, and independently discovered and significantly extended in 1968 by William Pohl [9] and his student James White. In 1974, James White and the author in [1] established a relationship between the self-linking numbers of the inverted images \(I_Q(X(t))\) of \(X(t)\), where \(I_Q\) represents inversion with respect to a sphere centered at \(Q\). Specifically, if \(Q(t)\) is a path that meets the osculating tube of \(X\) transversely, then \(\text{SL}(I_Q(1)X) - \text{SL}(I_Q(0)X)\) is the algebraic number of times
that $Q(t)$ intersects the (oriented) osculating tube of $X$. In this article, we will use an analogous argument to investigate self-linking numbers of curves projected stereographically into three-space from various points on the three-sphere.

**Self-Linking Numbers for Curves on the Three-Sphere**

Let $X(t)$ be a curve on the unit three-sphere in four-space, so $|X(t)| = 1$. Then as before, if $T'(t) \neq 0$, we have $T'(t) = \kappa(t)P(t)$. We may resolve this curvature vector as a multiple of the outward unit normal vector $N(t)$ to the three-sphere, namely $X(t)$ itself, plus a vector tangent to the three-sphere. If $P(t)$ is not a multiple of $X(t)$, then we may define the geodesic curvature $\kappa_g(t)$, a non-negative quantity, the normal curvature $\kappa_N(t)$, and a unit vector $U(t)$ in the tangent hyperplane to the three-sphere by the expression

$$\kappa(t)P(t) = \kappa_g(t)U(t) + \kappa_N(t)N(t).$$

Note that as in the case of a curve on the unit sphere in three-space, the normal curvature of a curve on the four-sphere is $-1$. This follows since

$$X''(t) = s''(t)T(t) + s'(t)^2\kappa_g(t)U(t) + s'(t)^2\kappa_N(t)N(t)$$

so

$$s'(t)^2\kappa_N = X''(t) \cdot N(t) = X''(t) \cdot X(t).$$
But \(X(t) \cdot X'(t) = 0\) yields \(X(t) \cdot X''(t) + X'(t) \cdot X'(t) = 0\) so \(s'(t)^2 \kappa_N = -s'(t)^2\). Therefore
\[
\kappa(t) P(t) = \kappa_g(t) U(t) - X(t).
\]
The geodesic curvature will be zero when \(P(t) = -X(t)\). In this case, the plane through \(X(t)\) spanned by \(T(t)\) and \(P(t)\) will pass through the origin, cutting out a great circle on the three-sphere which is the osculating circle at \(X(t)\).

If the geodesic curvature is non-zero for all \(t\), then we have a well-defined normal vector \(U(t)\) at each point of \(X(t)\) and we can define a push-off curve on the three-sphere by normalizing the push-off curve \(X(t) + \epsilon U(t)\).

As in the case of a pair of curves in three-space, two disjoint oriented curves on the three-sphere have a well-defined linking number given by the algebraic number of times that one of them intersects any oriented surface bounded by the other. We may compute this linking number by using stereographic projection from a point not on either curve to obtain a pair of curves in three-space with the same linking number as the curves on the sphere.

In analogy with the definition for a curve in three-space, we define the self-linking number \(\text{SL}(X)\) of a curve \(X(t)\) on the three-sphere to be the linking number of \(X(t)\) with the normalized push-off of \(X(t) + \epsilon U(t)\) where \(\epsilon\) is chosen small enough that the linking number will not change for smaller values. We can then compute \(\text{SL}(X)\) by stereographically projecting both curves into three-space and calculating the linking number there.

Our first observation is that the self-linking number of a curve \(X(t)\) on the three-sphere is not the same as the self-linking number of the curve \(\pi_Q X(t)\) for an arbitrary \(Q\) on the three-sphere not on \(X(t)\). The easiest way to see this is to consider a family of examples.

