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Polyhedral Catastrophe Theory I: Maps of the Line to the Line[†‡]

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Thom [1] has described seven differentiable catastrophes, and these have been elaborated by Zeeman [2] and by both Thom and Zeeman in this symposium on dynamic systems. In this paper, we present a beginning of the theory of polyhedral catastrophes and, in particular, we examine analogues of each of the four basic differentiable catastrophes which map the line to the line. The unfoldings of these polyhedral singularities are topologically equivalent to those of the differentiable category, so this alternative view of catastrophe theory can give some additional insight into the structure of differentiable unfoldings. The main advantage of the polyhedral theory is that there is a precise algorithm for determining the singular behavior as opposed to the progressively complicated algebraic calculations required in the differentiable theory.

1 Introduction: the quartic case

We begin with an examination of a case where the differentiable and polyhedral theories are in close correspondence. Subsequently we shall

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[‡] The author wishes to thank Thom for introducing him to the subject and Zeeman for several helpful conversations during the work on this paper.

resent more precise definitions and describe some more general examples.

In catastrophe theory, the object of study is an unfolding of f, i.e., a amily of functions $f_u: \mathscr{X} \to R$ from a manifold or manifold-with-oundary $\mathscr{X}$ to a manifold R, usually a subset of cartesian n-space R^n, vhere u ranges over a parameter space or control space $\mathscr{U}$, usually itself subset of a cartesian m-space for some m, and where $f = f_0$ for some nterior point 0 of $\mathscr{U}$.

We shall examine a pair of functions, $f(x) = x^4$ in the differentiable ategory and a polygonal function g defined on $-2 \le x \le 2$, which is iven by $g(-2) = 2, g(-1) = 0 = g(0) = g(1)$, and $g(2) = 2$, and which s linear on each subinterval. Such a polygonal function g corresponds to he sequence (2, 0, 0, 0, 2).

In each case $\mathscr{U}$ will be the unit square $\{(u_1, u_2) \mid -1 \le u_1 \le 1, -1 \le u_2 \le 1\}$ in R^2. For a given $u = (u_1, u_2)$, we set $f_u(x) = x^4 + u_1x^2 + u_2x$, and set g_u equal to the polygonal function with sequence (2, $\frac{1}{2}(u_1 - u_2)$, 0, $\frac{1}{2}(u_1 + u_2)$, 2).

In each case, the value $\mathbf{0} = (0, 0)$ yields the original function: $f_0 = f$, $g_0 = g$.

A smooth function on the real line is degenerate if it has a degenerate critical point, i.e., if for some $x, f_u'(x) = 0$ and $f_u''(x) = 0$. A polygonal function on the real line is degenerate if it has a horizontal inflection edge, i.e., for four successive vertices v_1, v_2, v_3, v_4, we have $g(v_1) \ge g(v_2) = g(v_3) \ge g(v_4)$, or $g(v_1) \le g(v_2) = g_3(v) \le g(v_4)$.

In the smooth case, we have a degenerate function f_u only when we can solve simultaneously the two conditions $f_u'(x) = 4x^3 + 2u_1x + u_2 = 0$ and $f_u''(x) = 12x^2 + 2u_1 = 0$. Eliminating x, we obtain the condition

$$-8{u_1}^3 = 27{u_2}^2.$$

In the polygonal case, we have a degenerate function g_u only when either $u_1 - u_2 = 0 \ge u_1 + u_2$, or $u_1 - u_2 \le 0 = u_1 + u_2$, i.e., either $u_1 = u_2 \le 0$, or $u_1 = -u_2 \le 0$. The graphs of these two conditions are topologically similar—each is topologically a cone from the origin over a two-point subset of the boundary of $\mathscr{U}$. In the smooth case, there is a cusp at the origin which does not appear in the polygonal case where the cone is an actual geometric cone from the origin (Figure 1). In each case, the shaded area indicates the values of u where f_u or g_u has three critical points; for u in the unshaded area, there is exactly one critical point.

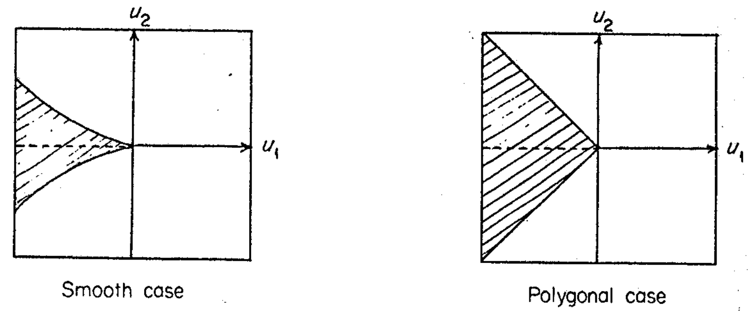

Figure 1

We now graph the location of the critical points of each function f_u or g_u in the space $\mathscr{X} \times \mathscr{U}$. In the smooth case, we obtain a cubic surface $u_2 = -4x^3 - 2u_1x$ (Figure 2).

In the polygonal case we can only have (interior) critical points at the vertices $x = -1$, 0, or 1, unless we have a degenerate function g_u, in which case we may have an entire interval of critical points. We obtain a polyhedral surface in $\mathscr{X} \times \mathscr{U}$ (Figure 3).

Next we graph the location of the critical values of functions f_u or g_u in the space $\mathscr{U} \times R$. In the smooth case, we must eliminate x from the

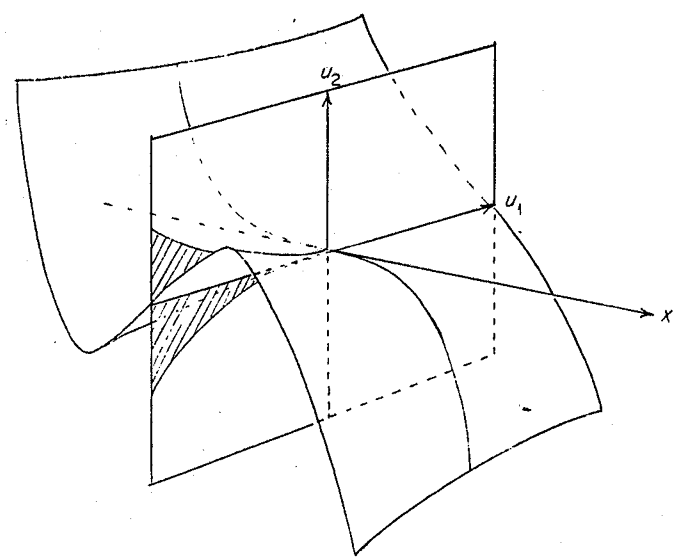

Figure 2

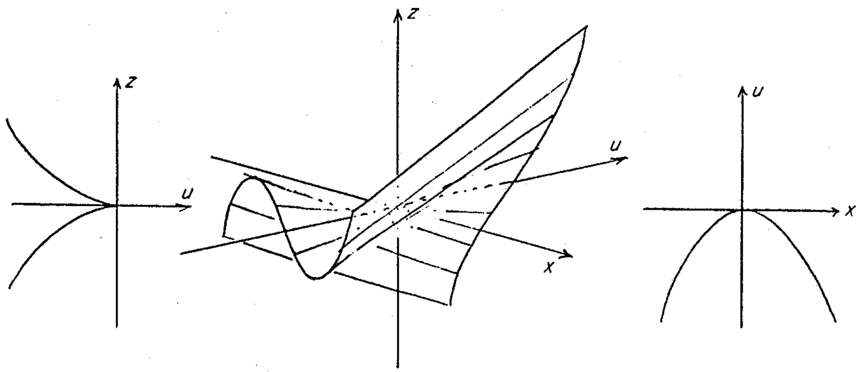

Figure 3

two expressions $z(x) = x^4 + u_1x^2 + u_2x$ and $4x^3 + 2u_1x + u_2 = 0$. (Figure 4).

In the polyhedral case the situation is somewhat simpler—the critical value for a function g_u can only occur at $\frac{1}{2}(u_1 + u_2)$, $\frac{1}{2}(u_1 - u_2)$, or 0, so we need only check out the points on a finite set of planes (Figure 5).

Each of these surfaces has an interval of self-intersection where there are two distinct critical points which have the same critical value. For each value $u_2 = 0$, $u_1 < 0$ in the smooth case, and for $u_1 < 0$, $u_2 = 0$ in the polygonal case, we have an even function $f_u(-x) = f_u(x)$ or $g_u(-x) = g_u(x)$, and the absolute minimum is taken on at two distinct points.