**Torus Knots on the Three-Sphere**

A \((m,n)\)-Torus Knot on the three-sphere is defined by
\[
X(t) = (\cos(a) \cos(mt), \cos(a) \sin(mt), \sin(a) \cos(nt), \sin(a) \sin(nt))
\]
where \(-\pi \leq t \leq \pi\) and \(a\) is an angle between 0 and \(\pi/4\). Then
\[
X'(t) = (-m \cos(a) \sin(mt), m \cos(a) \cos(mt), -n \sin(a) \sin(nt), n \sin(a) \cos(nt))
\]
with \(s'(t) = \sqrt{m^2 \cos^2(a) + n^2 \sin^2(a)}\) and
\[
X''(t) = (-m^2 \cos(a) \cos(mt), -m^2 \cos(a) \sin(mt), -n^2 \sin(a) \cos(nt), -n^2 \sin(a) \sin(nt)).
\]
Then \(|X''(t)| = \sqrt{m^4 \cos^2(a) + n^4 \sin^2(a)} = s'(t)^2\kappa(t)\) so the curvature
\[
\kappa(t) = \frac{\sqrt{m^4 \cos^2(a) + n^4 \sin^2(a)}}{m^2 \cos^2(a) + n^2 \sin^2(a)}
\]
is constant.

The normal curvature \(\kappa_N(t)\) of a curve on a unit sphere is \(-1\), so the only time that all of the curvature is in the normal direction is when \(\kappa(t) = 1\), i.e. when \(m^4 \cos^2(a) + n^4 \sin^2(a) = (m^2 \cos^2(a) + n^2 \sin^2(a))^2\).
Note that \( X''(t) \) has no \( T(t) \)-component since \( s''(t) = 0 \). The geodesic curvature of \( X(t) \) will be zero if \( X''(t) \) lies in the same direction as \( X(t) \), and this occurs exactly when \( m^2 = n^2 \). We wish to calculate the self-linking numbers of the curves for which the geodesic curvature is never zero, so from now on we will deal only with the case where \( m^2 \neq n^2 \).

**Self-Linking for Torus Knots on the Three-Sphere**

Since \( X''(t) \) has constant length for an \((m,n)\)-Torus Knot, we may obtain the push-off \( Y(t,s) \) of \( X(t) \) in the direction of \( P(t) \) by normalizing the vector \( X(t) + sX''(t) \) for a small value of \( s \). We thus obtain a curve

\[
Y(t, s) = \frac{1}{c}((1 - sm^2) \cos(a) \cos(mt), (1 - sm^2) \cos(a) \sin(nt),
(1 - sn^2) \sin(a) \cos(nt), (1 - sn^2) \sin(a) \sin(nt)).
\]

where \( c = \sqrt{(1 - sm^2)^2 \cos^2(a) + (1 - sn^2)^2 \sin^2(a)} \).

Note that

\[
Y(t, s) = (\cos(b(s)) \cos(mt), \cos(b(s)) \sin(nt),
\sin(b(s)) \cos(nt), \sin(b(s)) \sin(nt))
\]

for some angle \( b(s) \) which is small if \( s \) is small. The problem of computing the self-linking number of the original torus knot is then seen to be equivalent to finding the linking number of two curves \( X(t) \) for two different values of \( a \), namely \( a \) and \( a + h \) for some small \( h \).

To compute the linking numbers of these two curves, we may stereographically project the curves into three-space from the point \( Q = (0, 0, 0, 1) \) to get \( \pi_QX(a, t) \) and \( \pi_QX(a + h, t) \). Figure 4 shows the stereographic projection of a \((3,2)\)-torus knot (as well as the image of its pushoff in the normal direction, defined below). Figure 5 shows the analogous illustration for a \((2,3)\)-torus knot.