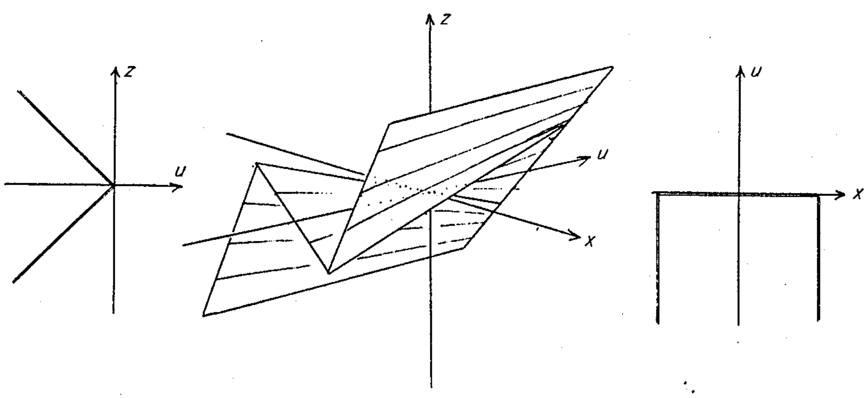

Figure 4

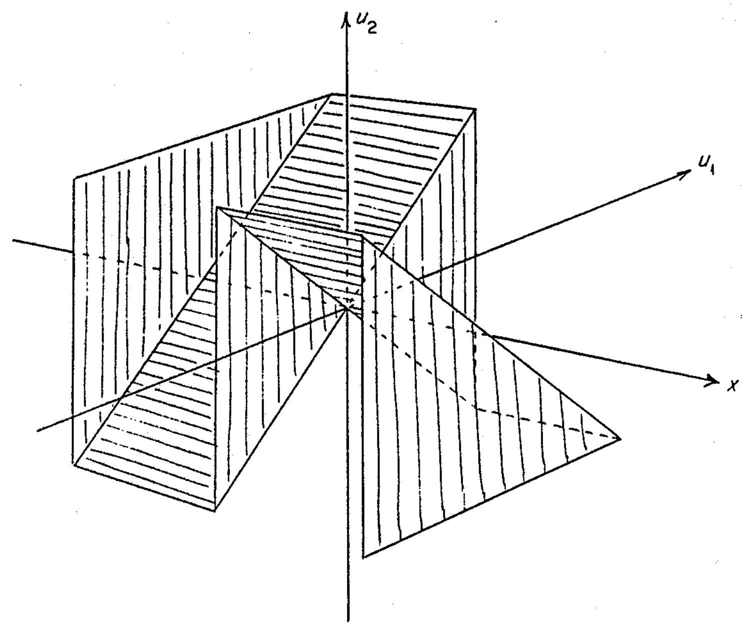

Figure 5

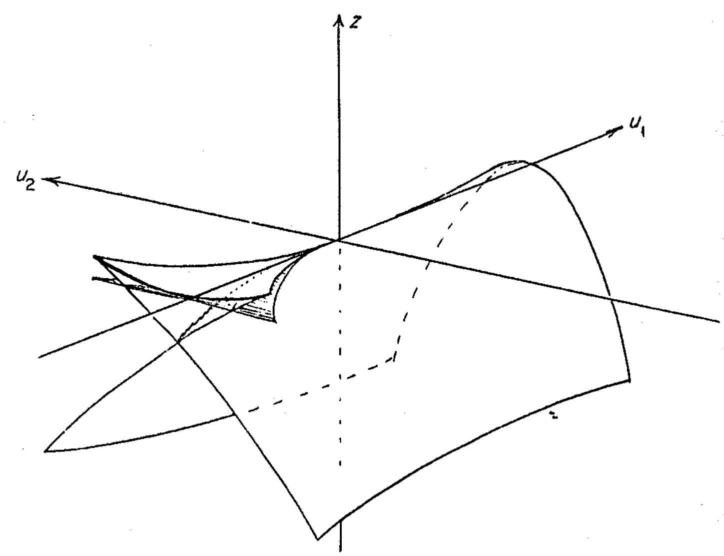

Figure 6

It is for precisely these points u in the control space that the Maxwell convention, which selects the lowest critical point of a function, fails to work unambiguously, and the set of such points u is called the *Maxwell set* of the family. More generally the *bifurcation set* of a family parametrized by $\mathscr{U}$ is the set of u for which the function has two critical points with the same critical value. The graph of control points vs. critical values will have self-intersections precisely above the control points in the bifurcation set of the family.

Just as the degeneracy set in $\mathscr{U}$ is topologically a cone from the origin over a 0-sphere (a two-point set) in the boundary square $\partial\mathscr{U}$, each of the graphs above in $\mathscr{U} \times R$ is topologically a cone over the cylinder $\partial\mathscr{U} \times R$.

Since the range in each case is a finite interval $-1 \leq z \leq 1$, we may consider this graph to lie in the cylinder $\partial\mathscr{U} \times [-1, 1]$ which we may then project stereographically from the center of $\mathscr{U} \times \{1\}$ to obtain an annulus in the plane (Figure 6).

2 The cubic and quadratic cases

We now return to two lower-dimensional cases and make specific some of the other notions in catastrophe theory. In catastrophe theory we begin with a function f and try to find a family $\{f_u \mid u \in \mathscr{U}\}$ of functions with a finite-dimensional parameter space $\mathscr{U}$ (the *control* space) so that this family $\mathscr{F}_{\mathscr{U}}$ is a factor in a product neighborhood N of f in some function space $\mathscr{F}$. This means that if $\pi: N \to \mathscr{F}_{\mathscr{U}}$ denotes projection to the family $\mathscr{F}_{\mathscr{U}}$ and if $\pi(g) = \pi(f_u) = (f_u, 0)$, then g and f_u are equivalent in $\mathscr{F}$, i.e., there are homeomorphisms h_1 of $\mathscr{X}$ and h_2 of R such that $g \circ h_1 = h_2 \circ f_u$. The minimum dimension of such a control neighborhood $\mathscr{U}$ is called the *codimension* of the function f.

In the case of a differentiable function f, the function space $\mathscr{F}$ is the space of all differentiable functions equipped with the C^∞ topology. For functions with domain and range in the real numbers, the codimension of a function f is one less than the maximum number of critical points of any function near f in $\mathscr{F}$. To see this in the example of a monomial $f(x) = x^n$, observe that we may find a neighborhood N of f in $\mathscr{F}$ so small that $|f^{(n)}(x) - g^{(n)}(x)| < \varepsilon < 1$ for all x in the domain. Since $f^{(n)}(0) = n! \neq 0$, for all x in a sufficiently small neighborhood of 0, we have $g^{(n)}(x) \neq 0$. But if g has k critical points, then g' has k zeros, so g'' has $k - 1$ zeros, and generally $g^{(j)}$ has $(k - j + 1)$ zeros. Thus if we have a function g with n critical points in the neighborhood N of f, we have at least $n - n + 1$ zeros of $g^{(n)}$, a contradiction. Thus the maximum number of critical points we can get is $n - 1$, so the codimension is predicted to be $n - 2$. We obtain a representation of $\mathscr{F}_{\mathscr{U}}$ by taking first the projection of a function g to its nth order jet, i.e.,

$$\pi(g) = \sum_{k=0}^{n} (1/k!)g^{(k)},$$

then multiplying by a constant to make $\pi(g)$ monic, then making a linear transformation in the domain to eliminate the $(n - 1)$th-order term and a translation in the range to eliminate the constant. In this way, we have a composite projection of g to a function

$$f_u(x) = x^n + u_1x^{n-2} + \cdots + u_{n-1}x^2 + u_{n-2}x$$

parametrized by points $u = (u_1, u_2, \ldots, u_{n-2})$ in $\mathscr{U}$.

For a cubic $g(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$, there is a unique inflection point with $g'(x) = 0 = 6ax + b$, and if we translate axes so that this inflection point goes to the origin, then the cubic takes the form

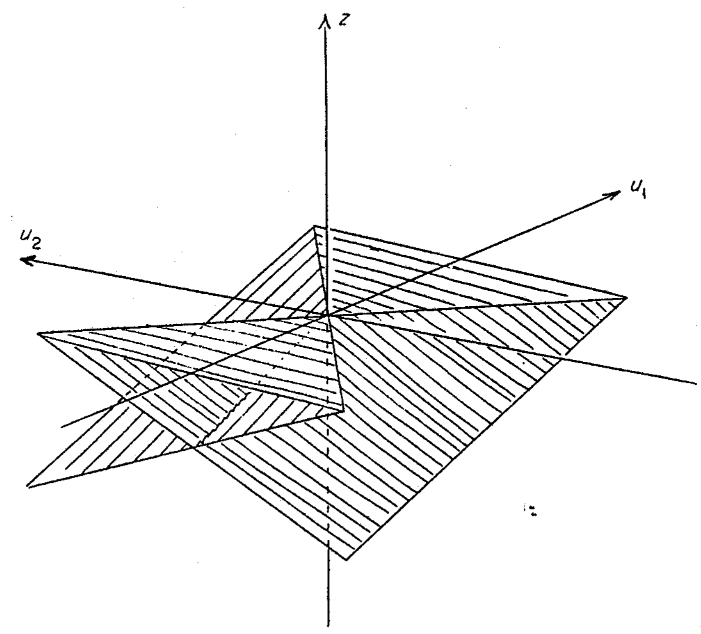

Figure 7

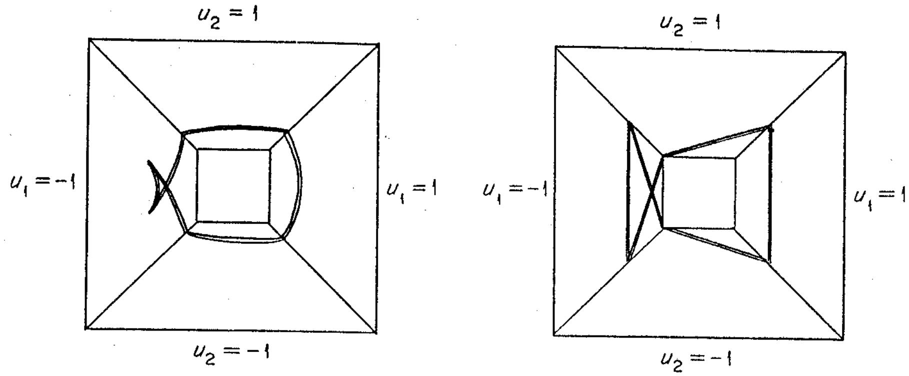

Figure 8

$f_u(x) = x^3 + ux$. The corresponding polynomial function is defined by $g(-2) = -2$, $g(-1) = -u$, $g(1) = u$, $g(2) = 2$.