Observe first of all that the projection \( \pi_z(\pi_QX(a, t)) \) of \( \pi_QX(a, t) \) into the horizontal plane is a locally convex curve, and for such a curve, the self-linking number is the algebraic number of self-crossings, in this case \( m \). However the linking number of \( \pi_z(\pi_QX(a, t)) \) and \( \pi_z(\pi_QX(a + h, t)) \) is \( m \) plus the algebraic number of values of \( t \) for which

\[
\lim_{h \to 0} \pi_z(\pi_QX(a + h, t) - \pi_QX(a, t)) = 0.
\]

The formula for stereographic projection from the point \( Q = (0, 0, 0, 1) \) is

\[
\pi_QX(a, t) = \frac{1}{1 - \sin(a) \sin(nt)}(\cos(a) \cos(mt), \cos(a) \sin(mt), \sin(a) \cos(nt)).
\]

The subsequent projection to the x-y-plane yields the curve

\[
Y(a, t) = \pi_z \pi_QX(a, t) = \frac{1}{1 - \sin(a) \sin(nt)}(\cos(a) \cos(mt), \cos(a) \sin(mt)),
\]
a locally convex plane curve with \( n(m - 1) \) crossings. This curve will intersect the curve \( Y(a + h, t) \) twice in the neighborhood of each of these crossings, for a total of \( 2n(m - 1) \) crossings.

There will also be crossings when \( Y(a + h, t) = Y(a, t) \), namely when

\[
\frac{\cos(a)}{1 - \sin(a) \sin(nt)} = \frac{\cos(a + h)}{1 - \sin(a + h) \sin(nt)}
\]

so

\[
\cos(a)(1 - \sin(a + h) \sin(nt)) = \cos(a + h)(1 - \sin(a) \sin(nt)).
\]
This simplifies to
\[
\cos(a + h) - \cos(a) \\
= -\cos(a + h) \sin(a) \sin(nt) + \sin(a + h) \cos(a) \sin(nt) \\
= \sin(a) \sin(nt).
\]

As \( h \) goes to 0, the condition for the point of intersection is \( 0 = \sin(a) \sin(nt) \) and, for non-zero \( a \), this gives \( 2n \) values for \( t \) between 0 and \( 2\pi \). The total of crossings is then \( 2n(m - 1) + 2n = 2nm \), and at exactly \( nm \) of them, the curve \( Y(a + h, t) \) crosses the curve \( Y(a, t) \), always in a positive direction. The self-linking number of \( X \) on the 3-sphere is therefore \( nm \), completing the proof of the Theorem.

It follows that the self-linking number of an \((m, n)\)-torus knot on the three-sphere is \( nm \), as predicted by Kuiper, and this is the major result of this paper.

**Self-Linking of a Torus Knot on the**
**Three-Sphere and Self-Linking Numbers**
**of Its Stereographic Projections**

We have observed that stereographic projection of an \((m, n)\)-torus knot from the point \((0, 0, 0, 1)\) on the three-sphere, subsequently projected to a plane, gives a locally convex curve with \(n(m - 1)\) crossings, all of the same sign, leading to a curve in three-space with self-linking number \( n(m - 1) \). Similarly stereographic projection from \((1, 0, 0, 0)\) on the three-sphere leads to a curve with self-linking number \( m(n - 1) \). The first observation is that the self-linking number of the
projection depends on the North Pole of the projection, and neither projection has self-linking number equal to the actual self-linking number on the three-sphere, which is $mn$. The relationship between the point of projection and the self-linking number of the projected curve is related to the geometry of the osculating tube of the torus knot on the three-sphere.

**The Osculating Tube for a Curve on the Three-Sphere**

Just as the osculating tube of a spherical curve $X(t)$ is given as the image of the osculating tube of $\pi_N X(t)$ under inverse stereographic projection, we may obtain the osculating tubes of a curve $X(t)$ on the three-sphere as the inverse stereographic images of the osculating tube of $\pi_N X(t)$. Note that if $N$ is not on any osculating circle of $X(t)$, then the projected curve $\pi_N X(t)$ will have curvature non-zero for all $t$.

If we take a path $Q(t)$ on the three-sphere from $Q(0) = (0, 0, 0, 1)$ to $Q(1) = (1, 0, 0, 0)$ then the self-linking number of the projected curve $\pi_Q(t)X(t)$ will change precisely when a point of this curve passes through the osculating tube of $X(t)$ on the three-sphere, and this is the second and final major result of this paper.

**References**


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