In each case, the degeneracy set consists of $\{u = 0\}$, and the bifurcation set is empty. We may then graph the entire space $\mathscr{X} \times \mathscr{U} \times R$, where $\mathscr{U} = [-1, 1]$. (See Figure 7.) In the first projection we map onto $\mathscr{U} \times R$ to get the critical values, i.e., the points (u, z), with $z = x^3 + ux$ and $0 = 3x^2 + u$. Eliminating x, we obtain $27z^2 = -8u^3$. For the second projection, we have the critical points $u = -3x^2$ in $\mathscr{X} \times \mathscr{U}$.

The polygonal case is shown in Figure 8.

Note that although each of the projection to $\mathscr{U} \times R$ and to $\mathscr{X} \times \mathscr{U}$ is polyhedral, the graph in $\mathscr{X} \times \mathscr{U} \times R$ is a collection of ruled surfaces, and is *not* polyhedral. This appears to be a significant disadvantage in the piecewise linear theory.

The quadratic case is trivial. Any quadratic function $g(x) = ax^2 + bx + c$ with $a > 0$ is equivalent to $\tilde{g}(x) = ax^2$ by a translation of axes taking the vertex of the parabola to the origin, and multiplication by $1/a$ transforms $\tilde{g}$ to the function $f(x) = x^2$. The codimension of $f(x) = x^2$ is, therefore, zero. The corresponding polygonal situation is given by the function with $g(-1) = 2$, $g(0) = 0$, $g(1) = 2$, and any polygonal function with exactly one critical point can be transformed into g by polygonal functions in $\mathscr{X}$ and R with no critical points.

3 The quintic

In order to study the quintic function $f(x) = x^5$, we must consider the family $f_u(x) = x^5 + u_1x^3 + u_2x^2 + u_3x$. Critical points occur when $f_u'(x) = 0 = 5x^4 + 3u_1x^2 + 2u_2x + u_3$, and the degeneracy set in the control space $\mathscr{U}$ is given by solving simultaneously $f_u'(x) = 0$ and $f_u''(x) = 0 = 20x^3 + 6u_1x + 2u_2$ to get a surface in $\mathscr{U}$.

In the polygonal case, the situation is considerably simpler. We may take g_u determined by the sequence $(-2, -, u_1, u_2, u_3, 2)$, and as before the degeneracy set consists of points u, where there is a horizontal inflection edge. This situation occurs in the following cases: (1) $u_1 = 0$, $u_2 > 0$; (2) $0 < u_1 = u_2 < u_3$; (3) $0 > u_1 = u_2 > u_3$; (4) $u_1 < u_2 = u_3 < 2$. We may describe this set precisely in cartesian three-space; see Figure 9.

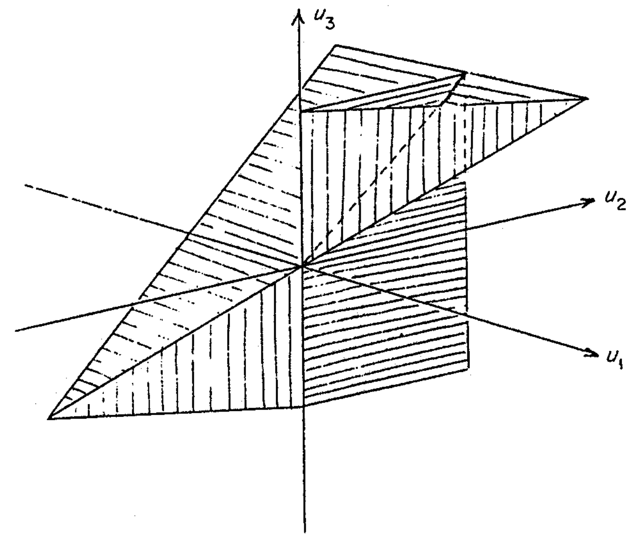

Figure 9

A more interesting parametrization occurs if we take $g_u = (-2, \frac{1}{2}(u_1 + u_2), u_3, -u_3, \frac{1}{2}(-u_1 + u_2), 2)$, so that the degeneracy set is given by:

(1) $\frac{1}{2}(u_1 + u_2) = u_3 < -u_3$, i.e., $\frac{1}{2}(u_1 + u_2) = u_3 < 0$,
(2) $\frac{1}{2}(u_1 + u_2) < u_3 = 0 < \frac{1}{2}(-u_1 + u_2)$,
(3) $\frac{1}{2}(u_1 + u_2) > u_3 = 0 > \frac{1}{2}(-u_1 + u_2)$,
(4) $u_3 < -u_3 = \frac{1}{2}(-u_1 + u_2)$, i.e., $\frac{1}{2}(u_1 - u_2) = u_3 < 0$.
(See Figure 10.)

With this parametrization, the degeneracy set for this fifth-order function is identical with the graph in $\mathscr{U} \times R$ for the fourth-order function

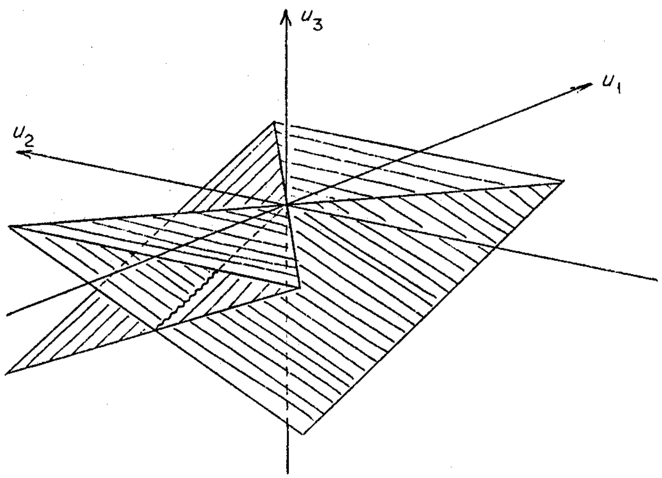

Figure 10

g given by (2, 0, 0, 0, 2), and this is the same situation that occurs with respect to the degeneracy set of $(2, u_1 + u_2, 0, u_1 - u_2, 2)$ compared with the graph in $\mathscr{U} \times R$ of the cubic function $(-2, -u, u, 2)$.

In order to clarify this relationship and to give a presentation which is independent of the parametrizations used, we switch from rectangular coordinates where the zero value has a privileged place to barycentric coordinates. In the quartic example g given by (2, 0, 0, 0, 2), for example, we consider the family g_u given by $(2, u_1, u_2, u_3, 2)$, where $u = (u_1, u_2, u_3)$, $0 \leq u_i \leq 1$, and $u_1 + u_2 + u_3 = 1$. With such a parametrization we have degeneracy only when $u_1 = u_2 > u_3$ or $u_1 < u_2 = u_3$, and we have two minima with the same critical value when $u_1 = u_3 < u_2$.

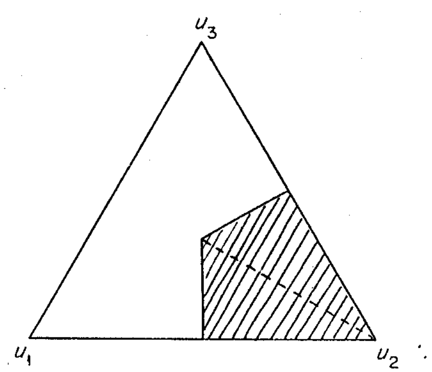

Figure 11

The control space is then a triangle (Figure 11), and the degeneracy set and bifurcation set are indicated. The shaded region corresponds to values u for which g_u has three critical points.

We may then graph the critical value set in $\mathscr{U} \times R$, as before, in a cartesian system (Figure 12).

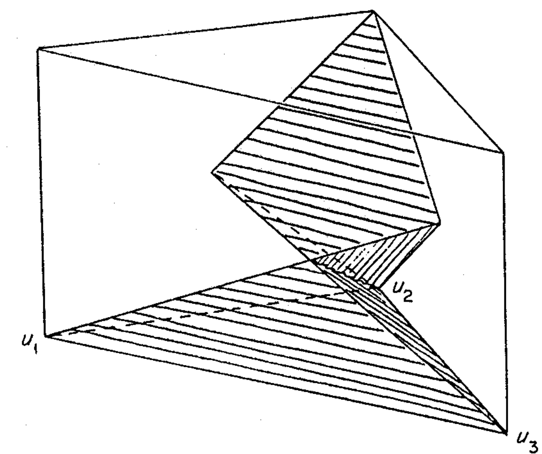

Figure 12

For the quintic example, we have g given by $(-2, u_1, u_2, u_3, u_4, 2)$, again with $0 \leq u_i \leq 1$, $\sum_{i=1}^{4} u_i = 1$. Here we have a tetrahedron as control space and degeneracy set given by: (1) $u_1 = u_2 < u_3$; (2) $u_1 < u_2 = u_3 < u_4$; (3) $u_1 > u_2 = u_3 > u_4$; (4) $u_2 < u_3 = u_4$. (See Figure 13.)

Again the degeneracy set is topologically and geometrically a cone from the barycenter over a self-intersecting curve on the boundary of the sim-

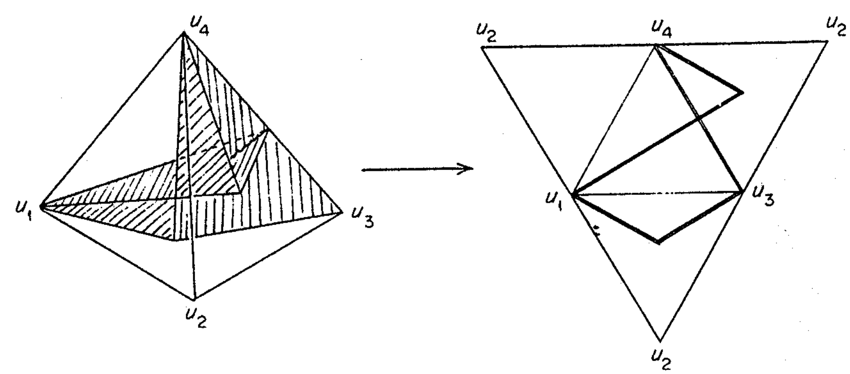

Figure 13

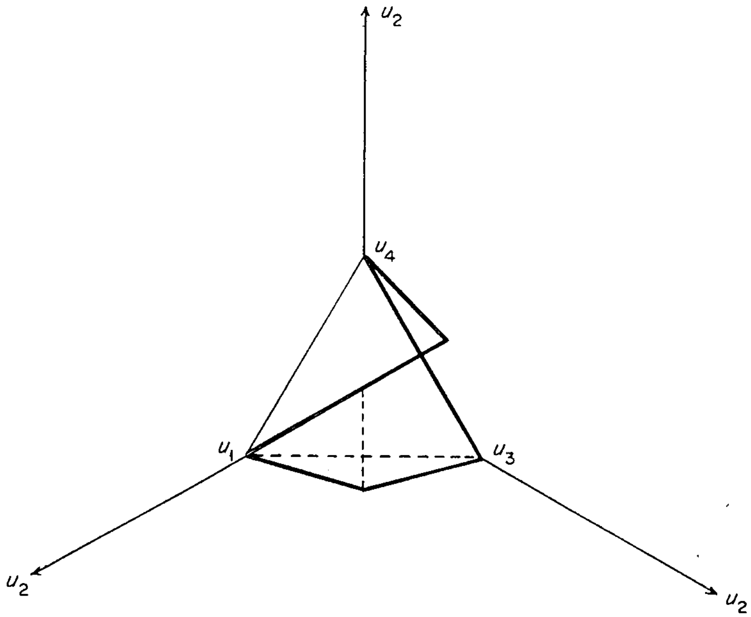

Figure 14

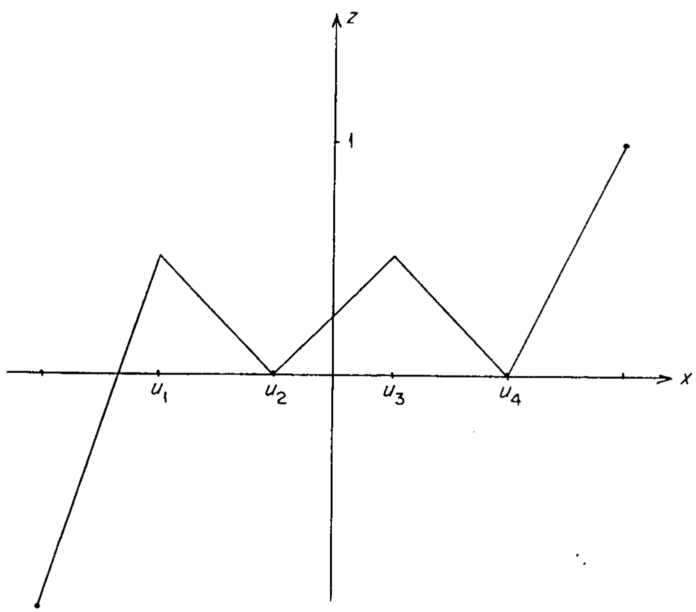

Figure 15

plex which is displayed in a disassembled form. In a central projection from the point u_2, we obtain Figure 14.

The bifurcation set for this quintic will then be $u_1 = u_3 \geq u_2, u_4$, and $u_2 = u_4 \leq u_1, u_3$, and again this will be a cone over the intersection of this set with the boundary of the tetrahedron (given by dotted lines in the previous diagram). There we can describe the sets more simply (Figure 15): If $u_2 = 0$, we get $u_1 = u_3 \geq u_4$, and if $u_4 = 0$, we get $u_1 = u_3 \geq u_2$. If $u_2 = 0 = u_4$, then we have the whole edge $u_1 + u_3 = 1$. The only points of the bifurcation set on the faces $u_1 = 0$ or $u_3 = 0$ are the ends of this segment, $u_1 = 1$ or $u_3 = 1$, and all others zero. Note that these two parts of the bifurcation set intersect at a point $u_1 = \frac{1}{2} = u_3$, $u_2 = 0 = u_4$, which corresponds to a function with two local maxima with the same value and two local minima also with the same value.

4 The sextic

In the case of a sextic equation $f(x) = x^6$ in the family $f_u(x) = x^6 + u_1x^4 + u_2x^3 + u_3x^2 + u_4x$. the control space will be four-dimensional, and the degeneracy set will be a cone over the origin in 4-space. We can get a topological picture of the degeneracy set then by eliminating x from the two equations:

$$f_u'(x) = 0 = 6x^5 + 4u_1x^3 + 3u_2x^2 + 2u_3x + u_4,$$

$$f_u''(x) = 0 = 30x^4 + 12u_1x^2 + 6u_2x + 2u_3.$$

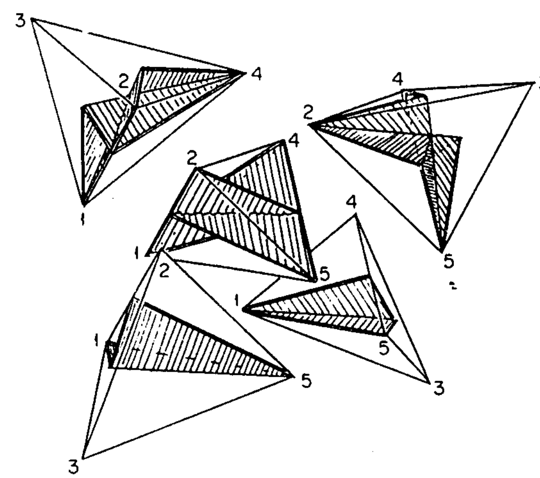

Figure 16

We intersect this set with the sphere $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$ to get a surface in this 3-sphere, which we could then study by means of stereographic projection.

Again in the polygonal case, the situation is considerably simpler. We consider the family of functions g_u corresponding to $(2, u_1, u_2, u_3, u_4, u_5, 2)$ with $0 \leq u_i \leq 1$, $\sum_{i=1}^{5} u_i = 1$.

We describe the degeneracy set for this sextic function by giving the intersection with each of the faces of the control simplex:

$u_1 = 0$: $u_2 = u_3 < u_4$, $u_2 < u_3 = u_4 < u_5$, $u_2 > u_3 = u_4 > u_5$, $u_3 < u_4 = u_5$,

$u_2 = 0$: $u_3 = u_4 < u_5$, $u_3 < u_4 = u_5$,

$u_3 = 0$: $u_1 = u_2$, $u_4 = u_5$,

$u_4 = 0$: $u_3 = u_2 < u_1$, $u_3 < u_2 = u_1$,

$u_5 = 0$: $u_4 = u_3 < u_2$, $u_4 < u_3 = u_2 < u_1$, $u_4 > u_3 = u_2 > u_1$, $u_3 > u_2 = u_1$.

These components fit together to form a 2-sphere in the 3-sphere with a curve of self-intersection and a pair of points at which the sphere has local self-intersections. (See Figure 16.)

The bifurcation set in this case consists of pieces:

$u_1 = 0$: $u_2 = 0$, $u_4 = 0$, $u_2 = u_4 > u_3$, u_5; $u_3 = u_5 < u_4$, u_2, $u_3 < u_2 = u_5 < u_4$,

$u_2 = 0$: $u_4 = 0$, $u_5 = 0$,

$u_3 = 0$: $u_1 = 0$, $u_5 = 0$, $u_5 < u_1 = u_4 < u_2$; $u_1 < u_2 = u_5 < u_4$,

$u_4 = 0$: $u_2 = 0$, $u_1 = 0$,

$u_5 = 0$: $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_2 = u_4 > u_3$, u_1, $u_1 = u_3 < u_2$, u_4.

The self-intersection points of the bifurcation set include points with two maxima at the same level and two minima at the same level, e.g., $(2, u_1, u_2, u_1, u_2, u_5, 2)$ with u_1 and $u_5 < u_2$ and $(2, u_1, u_2, u_3, u_2, u_3, 2)$ with u_1 and $u_3 < u_2$ or $(2, u_1, u_2, u_3, u_2, u_1, 2)$ with u_1 and $u_3 < u_2$, and a family of points with three local minima at the same level $(2, u_1, u_2, u_1, u_3, u_1, 2)$. We have as well a triple point set in the bifurcation set

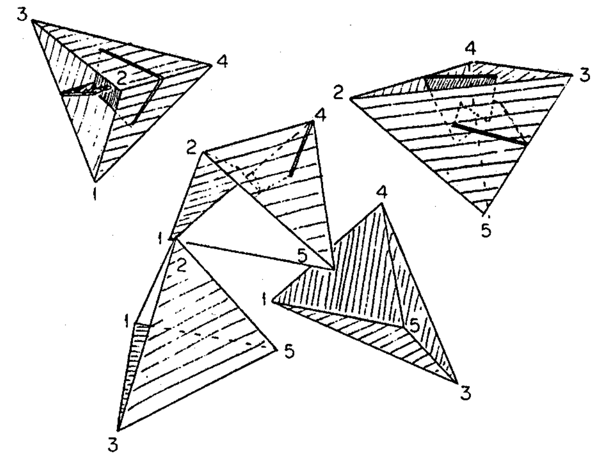

Figure 17

corresponding to points $(2, u_1, u_2, u_1, u_2, u_1, 2)$ with $u_1 < u_2$, with three minima at the same level and three maxima at the same level. This last set meets the boundary of the simplex $\mathscr{U}$ at a single point, $(2, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 2)$. (See Figure 17.)

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